HOMEWORK 2

1. **Exercise.**
   (a) Let A be a $\mathbb{Z}$-graded associative super algebra such that the $\mathbb{Z}/2\mathbb{Z}$ grading is induced from the $\mathbb{Z}$-grading. Let $\mathfrak{g} = A$ as a graded vector superspace with the bracket defined by $[a, b] = ab - (-1)^{ab}ba$. Show that $\mathfrak{g}$ is naturally a $\mathbb{Z}$-graded Lie superalgebra.
   (b) Let V be a $\mathbb{Z}$ graded super vector space such that the $\mathbb{Z}/2\mathbb{Z}$ grading is induced from the $\mathbb{Z}$-grading. Show that $\mathfrak{gl}(V) = \mathrm{End}(V)$ is naturally a $\mathbb{Z}$-graded Lie superalgebra.
   (c) Let A as in (a) be also (super)commutative. Show that $\mathrm{Der}(A, A) \subset \mathfrak{gl}(A)$ is a $\mathbb{Z}$-graded sub Lie superalgebra. Where did you use supercommutativity?

2. **Exercise.** A little bit more generally, Let us define first a dga. That is A is a $\mathbb{Z}$ graded associative superalgebra (the $\mathbb{Z}/2\mathbb{Z}$ grading does not need to be compatible with the $\mathbb{Z}$-grading). Denote by $A[1]$ the vector space A with simultaneous shifts on the $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ gradings. That is $A[1] = k[1]\otimes A$, where $k[1]$ is an odd copy of the field k put in degree -1. A is endowed with an odd derivation of degree 1, that is $\delta : A \to A[1]$ preserves degrees and satisfies $\delta(ab) = (\delta a)b+(-1)^a a\delta b$ for all $a, b \in A$. We require $\delta^2 = 0$.

Similarly define a dgla $\mathfrak{g}$ as a $\mathbb{Z}$-graded Lie superalgebra (the $\mathbb{Z}/2\mathbb{Z}$ grading does not need to be compatible with the $\mathbb{Z}$-grading) endowed with a degree 1 odd derivation δ, that is $\delta[a, b] = [\delta a, b] + (-1)^a[a, \delta b]$. We require $\delta^2 = 0$.

Finally define a complex as a $\mathbb{Z}$-graded supervector space (the $\mathbb{Z}/2\mathbb{Z}$ grading does not need to be compatible with the $\mathbb{Z}$-grading) endowed with an odd degree 1 map δ, that is $\delta : V \to V[1]$ such that $\delta^2 = 0$. Equivalently, a complex is a $\mathbb{Z}$-graded vector superspace with an action of the commutative Lie superalgebra $k[-1]$.

   (a) Show that if A is a dga, then A^{Lie} (that is A with the usual commutator) is a dgla.
   (b) Show that if V is a complex then $\mathrm{End}_k(V)$ is a dgla.
   (c) Show that if A is a dga then $\mathrm{Der}(A, A)$ is a dgla [Hint. Notice that if $a \in A$ has $\mathbb{Z}$-degree k and $\mathbb{Z}/2\mathbb{Z}$ degree $\bar{l}$ then $\mathrm{ad}(a) \in \mathrm{End}(A)$ is a derivation of $\mathbb{Z}$ degree k and $\mathbb{Z}/2\mathbb{Z}$ degree $\bar{l}$.]

3. **Exercise.** Even more generally a $\mathbb{Z}$ graded superalgebra A is said to be *left symmetric* if it satisfies

$$(ab)c - a(bc) = (-1)^{ab}\Big((ba)c - b(ac)\Big).$$

Show that A^{Lie} with the usual commutator $[a, b] = ab-(-1)^{ab}ba$ is a $\mathbb{Z}$-graded Lie superalgebra. If A is endowed with a differential δ as before, so is A^{Lie}.

4. **Exercise.** Let $T : V \to W$ be a map of complexes. That is $T\delta_V = \delta_W T$ and T preserves degrees. We define the complex $\mathrm{cone}(T) = V[1] \oplus W$ with the differential given by

$$\delta_{\mathrm{cone}(T)} = \begin{pmatrix} \delta_{V[1]} & 0 \\ T[1] & \delta_W \end{pmatrix}$$

   (a) Show that $\mathrm{cone}(T)$ is indeed a complex, that is $\delta^2 = 0$.
   (b) If $\varphi : \mathfrak{g} \to \mathfrak{g}'$ is a morphism of dgla's (that is a map of complexes preserving the brackets) then show that $\mathrm{cone}(T)$ is naturally a dgla. Let $\mathfrak{g}$ be a usual Lie algebra concentrated in degree 0, describe $\mathrm{cone}(id)$.

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5. **Exercise.** Let $\mathfrak{g} = \mathfrak{g}_{\bar{1}} = \mathbb{C}$ be the one dimensional purely odd commutative Lie superalgebra. Let $\mathfrak{g}((t))$ be the corresponding infinite dimensional commutative super Lie algebra. Let $F(\mathfrak{g}) = \mathfrak{g}((t)) \oplus \mathbb{C}K$ be its central extension as follows, choose a skew-supersymmetric bilinear pairing on $\mathfrak{g}$ given by $(a, b) = ab$ and define the bracket

$$[a \otimes f(t), b \otimes g(t)] = (a, b) \oint f(t)g(t)dt$$

while K is central. Show that $F(\mathfrak{g})$ is an infinite dimensional super Lie algebra.

Define the basis $\left\{\psi_n = 1 \otimes t^{n-1/2}, K\right\}_{n \in 1/2+\mathbb{Z}}$ and show that $F(\mathfrak{g})$ is naturally a $\frac{1}{2}\mathbb{Z}$-graded super Lie algebra.

6. **Exercise.** Let $F(\mathfrak{g})$ be the super Lie algebra of the previous exercise. Consider its Fock representation V, that is a representation with a vector $|0\rangle$ such that $\varphi_n|0\rangle = 0$ for all $n > 0$ and $K|0\rangle = |0\rangle$. Define the operators

$$L_m = -\frac{1}{2} \sum_{k \in 1/2+\mathbb{Z}} \left(k + \frac{1}{2}\right) \varphi_k \varphi_{m-k}, \qquad m \in \mathbb{Z}.$$

Show that they produce a representation of the Virasoro Lie algebra of central charge $c = 1/2$.

7. **Exercise.** We saw in the previous lecture that given a commutative Lie algebra $\mathfrak{g} = \mathbb{C}$ we would attach a representation of the Virasoro Lie algebra of central charge 1. Show that if $\mathfrak{g}$ is a two dimensional commutative Lie algebra then we obtain a representation of the Virasoro Lie algebra of central charge $c = 2$. If $\mathfrak{g}$ is a two dimensional commutative purely odd Lie algebra we can use this to produce a representation of the Virasoro Lie algebra of which central charge? Compare this construction with the previous one in exercise 3.