HOMEWORK 1

1. **Exercise.** Correct all the statements in the exercises below before solving them.

2. **Exercise.** Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field k and let $(,) \in \mathfrak{g}^* \otimes \mathfrak{g}^*$. Consider the field $\mathscr{K} = k((t))$ and the ring $\mathscr{O} = k[[t]] \subset \mathscr{K}$. On the vector space $\hat{\mathfrak{g}} := \mathfrak{g} \otimes_k \mathscr{K} \oplus k$ define the following bilinear operation

$$[a \otimes f, b \otimes g] = [a, b] \otimes fg + (a, b)\frac{1}{2\pi i}\oint f dg, \quad [k, \hat{\mathfrak{g}}] = 0, \qquad a, b \in \mathfrak{g},\ f, g \in \mathscr{K},$$

Here the symbol $\frac{1}{2\pi i}\oint f(t)dt$ means the coefficient of t^{-1}.

(a) Show that $\hat{\mathfrak{g}}$ with this bracket is a Lie algebra if and only if $(,) \in (\mathrm{Sym}^2\, \mathfrak{g}^*)^{\mathfrak{g}}$ (ie. the bilinear form is symmetric and invariant).

(b) Show that $\hat{\mathfrak{g}}_+ := \mathfrak{g} \otimes \mathscr{O} \oplus k$ and $\hat{\mathfrak{g}}_- := \mathfrak{g} \otimes t^{-1}k[t^{-1}]$ are subalgebras of $\hat{\mathfrak{g}}$.

3. **Exercise.** Let $k = \mathbb{C}$.

(a) Show that $k((t))$ is a field.

(b) Let $k(t, w)$ be the field of fractions of $k[t, w]$ and let $k((t, w))$ be the field of fractions of $k[[t, w]]$. Define

$$k((t))((w)) \xleftarrow{i_{|z|>|w|}} k((t, w)) \xrightarrow{i_{|w|>|z|}} k((w))((t))$$

by expanding a series in $k((t, w))$ in the respective domain. Show that each of these maps is an extension of fields.

(c) Notice that $k((t))((w)) \subset k[[t, t^{-1}, w, w^{-1}]] \supset k((w))((t))$. Show that

$$k\left((t)\right)\left((w)\right) \cap k\left((w)\right)\left((t)\right) = k[[t, w]][t^{-1}, w^{-1}].$$

(d) In particular, let $(t - w)^{-1} \in k((t, w)) \setminus k[[t, w]][t^{-1}, w^{-1}]$ and define

$$\delta(t, w) = i_{|z|>|w|}(t - w)^{-1} - i_{|w|>|z|}(t - w)^{-1} \in k[[t, t^{-1}, w, w^{-1}]].$$

Show that $(\partial_t + \partial_w)\delta = (t - w)\delta = 0$.

4. **Exercise.** Let $\mathfrak{g} = k$ be a one dimensional commutative algebra with generator α and let $(,)$ be defined so that $(\alpha, \alpha) = 1$. Consider the algebra $\hat{\mathfrak{g}}$ of Exercise 2 and denote $\alpha_n := \alpha \otimes t^n$. This algebra is the *Heisenberg* Lie algebra.

(a) Show that the projection $\pi : \hat{\mathfrak{g}}_+ \to k$ is a morphism of Lie algebras (ie. $\mathfrak{g} \otimes \mathscr{O} \subset \hat{\mathfrak{g}}_+$ is an ideal.

(b) Consider the Fock representation of $\hat{\mathfrak{g}}$ constructed as follows. Start with the standard representation of $k = \mathfrak{gl}(1)$ on k, compose it with π to obtain a 1 dimensional representation of $\hat{\mathfrak{g}}_+$. Define

$$V := U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_+} k.$$

Show that as vector spaces $V \simeq U(\hat{\mathfrak{g}}_-)$.

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(c) Abuse notation and think of $\alpha_n \in \mathrm{End}(V)$. For each pair n, m define

$$\mathrm{End}(V) \ni : a_n a_m := \begin{cases} a_n a_m & n < 0 \\ a_m a_n & n \geq 0 \end{cases}$$

Now define for each $n \in \mathbb{Z}$

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} : \alpha_n \alpha_{n-m} :$$

Prove that L_m is a well defined endomorphism of V and moreover that

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n} \frac{m^3 - m}{12} \mathrm{Id}_V \,. \tag{0.1}$$

(d) For any $\hat{\mathfrak{g}}$ as in Exercise 2 we will say that its module W is *smooth* if given any vector $w \in W$ and $a \in \mathfrak{g}$ we have $a_n w = 0$ for $n \gg 0$. Show that for any smooth representation of the Heisenberg algebra the operators L_m defined above are well defined. Is equation (0.1) still true?

Note however that L_m so defined does not belong to $U(\hat{\mathfrak{g}})$.

5. **Exercise.**

(a) Consider $\hat{\mathfrak{g}}$ as in Exercise 2 and for each $a \in \mathfrak{g}$ define

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-1-n} = \sum_{n \in \mathbb{Z}} a \otimes t^n z^{-1-n}.$$

Prove that $(z-w)^2[a(z), b(w)] = 0$ for all $a, b \in \mathfrak{g}$.

(b) Define $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-2-n}$ where L_n are the endomorphisms of the previous exercise. Show that

$$(z-w)^3[L(z), L(w)] \neq 0, \qquad (z-w)^4[L(z), L(w)] = 0.$$