# Lecture 1: Definitions and examples

**1.1 Definition.** Let $k$ be a field. An *algebra over* $k$ is a $k$-vector space $A$ together with a bilinear operation

$$A \otimes_k A \to A, \qquad a \otimes b \mapsto a \cdot b.$$

A *morphism* of $k$-algebras is a linear map $\varphi \in \operatorname{Hom}_k(A, B)$ such that

$$\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b), \qquad \forall\, a, b \in A.$$

A *subalgebra* $B \subset A$ is a subspace such that $B \cdot B \subset B$.

**1.2 Notation.** *The bilinear operation* $\cdot$ *might be denoted* $[\,,\,]$, $\triangleleft$, $\{\,,\,\}$, *etc. depending on the context.*

**1.3 Definition.** Let $A$ be a $k$-algebra. $A$ is said

a) *commutative* if $a \cdot b = b \cdot a$ for every $a, b$ in $A$,

b) *skew-symmetric* if $[a, a] = 0$ for every $a \in A$,

c) *associative* if $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for every $a, b, c$ in $A$,

d) *right-symmetric* if
$$(a \triangleleft b) \triangleleft c - a \triangleleft (b \triangleleft c) = (a \triangleleft c) \triangleleft b - a \triangleleft (c \triangleleft b),$$
for every $a, b, c$ in $A$,

e) *left-symmetric* if
$$(a \triangleleft b) \triangleleft c - a \triangleleft (b \triangleleft c) = (b \triangleleft a) \triangleleft c - b \triangleleft (a \triangleleft c),$$
for every $a, b, c$ in $A$,

f) *finite dimensional* if $A$ is finite dimensional as a $k$-vector space,

g) *finitely generated* if there exists a finite subset $I \subset A$ such that every element of $A$ can be written as a linear combination of elements of the form
$$a_1 \cdot a_2 \cdots a_n,$$
for $\{a_i\} \subset I$ and arbitrary parenthesis assignments in the product,

h) *unital* if there exist an element $\mathbb{1} \in A$ such that
$$\mathbb{1} \cdot a = a \cdot \mathbb{1} = a,$$
for all $a \in A$.

**1.4 Remark.** Any associative algebra is a left-symmetric and right-symmetric algebra. The converse is not true (see example 1.5 f) below)

**1.5 Examples.**

a) Let $M$ be a topological space, the algebra $C(M)$ of continuous functions is commutative, associative and unital.

b) The algebra $C^\infty(\mathbb{R}^n)$ of smooth functions on $\mathbb{R}^n$ is commutative associative and unital.

c) The subalgebra $A \subset C^\infty(\mathbb{R}^n)$ of Lebesgue integrable smooth functions is commutative, associative but non-unital.

d) Let $A$ be a $k$-vector space. The *midpoint operation*

$$a \otimes b \mapsto \frac{1}{2}(a+b),$$

endows $A$ with a commutative algebra structure which is not associative nor unital.

e) Let $V$ be a $k$-vector space, its *endomorphism algebra* $\mathrm{End}_k(V)$ is defined as the set of endomorphisms with multiplication being composition. It is an associative unital algebra, which is not commutative if $\dim_k V > 1$.

f) Let $A = \mathfrak{X}(\mathbb{R}^n)$ be the set of smooth vector fields on $\mathbb{R}^n$, that is expressions of the form

$$X = \sum_{i=1}^n f_i \frac{\partial}{\partial x^i},$$

with $f_i \in C^\infty(\mathbb{R}^n)$ for every $i$. Then $A$ with the operation

$$X \triangleleft Y = \sum_{j=1}^n \left( \sum_{i=1}^n f_i \cdot \frac{\partial g_j}{\partial x^i} \right) \frac{\partial}{\partial x^j},$$

for $Y = \sum_{i=1}^n g_i \frac{\partial}{\partial x^i}$, is a left-symmetric algebra. Indeed, for $Z = \sum_{i=1}^n h_i \frac{\partial}{\partial x^i}$ we have

$$(X \triangleleft Y) \triangleleft Z = \sum_{ijk} f_i \cdot \frac{\partial g_j}{\partial x^i} \frac{\partial h_k}{\partial x^j} \frac{\partial}{\partial x^k}, \tag{1.5.1}$$

and

$$X \triangleleft (Y \triangleleft Z) = \sum_{ijk} f_i \frac{\partial}{\partial x^i} \left( g_j \frac{\partial h_k}{\partial x^j} \right) \frac{\partial}{\partial x^k}. \tag{1.5.2}$$

