Homework 2

Due 2 September 2019

Exercise 1. *Compute the coordinate representation for each of the following maps in stereographic coordinates (as in Exercise 1 of the previous homework), use this to prove that each map is smooth:*

1. *For each $n \in \mathbb{Z}$, the n-th power map $p_n : S^1 \to S^1$ is given in complex notation by $p_n(z) = z^n$ (recall also Excersise 3 of the previous homework).*
2. *The antipodal map $S^n \to S^n$, $x \mapsto -x$.*
3. *The map $S^3 \to S^2$ given by*

$$(z, w) \mapsto (z\overline{w} + w\overline{z}, iw\overline{z} - iz\overline{w}, z\overline{z} - w\overline{w}$$

where we think of S^3 as

$$S^3 \simeq \left\{(w, z) \in \mathbb{C}^2 \mid |w|^2 + |z|^2 = 1\right\} \subset \mathbb{C}^2.$$

Exercise 2. *Let $f : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{k+1} \setminus \{0\}$ be a smooth map and suppose that for some $d \in \mathbb{Z}$ we have $f(\lambda x) = \lambda^d f(x)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}^{n+1} \setminus \{0\}$. Show that the map $\overline{f} : \mathbb{RP}^n \to \mathbb{RP}^k$ induced by f is well defined and smooth.*

Exercise 3. *For any topological space M let $C(M)$ be the algebra of continuous functions $M \to \mathbb{C}$. If $\varphi : M \to N$ is a continuous map defined $\varphi^* : C(N) \to C(M)$ by $\varphi^*(f) = f \circ \varphi$.*

1. *Show that φ^* is a linear map.*
2. *If M, N are smooth manifolds, show that φ is smooth if and only if $\varphi^*(C^\infty(N)) \subset C^\infty(M)$.*
3. *If $\varphi : M \to N$ is a homemorphism between smooth manifolds, show that it is a diffeomorphism if and only iff φ^* restricts to an isomorphism $C^\infty(N) \simeq C^\infty(M)$.*

Exercise 4*. *Let G be a connected Lie group and $U \subset G$ be any neighborhood of the identity. Show that every element of G can be written as a finite product of elements of U.*

Exercise 5. *Let G be a connected Lie group. Show that the universal covering $\tilde{G}$ is unique in the following sense: if $G' \to G$ is any other Lie group homomorphism that is a simply connected covering, then there exists a unique Lie group homomorphism $\varphi : \tilde{G} \to G'$ making the following diagram commute:*

$$\begin{array}{ccc} \tilde{G} & \overset{\exists!}{\dashrightarrow} & G' \\ \downarrow & \swarrow & \\ G & & \end{array}$$

Exercise 6. *Let M be a topological manifold and $\mathfrak{A}$ be an open cover by precompact sets. Show that $\mathfrak{A}$ is locally finite if and only if every open $U \in \mathfrak{A}$ intersects non-trivially only finitely many open sets $V \in \mathfrak{A}$.*

Exercise 7. *Show that $SO(3)$ is a connected Lie group and that $SU(2)$ is its universal cover.*