# Homework 6

Due 13 November 2019

**Exercise 1.** *Let $M$ be a smooth manifold. Two embedded submanifolds $S_1, S_2 \subset M$ are said to be transverse if for each $p \in S_1 \cap S_2$, $T_pS_1 + T_pS_2 = T_pM$.*

- *Let $M = \mathbb{R}^3$. $S_1 = \left\{(x, y, z) \in \mathbb{R}^3 | x = z^2\right\}$, $S_2 = \left\{(x, y, z) \in \mathbb{R}^3 | y = z^3\right\}$. Are $S_1$ and $S_2$ transverse?*
- *Show that if $S_1$ and $S_2$ are transverse, then the intersection $S_1 \cap S_2$ is an embedded submanifold of dimension $\dim S_1 + \dim S_2 - \dim M$.*

**Exercise 2.** *Let $G$ be a Lie group acting smoothly on a manifold $M$. Show that each orbit is immersed. Under what conditions on the action are the orbits embedded?*

**Exercise 3.** *Give an example of a smooth proper action of a group on a manifold $M$ such that the quotient space is not a topological manifold.*

**Exercise 4.** *Show that $U(n)$ is diffeomorphic to $U(1) \times SU(n)$ for every $n$ but they are not isomorphic Lie groups if $n > 1$.*

**Exercise 5.** *Let $V$ be a finite dimensional vector space and $F_\bullet = \left\{0 \subset F_1 \subset F_2 \subset \cdots \subset F_k \subset V\right\}$ a sequence of sub-vector spaces each one included in the next. Show that the set*

$$\left\{F'_\bullet = \left\{0 \subset F'_1 \subset \cdots \subset F'_k \subset V\right\} \mid \exists \phi \in GL(V),\ \phi F_i = F'_i\ \forall i\right\}$$

*is naturally a smooth manifold.*

**Exercise 6.** *Let $M$ be the manifold of the previous example.*

- *Show that $M$ comes equipped with a sequence of vector bundles $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \cdots \subset \mathscr{F}_k$ each one a sub-bundle of the next one.*
- *Show that all of them are included in a trivial vector bundle of rank $\dim V$.*

**Exercise 7.** *Let $V \subset W$ be a subvector space of a finite dimensional vector space. Consider it's quotient $Q$ and the sequence*

$$0 \to V \xrightarrow{\iota} W \xrightarrow{\pi} Q \to 0.$$

*There always exists a section $\sigma : Q \to W$ and $j : W \to V$ such that $\pi \circ \sigma = \mathrm{Id}_Q$, $j \circ \iota = \mathrm{Id}_V$ and $\ker j = \mathrm{im}\sigma$ (simply take a complementary subspace to $V \subset W$).*

*Show that this is not the case if $V, W$ and $Q$ are vector bundles on a manifold $M$ instead of vector spaces [Hint: the previous excercise provides counterexamples]*

**Exercise 8.** *Show that $SL(n, \mathbb{R})$ and $SL(n, \mathbb{C})$ are diffeomorphic to $SO(n) \times \mathbb{R}^{n(n+1)/2}$ and $SU(n) \times \mathbb{R}^{n^2}$ respectively.*