Subtracting (1.5.1) from (1.5.2) and using the Leibniz rule we obtain

$$X \triangleleft (Y \triangleleft Z) - (X \triangleleft Y) \triangleleft Z = \sum_{ijk} f_i g_j \frac{\partial h_k}{\partial x^i \partial x^j} \frac{\partial}{\partial x^k}, \tag{1.5.3}$$

which is manifestly invariant by the exchange $X \leftrightarrow Y$. Since it is easy to find $X, Y, Z$ such that the RHS of (1.5.3) is not zero (think of a quadratic expression for $h_k$ for example) this algebra is not associative.

g) The examples above have an algebraic counterpart by considering the algebra $\mathbb{R}[x^1, \cdots, x^n]$ of polynomial functions on $\mathbb{R}^n$ in place of $C^\infty(\mathbb{R}^n)$. The corresponding algebras are infinite dimensional but finitely generated.

**1.6 Definition**. A *Lie algebra* is a skew-symmetric algebra $(\mathfrak{g}, [\,,\,])$ that satisfies the following identity called the *Jacobi identity*:

$$[a, [b, c]] = [[a, b], c]] + [a, [b, c]],$$

for all $a, b, c$ in $\mathfrak{g}$.

**1.7 Lemma**. *Let $(A, \triangleleft)$ be a left-symmetric or right-symmetric algebra. Let $A^{\mathrm{Lie}}$ be the vector space $A$ endowed with the operation*

$$[a, b] = a \triangleleft b - b \triangleleft a.$$

*Then $A^{\mathrm{Lie}}$ is a Lie algebra.*

*Proof.* Let $A$ be a left-symmetric algebra. $A^{\mathrm{Lie}}$ is obviously a skew-symmetric algebra, we only need to check the Jacobi identity. We have

$$\begin{aligned}
[a,[b,c]] - [[a,b],c] - [b,[a,c]] &= a \triangleleft (b \triangleleft c - c \triangleleft b) - (b \triangleleft c - c \triangleleft b) \triangleleft a - (a \triangleleft b - b \triangleleft a) \triangleleft c \\
&\quad + c \triangleleft (a \triangleleft b - b \triangleleft a) - b \triangleleft (a \triangleleft c - c \triangleleft a) + (a \triangleleft c - c \triangleleft a) \triangleleft b \\
&= \Big( a \triangleleft (b \triangleleft c) - (a \triangleleft b) \triangleleft c \Big) - \Big( a \triangleleft (c \triangleleft b) - (a \triangleleft c) \triangleleft b \Big) - \Big( (b \triangleleft c) \triangleleft a - b \triangleleft (c \triangleleft a) \Big) \\
&\quad + \Big( (c \triangleleft b) \triangleleft a - c \triangleleft (b \triangleleft a) \Big) + \Big( (b \triangleleft a) \triangleleft c - b \triangleleft (a \triangleleft c) \Big) + \Big( c \triangleleft (a \triangleleft b) - (c \triangleleft a) \triangleleft b \Big) = 0,
\end{aligned}$$

were We used that the first three terms on the RHS cancel the last three terms by the left-symmetric property. □

**1.8 Examples.**

a) Any vector space $V$ with the zero bracket is a Lie algebra. These are called *Abelian Lie algebras*. In particular, the one dimensional vector space $k$ is an Abelian Lie algebra.

b) Let $V$ be a $k$-vector space. The *general linear* Lie algebra $\mathfrak{gl}(V)$ is the Lie algebra $A^{\mathrm{Lie}}$ given by Lemma 1.7 applied to $A = \mathrm{End}_k(V)$ from 1.5 e). When $V = k^{\oplus n}$ this Lie algebra is denoted by $\mathfrak{gl}_n$.

c) The Lie algebra $\mathfrak{X}(\mathbb{R}^n)$ of smooth vector fields on $\mathbb{R}^n$ is the Lie algebra $A^{\mathrm{Lie}}$ given by Lemma 1.7 applied to the left symmetric algebra $A = \mathfrak{X}(\mathbb{R}^n)$ from 1.5 f).

