HOMEWORK 3

1. **Exercise.** Let V be a vector space and $\mathcal{F}(V) := \mathrm{Hom}_k\big(V, V((t))\big)$ be the space of quantum fields in V. Let $T \operatorname{End}(V)$ be an even endomorphism and let $a(z)\mathcal{F}(V)$ be translation covariant, that is $[T, a(z)]\partial_z a(z)$. Suppose there exists a vector $|0\rangle \in V$ such that $T|0\rangle = 0$. Show that
   (a) $a(z)|0\rangle \in V[[z]]$, so we can define $V \ni a := a(z)|0\rangle|_{z=0}$.
   (b) $a(z)|0\rangle = e^{zT}a$.
   (c) Deduce that if $a = 0$ then $a(z)|0\rangle = 0$.

2. **Exercise.** Let V be a vector space and a, b, c be quantum fields. Define the n-th product of fields to be:

$$(a_{(n)}b)(w) = \mathrm{res}_z\Big(i_{|z|>|w|}(z-w)^n a(z)b(w) - (-1)^{ab} i_{|w|>|z|}(z-w)^n b(w)a(z)\Big).$$

Prove
   (a) $a_{(n)}b \in \mathcal{F}(V)$.
   (b) If a, b are translation covariant then $\partial_z a(z)$ and $a_{(n)}b$ are translation covariant.
   (c) if a, b, c are pairwise local then $a_{(n)}b$ and c are a local pair.
   (d) Show that $\partial_w a(w) = (a_{(-2)} \mathrm{Id}_V)(w)$, deduce that if a, b is a local pair then $\partial_z a(z), b(z)$ is a local pair

3. **Exercise.** Let V be a vertex algebra. Consider the quotient $\mathfrak{g} := V((t))/\sim$ where the equivalence relation is defined by $Ta \otimes f(t) \sim -a \otimes f'(t)$. Define the bracket

$$[a \otimes t^m, b \otimes t^n] = \sum_{j \geq 0} \binom{m}{j} \big(a_{(j)}b\big) \otimes t^{m+n-j}.$$

Show that $\mathfrak{g}$ with this bracket is a Lie (super)algebra. Notice that we do not need the whole structure of vertex algebra but just the positive products $a_{(j)}b$ with $j \geq 0$.

4. **Exercise.** Let $\mathscr{L}$ be a Lie algebra over a commutative algebra $\mathscr{O}$, define $\mathscr{L}^* := \mathcal{H}om_{\mathscr{O}}(\mathscr{L}, \mathscr{O})$ the dual $\mathscr{O}$-module. Let $\mathscr{A} := \mathrm{Sym}_{\mathscr{O}} L^*[-1]$ this is naturally a $\mathbb{Z}$-graded commutative (super) algebra. The Lie algebra $\mathscr{L}$ acts on $\mathscr{L}^*$ via the adjoint action and by the Leibniz rule we have an action of $\mathscr{L}$ on $\mathscr{A}$ **by derivations** of the commutative algebra structure. Recall the dgla $L_\dagger := cone(id_L)$ from Exercise 3(b) in the previous homework.
   (a) Forgetting about the differential of $L_\dagger$ (that is consider $L_\dagger$ as a $\mathbb{Z}$-graded Lie superalgebra), show that the action of L on $\mathscr{A}$ extends to an action of $L_\dagger$. [Hint: the copy $L[1] \subset L_\dagger$ in degree -1 acts by contractions.
   (b) Show that there exists a differential $\delta : \mathscr{A} \to \mathscr{A}[-1]$ odd of degree 1 such that $\delta^2 = 0$ such that the action of $L_\dagger$ on $\mathscr{A}$ is compatible with the differentials, namely $\mathscr{A}$ is a commutative dga with an actions by derivations of the dgla $L_\dagger$. [Hint: read the next exercise]

5. **Exercise.** Think about how to state the previous exercise when $\mathscr{O}$ is the algebra of functions on a space, $\mathscr{L}$ is the (Lie) algebra of vector fields on this space and $\mathscr{A}$ is the de Rham complex of that space. Deduce that (b) above is equivalent to Cartan's magic formula

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Date: Due: April 1st.