HOMEWORK 4

1. **Exercise.** Once more: Show that the free boson

$$[\alpha_\lambda \alpha] = \lambda$$

is not associative $\alpha(\alpha\alpha) \neq (\alpha\alpha)\alpha$ while the free Fermion is:

$$[\varphi_\lambda \varphi] = 1 \Rightarrow \varphi(\varphi\varphi) = (\varphi\varphi)\varphi$$

2. **Exercise.** Let $\psi^\pm$ form a system of *charged Fermions*, that is the non-vanishing brackets are given by $[\psi^\pm_\lambda \psi^\mp] = 1$. Define $\alpha := \psi^+\psi^-$ and $L = \frac{1}{2}(\partial\psi^+)\psi^- + \frac{1}{2}(\partial\psi^-)\psi^+$ and $\tilde{L} = \frac{1}{2}\alpha\alpha$. Show that

(a) L and $\tilde{L}$ are Virasoro fields of central charge 1.
(b) With respect to both L and $\tilde{L}$ $\psi^\pm$ are primary of conformal weight 1/2 and α is primary of conformal weight 1.
(c) $L = \tilde{L}$.

3. **Exercise.** Let V be a Lie conformal algebra finitely generated as a $k[\partial]$ module and let $\mathfrak{g} = \mathrm{Lie}(V)$ be the corresponding Lie algebra.

(a) Show that there exists a filtration

$$\cdot \supset \mathfrak{g}_{-1} \supset \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset$$

in $\mathfrak{g}$ such that $\mathfrak{g} = \cup \mathfrak{g}_j$, $0 = \cap \mathfrak{g}_j$ and $[\mathfrak{g}_j, \mathfrak{g}_k] \subset \mathfrak{g}_{j+k}$.
(b) Show that the associated graded $\mathrm{gr}\mathfrak{g} = \oplus \mathfrak{g}_j/\mathfrak{g}_{j+1}$ has finite dimensional graded components (check if you need to add the adjective "free" in the statement, or free modulo some central elements).

This filtrations allow us to define the completion $\bar{\mathfrak{g}}$ of $\mathfrak{g}$

4. **Exercise.** Let $V = \mathrm{Cur}(\mathfrak{g})$ be the current algebra of a finite dimensional simple Lie algebra $\mathfrak{g}$, that is

$$[a_\lambda b] = [a, b] + \lambda(a, b).$$

(a) Show that $Lie(V) \simeq \hat{\mathfrak{g}} := \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ the affine Kac-Moody Lie algebra.
(b) Consider the filtration obtained in the previous exercise and the corresponding completion, show that it is $\mathfrak{g}((t)) \oplus \mathbb{C}$.
(c) Consider now the universal enveloping algebra $U = U(\hat{fg})$. Show that the filtration $U_i \supset U_{i+1}$ so that an element of U_i consists of an infinite series $\sum_n u_n$ where all but finitely many elements lie in $U\mathfrak{g}_i$ is compatible with the algebra structure and satisfies the usual $U = \cup U_i$, $0 = \cap U_i$.
(d) Consider $\bar{U}$ the completion of U with respect to this filtration. Show that the Sugawara construction

$$L_m := \frac{1}{2(k + h^\vee)} \sum_n : a^i_n a^i_{m-n}$$

produces elements in U^{comp} which are not in U.

Representations of $\hat{\mathfrak{g}}$ that can be extended to U^{comp} are called *smooth*.

Date: Due: April 8th.