**1.9 The trace morphism**. Let $V$ be a $k$-vector space. We have a canonical map defined by the linear extension of

$$V^* \otimes V \to \mathrm{End}_k(V), \qquad \zeta \otimes w \mapsto \{v \mapsto \zeta(v) w\}. \tag{1.9.1}$$

When $V$ is finite dimensional this map is an isomorphism. In this case we define the *trace* to be the linear composition

$$tr : \mathrm{End}_k(V) \xrightarrow{\sim} V^* \otimes V \to k, \tag{1.9.2}$$

where the first arrow is the isomorphism (1.9.1) and the second arrow is the canonical pairing

$$\zeta \otimes v \mapsto \zeta(v).$$

It follows from Exercise 1.4 that tr is a morphism of Lie algebras $\mathfrak{gl}(V) \to k$.

**1.10 Definition**. An *ideal* of a Lie algebra $I \subset \mathfrak{g}$ is a subspace such that $[I, \mathfrak{g}] \subset I$. A Lie algebra $\mathfrak{g}$ is called *simple* if it is not Abelian and its only ideals are 0 and $\mathfrak{g}$.

**1.11 Remark**. One of the main topics of these lectures will be to classify all simple finite dimensional Lie algebras over the complex numbers up to isomorphism.

**1.12 Lemma**. *Let $\varphi : \mathfrak{g} \to \mathfrak{h}$ be a morphism of Lie algebras.*

*a) Its kernel,* $\ker \varphi \subset \mathfrak{g}$ *is an ideal.*

b) *Its image,* $\operatorname{Im}\varphi \subset \mathfrak{h}$ *is a subalgebra.*

*Proof.* Let $a, b \in \mathfrak{g}$ with $\varphi(a) = 0$, we have

$$0 = [0, \varphi(b)] = [\varphi(a), \varphi(b)] = \varphi([a, b]),$$

from where $[a, b] \in \ker\varphi$, proving a). b) is proved similarly. □

**1.13 Example**. Let $V$ be a $k$-vector space, the *special linear* Lie algebra $\mathfrak{sl}(V) \subset \mathfrak{gl}(V)$ is defined as $\ker \operatorname{tr}$. When $V = k^{\oplus n}$ we denote this Lie algebra by $\mathfrak{sl}_n$.

**1.14.** Let $V, W$ be vector spaces, and $a \in \operatorname{Hom}_k(V, W)$, its transpose $a^* \in \operatorname{Hom}_k(W^*, V^*)$ is defined by

$$\Big(a^*(\zeta)\Big)(v) = \zeta(a(v)).$$

When $V$ is finite dimensional, the map $a \mapsto a^*$ is given by the composition

$$\operatorname{End}_k(V) \xrightarrow{\sim} V^* \otimes V \xrightarrow{\sim} V \otimes V^* \xrightarrow{\sim} \operatorname{End}_k(V^*),$$

where the first and last arrows are the isomorphisms (1.9.1) and the middle arrow is the isomorphism obtained by linearly extending $\zeta \otimes v \mapsto v \otimes \zeta$. It follows easily from this that $\operatorname{tr}(a) = \operatorname{tr}(a^*)$.

When $V = k^{\oplus n}$ we have a canonical isomorphism $V \simeq V^*$ given by identifying their dual bases. Endomorphisms of these spaces are given by $n \times n$ matrices with entries in $k$ (where $\dim V = \dim V^* = n$). In this case the transpose map is given by $a \mapsto a^{tr}$ the usual transpose of a matrix.

