Homework 2

Due 26 August 2019

Exercise 1. *Let* $p^{\pm} = (0, \cdots, 0, \pm 1) \in S^n \subset \mathbb{R}^{n+1}$. *Define the* stereographic projections $\sigma^{\pm} : S^n \backslash \{p^{\pm}\} \to \mathbb{R}^n$ *by*

$$\sigma^{\pm}(x^0, \cdots, x^n) = \frac{(x^0, \cdots, x^n)}{1 \mp x^n}.$$

Show that $\sigma^{\pm}$ *defines an Atlas on* S^n *and that the corresponding smooth structure is the same as the one defined in class.*

Exercise 2. *Denote by* $\mathbb{CP}^n$ *the set of complex linear subspaces of* $\mathbb{C}^{n+1}$ *with the quotient topology* $\pi : \mathbb{C}^{n+1} \setminus \{0\} \twoheadrightarrow \mathbb{CP}^n$. *Show that* $\mathbb{CP}^n$ *is naturally a compact* $2n$ *dimensional smooth manifold.*

Exercise 3. *Show that* $\mathbb{RP}^1 \simeq S^1$ *as smooth manifolds. Let* $[x_0 : x_1]$ *be homogeneous coordinates of* $\mathbb{RP}^1$ *and* $[y_0 : y_1]$ *be homogeneous coordinates of another copy of* $\mathbb{RP}^1$ *and let* $[z_0 : z_1 : z_2 : z_3]$ *be homogeneous coordinates on* $\mathbb{RP}^3$. *Show that the map*

$$\big([x_0 : x_1], [y_0 : y_1]\big) \to [x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1],$$

is a well defined smooth map $S^1 \times S^1 \hookrightarrow \mathbb{RP}^3$, *that is the two torus embeds as a submanifold of* $\mathbb{RP}^3$.

Exercise 4. *Show that* $\mathbb{CP}^1 \simeq S^2$ *as smooth manifolds. Conclude that with the same notation as in the previous exercise, there is an embedding* $S^2 \times S^2 \hookrightarrow \mathbb{CP}^3$.

Exercise 5. *Let* $Heis_3(\mathbb{R})$ *be the set of upper triangular* 3×3 *matrices with real entries and* 1 *on the diagonal. Let* $Heis_3(\mathbb{R})$ *be the subset of matrices with integer entries.*

1. *Show that they are both smooth manifolds of dimension* 3 *and* 0 *respectively.*
2. *Let* $X = Heis_3(\mathbb{R})/Heis_3(\mathbb{Z})$. *Show that the map*

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mapsto \left(e^{2\pi i a}, e^{2\pi i b}\right),$$

   descends to a well defined map $\pi : X \to \mathbb{T}^2$.

3. *For any* $x \in \mathbb{T}^2$ *show that* $\pi^{-1}(x) \simeq S^1$.

4*. *Show that* X *is a smooth manifold and that* π *is a smooth map. Such an* X *is called an* S^1 fibration over the torus.