Homework 7

Exercise 1. *Find the area of the paralellogram in* $\mathbb{R}^4$ *with vertices* $(1,1,1,1)$, $(3,1,1,2)$, $(3,2,3,5)$ *and* $(1,2,3,4)$.

Exercise 2. *evaluate the integral*

$$\int_S x_4 dA,$$

where S *is the surface in* $\mathbb{R}^4$ *defined by the parametrization*

$$(x_1, x_2, x_3, x_4) = (u, v, u\cos v, u\sin v)$$

for $0 \leq u \leq 1$ *and* $0 \leq v \leq \pi/2$. *Here* dA *is the volume form on* S *induced from the Riemmanian metric (which in turn is induced from the standard Riemannian metric of* $\mathbb{R}^4$*)*

Exercise 3. *Let* $f : \mathbb{C} \to \mathbb{C}$ *be a function. Let us say (this is not the usual definition) that* f *is* analytic *in an open set* $U \subset \mathbb{C}$ *if all the complex derivatives of* f *exists at every point of* U *and they are continuous.*

1. *if* $f = u + iv$ *is analytic on an open set* U *prove that*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

   where $z = x + iy$.

2. *Prove that if* f *is analytic on* U *then* $\int_c f dz = \int_c f(dx + i dy) = 0$ *for every closed curve* c *such that* $c = \partial U$ *for a* 2*-chain* U *(prove this just for a circle if you don't know what a chain is)*

Exercise 4. *If* $f : [a,b] \to \mathbb{R}$ *is non-negative and the graph of* f *in the* xy*-plane is revolved around the* x*-axis in* $\mathbb{R}^3$ *to yield a surface* S. *Show that the area of* S *is*

$$\int_a^b 2\pi f \sqrt{1 + (f')^2}.$$

Exercise 5. *Let*

$$\omega = \frac{x dy \wedge dz + y dz \wedge dx + z dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}},$$

be a two form on $\mathbb{R}^3 \setminus \{0\}$. *Show that* ω *is closed and it is not exact.*

Show that $H^2(S^2, \mathbb{R}) \neq 0$.

Exercise 6. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be a smooth function and consider its graph* $S \subset \mathbb{R}^3$. $g : \mathbb{R}^2 \to S$, $x \mapsto (x, f(x))$ *is a diffeomorphism. Let* dv_S *be the volume form given by the Riemmanian metric as embedded surface of* $\mathbb{R}^3$. *And let* $dv_0 = dx \wedge dy$ *be the standard volume form on* $\mathbb{R}^2$ *given by the standard Riemmanian metric of* $\mathbb{R}^2$. *Define* h *by*

$$g^* dv_S = h \times dv_0.$$

Find h.

Exercise 7. *Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere and let*

$$X = \sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i} \in \mathfrak{X}(\mathbb{R}^{n+1}).$$

Let $dv_0 = dx_0 \wedge \cdots \wedge dx_n$ be the standard volume form on $\mathbb{R}^{n+1}$ and let dv_S be the Riemmanian volume form induced on S^n. Show that

$$dv_S = \iota_X(dv_0)|_S.$$

Exercise 8. *Let T^2 be the 2-torus in $\mathbb{R}^4$ defined by the equations $w^2 + x^2 = y^2 + z^2 = 1$. Compute $\int_{T^2} xyz\, dw \wedge dy$.*