**1.15.** Let $V$ be a $k$-vector space and $B : V \otimes V \to k$ a bilinear map. We define the Lie subalgebra

$$\mathfrak{o}(V, B) = \left\{ a \in \mathfrak{gl}(V) \,\middle|\, B(a \cdot v, w) + B(v, a \cdot w) = 0,\ \forall v, w \in V \right\} \subset \mathfrak{gl}(V). \tag{1.15.1}$$

If $B$ and $B'$ are two isomorphic bilinear maps, that means there exists an isomorphism $S \in Aut(V)$ such that $B(Sv, Sw) = B'(v, w)$. Then $a \mapsto S^{-1}aS$ is an isomorphism of Lie algebras $\mathfrak{o}(V, B) \simeq \mathfrak{o}(V, B')$. Indeed for $a \in \mathfrak{o}(V, B)$ and $v, w$ in $V$, we have

$$B'(S^{-1}aSv, w) = B(aSv, Sw) = -B(Sv, aSw) = -B'(v, S^{-1}aSw).$$

**1.16 Examples.**

a) When $B$ is symmetric and non-degenerate, the Lie algebra $\mathfrak{o}(V, B)$ is called the *special orthogonal* Lie algebra and is denoted $\mathfrak{so}(V, B)$. In the particular case that $V = k^{\oplus n}$ and $B$ is the canonical bilinear given by the Gramm matrix $\operatorname{Id}_{n \times n}$, this Lie algebra is denoted $\mathfrak{so}_n$ and is described explicitly as

$$\mathfrak{so}_n = \left\{ a \in \mathfrak{gl}_n \,\middle|\, a + a^{tr} = 0 \right\}.$$

Notice that when $k$ is an algebraically closed field of characteristic zero, there is only one symmetric non-degenerate bilinear pairing modulo isomorphisms as in 1.15.

b) When $B$ is anti-symmetric and non-degenerate, the Lie algebra $\mathfrak{o}(V, B)$ is called the *special symplectic* Lie algebra and is denoted $\mathfrak{sp}(V, B)$. In this case it is easy to see that $V$ has to be even-dimensional. In the particular case that $V = k^{\oplus 2n}$ and $B$ is the bilinear map given by the Gramm matrix

$$\begin{pmatrix} 0 & \operatorname{Id}_{n \times n} \\ -\operatorname{Id}_{n \times n} & 0 \end{pmatrix}$$

this Lie algebra is denoted by $\mathfrak{sp}_{2n}$.

**1.17 Definition.** A *derivation* of an algebra $A$ is an endomorphism $D \in \mathrm{End}_k(A)$ such that the following *Leibniz rule* holds for every $a, b$ in $A$.

$$D(a \cdot b) = D(a) \cdot b + a \cdot D(b).$$

Derivations of an algebra $A$ form a vector space denoted by $\mathrm{Der}_k(A, A)$.

A derivation $d \in \mathrm{Der}_k(A, A)$ is called a *differential* if $d^2 = 0$.

**1.18 Lemma.** *Let $A$ be an algebra, then $\mathrm{Der}_k(A, A) \subset \mathrm{End}_k(A)$ is a sub Lie algebra.*

*Proof.* We need to show that $\mathrm{Der}_k(A, A)$ is closed under the commutator bracket. Let $D, D' \in \mathrm{Der}_k(A, A)$ and $a, b \in A$. We have

$$\begin{aligned}[D, D'](a \cdot b) = \left(DD' - D'D\right)(a \cdot b) = D\left(D'(a) \cdot b + a \cdot D'(b)\right) - D'\left(D(a) \cdot b + a \cdot D(b)\right) = \\ DD'(a) \cdot b + D'(a) \cdot D(b) + D(a) \cdot D'(b) + a \cdot DD'(b) \\ - D'D(a) \cdot b - D(a) \cdot D'(b) - D'(a) \cdot D(b) - a \cdot D'D(b) = \\ [D, D'](a) \cdot b + a \cdot [D, D'](b),\end{aligned}$$

proving that $[D, D'] \in \mathrm{Der}_k(A, A)$. □

**1.19 Example.** This example generalizes 1.8 c). Let $A = C^\infty(M)$ be the algebra of differentiable functions on a manifold $M$, then $\mathfrak{X}(M) = \mathrm{Der}_k(A, A)$ is the Lie algebra of *smooth vector fields* on $M$.

## Exercises

**1.1.** Is the algebra $\mathfrak{X}\left(\mathbb{R}^n\right)$ from 1.5 f) right-symmetric?

**1.2.** Prove Lemma 1.7 for right-symmetric algebras.

**1.3.** Show that when $V$ is finite dimensional the canonical map (1.9.1) is an isomorphism.

**1.4.** Show that the trace satisfies $\mathrm{tr}\,(a \cdot b) = \mathrm{tr}\,(b \cdot a)$.

**1.5.** Show that the map $a \mapsto S^{-1}aS$ in 1.15 is an isomorphism of Lie algebras $\mathfrak{o}(V, B) \simeq \mathfrak{o}(V, B')$.

**1.6.** Show that $\mathfrak{so}(V, B) \subset \mathfrak{sl}(V)$.

**1.7.** Show that both definitions of the Lie algebras $\mathfrak{X}\left(\mathbb{R}^n\right)$ given in 1.8 c) and in 1.19 coincide.

# Lecture 2: Representations

**2.1 Definition.** Let $\mathfrak{g}$ be a Lie algebra over $k$. A *representation* of $\mathfrak{g}$ (also called a $\mathfrak{g}$-module) is a morphism of Lie algebras $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ for some $k$-vector space $V$. Often times we will drop $\rho$ from the notation and we will simply say $V$ is a $\mathfrak{g}$-module and write this as $V \in \mathfrak{g}\text{-mod}$. For a vector $v \in V$ and an element $a \in \mathfrak{g}$ we will write

$$av = a \cdot v = \rho(a)\big(v\big).$$

Given two $\mathfrak{g}$-modules $V$ and $W$, a *morphism* of representations is a linear map $\varphi \in \operatorname{Hom}_k(V, W)$ such that

$$a \cdot \varphi(v) = \varphi\left(a \cdot v\right),$$

for every $a \in \mathfrak{g}$ and $v \in V$. An *isomorphism* is a morphism that is invertible as a linear map. The set of all morphisms is a vector space denoted $\operatorname{Hom}_{\mathfrak{g}}(V, W)$.

**2.2 Example.** All the classical Lie algebras $\mathfrak{gl}_n$, $\mathfrak{sl}_n$, $\mathfrak{so}_n$ and $\mathfrak{sp}_n$ come with a *defining* representation $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n$.

**2.3 Example.** Let $\mathfrak{g}$ be a Lie algebra. For $a \in \mathfrak{g}$ we let $ad_a \in \operatorname{End}_k(\mathfrak{g})$ be given by

$$\operatorname{ad}_a(b) = [a, b].$$

Then the map $a \mapsto \operatorname{ad}_a$ makes $\mathfrak{g}$ into an $\mathfrak{g}$-module. This is called the *adjoint representation*. Notice that indeed $\operatorname{ad}\mathfrak{g} \subset \operatorname{Der}_k(\mathfrak{g}, \mathfrak{g})$ since by the Jacobi identity:

$$\operatorname{ad}_a[b, c] = [\operatorname{ad}_a b, c] + [b, \operatorname{ad}_a c].$$

The kernel of $\mathfrak{g}$ under the adjoint map is called the *center* of $\mathfrak{g}$ and is denoted $Z(\mathfrak{g})$. The derivations of $\mathfrak{g}$ that are in the image of $\mathfrak{g}$ under the adjoint map are called *inner derivations*.

**2.4 Lemma.** *Let $\mathfrak{g}$ be a Lie algebra and $I \subset \mathfrak{g}$ be an ideal,*

*a) The quotient $\mathfrak{g}/I$ is naturally a Lie algebra with the bracket given by*

$$[a + I, b + I] = [a, b] + I. \tag{2.4.1}$$

*b) Given a morphism $\varphi : \mathfrak{g} \to \mathfrak{h}$ of Lie algebras we have a canonical isomorphism of Lie algebras*

$$\operatorname{Im}\varphi \simeq \mathfrak{g}/\ker\varphi.$$

*Proof.* The bracket (2.4.1) is well defined since $[I, b + I] \subset I$, and $[a, b + I] \subset I$. Skew-symmetry and the Jacobi condition hold on $\mathfrak{g}/I$ because they hold on $\mathfrak{g}$. The proof of b) is straightforward. □

**2.5 Lemma.** *Inner derivations form an ideal of* $\operatorname{Der}_k(\mathfrak{g}, \mathfrak{g})$.

*Proof.* Let $a \in \mathfrak{g}$ and $D \in \operatorname{Der}_k(\mathfrak{g}, \mathfrak{g})$. We have

$$[\operatorname{ad}_a, D]b = \operatorname{ad}_a D(b) - D\operatorname{ad}_a b = [a, D(b)] - D([a, b]) = \\ [a, D(b)] - [D(a), b] - [a, D(b)] = \operatorname{ad}_{-D(a)} b,$$

for every $b \in \mathfrak{g}$, therefore $[\operatorname{ad}_a, D] = \operatorname{ad}_{-D(a)} \in \operatorname{ad}\mathfrak{g}$. □

**2.6 Notation.** *We denote the quotient Lie algebra by*

$$H^1(\mathfrak{g},\mathfrak{g}) = \mathrm{Der}_k(\mathfrak{g},\mathfrak{g})/\operatorname{ad}\mathfrak{g}. \tag{2.6.1}$$

**2.7 Definition.** Let $V$ be a $\mathfrak{g}$-module. A *submodule* or *subrepresentation* is a vector subspace $U \subset V$ stable under the action of $\mathfrak{g}$, that is $\mathfrak{g}\cdot U \subset U$.

A representation $V$ is called *irreducible* if its only submodules are 0 and $V$ itself.

**2.8 Example.** Consider $\mathfrak{g}$ as an $\mathfrak{g}$-module via the adjoint representation. Then a submodule $I \subset \mathfrak{g}$ is the same as an ideal. Therefore for a non-abelian Lie algebra, its adjoint representation is irreducible if and only if $\mathfrak{g}$ is simple.

**2.9.** Let $V$ and $W$ be two $\mathfrak{g}$-modules. Then $V \oplus W$ is naturally a $\mathfrak{g}$-module, the action is given by

$$a\cdot(v+w) = a\cdot v + a\cdot w.$$

**2.10 Definition.** A representation $V$ is called *decomposable* if it is isomorphic to a direct sum $V \simeq W_1 \oplus W_2$ with $W_i \neq 0$, $i = 1, 2$. $V$ is called *indecomposable* if it is not decomposable.

**2.11.** Let $V = W_1 \oplus W_2$ be a decomposable representation, then $W_1 \subset W_1 \oplus W_2$ is a subrepresentation, hence $V$ is not irreducible. We see that irreducible representations are indecomposable. The converse is not true as the following example shows.

**2.12 Example.** The two dimensional solvable Lie algebra $\mathfrak{g}$ has as $k$-basis $a, b$ and commutation relations given by $[a,b] = b$. Its adjoint representation is indecomposable but it is not irreducible. Indeed $k\cdot b \subset \mathfrak{g}$ is an ideal, hence a non-trivial subrepresentation. On the other hand, suppose $\mathfrak{g}$ is decomposable, so that $\mathfrak{g} \simeq V \oplus W$ as representations. Then it must be $\dim V = \dim W = 1$. Let $v$ be a basis of $V$ and $w$ be a basis of $W$. Since it is a direct sum we would get $[v,w] = 0$, in this case $\mathfrak{g}$ would be abelian, which is a contradiction.

**2.13 Lemma.** *Let $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(V, W)$ be a morphism of representations, then*

*a)* $\ker\varphi \subset V$ *is a sub-representation.*

*b)* $\operatorname{coker}\varphi = W/\operatorname{Im}\varphi$ *is a representation and the quotient map* $W \mapsto \operatorname{coker}\varphi$ *is a morphism of representations.*

*Proof.* Let $\varphi v = 0$ and $a \in \mathfrak{g}$, then $\varphi a v = a\cdot \varphi v = 0$ so that $av \in \ker\varphi$ and a) is proved. The proof of b) is left as an exercise. ☐

**2.14.** Let $V \in \mathfrak{g}$-mod and $B : V \otimes V \to k$ a symmetric bilinear paring. We say that $B$ is $\mathfrak{g}$-*invariant* if

$$B(av, w) + B(v, aw) = 0.$$

for every $v, w \in V$ and every $a \in \mathfrak{g}$. This is equivalent to saying that the morphism $\mathfrak{g} \to \mathfrak{gl}(V)$ factors through $\mathfrak{so}(V, B) \subset \mathfrak{gl}(V)$. Suppose moreover that $B$ is non-degenerate and let $U \subset V$ be a submodule. Then $U^\perp \subset V$ is a submodule (see Exercise 2.3) and we have $V \simeq U \oplus U^\perp$ as representations. Indeed we have a direct sum as vector spaces by Exercise 2.3 so it is enough to prove that $U^\perp \subset V$ is a submodule. Let $v \in U^\perp$, $u \in U$ and $a \in \mathfrak{g}$. We have

$$B(av, u) = -B(v, au) = 0,$$

since $au \in U$ and $v \in U^\perp$. It follows that $av \in U^\perp$ and the claim follows. As a corollary of this discussion we obtain

**2.15 Proposition.** *Let $V \in \mathfrak{g}$-mod be a finite dimensional representation endowed with a symmetric, non-degenerate and invariant bilinear map $B : V \otimes V \to k$. Then $V \simeq V_1 \oplus \cdots \oplus V_k$ for some irreducible representations $V_1, \cdots, V_k$.*

*Proof.* We proceed by induction in the dimension of $V$. The case being trivial if $\dim V = 1$. If $V$ is irreducible there is nothing to prove. If on the other hand $U \subset V$ is a nontrivial subrepresentation, we have $V \simeq U \oplus U^\perp$ and $\dim U, \dim U^\perp < \dim V$. The restriction of $B$ to $U$ and $U^\perp$ is an invariant non-degenerate symmetric pairing by Exercise 2.3, hence we can apply induction to decompose each. □

**2.16 Remark.** The above proposition is false without the assumption of the existence of $B$ (Exercise 2.4). For a certain class of Lie algebras (semisimple) it goes by the name of *Weyl's complete irreducibility Theorem.* It allows reducing the problem of classifying finite dimensional representations to the problem of classifying irreducible ones. This is the second objective of these lectures.

In other contexts (finite groups, compact groups, etc) one shows the existence of such a $B$ by considering any symmetric nondegenerate (not necessarily invariant) pairing $B_0$ and then constructing an invariant one by acting on $B_0$ and *averaging*. This requires the notion of "sum" or "integral" over the finite group, compact group, etc.

**2.17.** Let $V, W \in \mathfrak{g}$-mod, then $\mathrm{Hom}_k(V, W)$ is naturally a representation. For $a \in \mathfrak{g}$, $\varphi \in \mathrm{Hom}_k(V, W)$ the action is defined by

$$(a \cdot \varphi)(v) = a\varphi(v) - \varphi(av).$$

Similarly $V \otimes W$ is a representation with

$$a(v \otimes w) = (av) \otimes w + v \otimes (aw).$$

## Exercises

**2.1.** Let $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(V, W)$ be an isomorphism of two representations $V$ and $W$ as in 2.1. Show that its linear inverse $\varphi^{-1}$ is also a morphism of representations.

**2.2.** Prove Lemma 2.13 b).

**2.3.** Let $V$ be a vector space and $B : V \otimes V \to k$ a symmetric non-degenerate bilinear map. Let $U \subset V$ be any subspace, then define

$$U^\perp = \left\{ v \in V \,\middle|\, B(u, v) = 0 \,\forall\, u \in U \right\}.$$

a) Prove that $V \simeq U \oplus U^\perp$.

b) Prove that the restriction of $B$ to $U$ and $U^\perp$ is still non-degenerate.

**2.4.** Let $\mathfrak{g}$ be the three dimensional Heisenberg Lie algebra. That is the algebra with basis $p, q, \hbar$ and commutation relations

$$[p, q] = \hbar, \qquad [p, \hbar] = [q, \hbar] = 0.$$

Show that its adjoint representation does not admit a symmetric, non-degenerate, invariant bilinear form.

**2.5.** Check that $\mathrm{Hom}_k(V, W)$ and $V \otimes W$ as defined in 2.17 are well defined representations.

**2.6.** For a $\mathfrak{g}$-module $V$, we define the *invariant* subspace by

$$V^{\mathfrak{g}} = \left\{ v \in V \,\middle|\, av = 0 \,\forall\, a \in \mathfrak{g} \right\}.$$

a) Let $V, W \in \mathfrak{g}$-mod, show that

$$\mathrm{Hom}_{\mathfrak{g}}(V, W) = \mathrm{Hom}_k(V, W)^{\mathfrak{g}}.$$

b) Let $V, W, Z \in \mathfrak{g}$-mod be finite dimensional, show that there is a natural linear isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(V \otimes W, Z) \simeq \mathrm{Hom}_{\mathfrak{g}}(V, \mathrm{Hom}_k(W, Z)).$$

**2.7.** Let $V \in \mathfrak{g}$-mod be finite dimensional. Let $\mathbb{1}$ be the trivial one dimensional representation, that is $\mathbb{1}$ is $k$ as a vector space and the action is zero. The construction in 2.17 endows

$$V^* \simeq \mathrm{Hom}_k(V, \mathbb{1}),$$

with a $\mathfrak{g}$-module structure.

Let $W \in \mathfrak{g}$-mod be finite dimensional. Show that the linear isomorphism

$$\mathrm{Hom}_k(V, W) \simeq V^* \otimes W,$$

is an isomorphism of representations.

# Lecture 3: Engel's Theorem

**3.1.** Let $V$ be a finite dimensional $k$-vector space, a *flag* $F_\bullet$ in $V$ is a sequence of vector spaces

$$0 \subsetneq F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = V.$$

A *complete flag* is a flag such that the successive quotients $F_{k+1}/F_k$ are one dimensional. We say that an endomorphism $a \in \mathrm{End}_k(V)$ *preserves* the flag if $aF_i \subset F_i$ for all $i = 1, \cdots, n$. The vector space $\mathfrak{p}_F \subset \mathfrak{gl}(V)$ of all endomorphisms preserving the flag $F$ is a Lie subalgebra. It is called a *parabolic subalgebra*. When $F$ is a complete flag it is called a *Borel* subalgebra and is denoted $\mathfrak{b} = \mathfrak{b}_F$.

Let $F_\bullet \subset V$ be a flag and $\mathfrak{p}_F \subset \mathfrak{gl}(V)$ be the corresponding parabolic subalgebra. Define the subset

$$\mathfrak{n} = \mathfrak{n}_F = \left\{ a \in \mathfrak{p}_F \,\middle|\, aF_i \subset F_{i-1} \; \forall\, i = 1, \cdots, n \right\}.$$

Then $\mathfrak{n} \subset \mathfrak{p}_F$ is a Lie algebra ideal.

**3.2 Definition.** An operator $a \in \mathrm{End}_k(V)$ is called *nilpotent* if there exists $n \geq 0$ such that $a^n = 0$.

**3.3 Example.** In the situation of 3.1, every $a \in \mathfrak{n}_F$ is a nilpotent endomorphism of $V$.

**3.4 Lemma.** *Let $a \in \mathrm{End}_k(V)$ be a nilpotent operator, then there exists a non-zero vector $v \in V$ such that $av = 0$.*

*Proof.* If $a = 0$ then any vector of $V$ would do. Otherwise let $m \geq 0$ be such that $a^m \neq 0$ and $a^{m+1} = 0$. Since $a^m \neq 0$ there exists a non-zero vector $v_0 \in V$ such that $v = a^m v_0 \neq 0$. It follows that $av = a^m v_0 = 0$. □

**3.5 Lemma.** *Let $a \in \mathrm{End}_k(V)$ be a nilpotent operator. Then so is* $\mathrm{ad}_a$.

*Proof.* Consider the associative algebra $A = \mathrm{End}_k(V)$. Let $b, c \in A$, and denote by $L_b$ (resp. $R_c$) the operator of *left multiplication by* $b$ (resp. *right multiplication by* $c$). That is $L_b(d) = b{\cdot}d$ (resp. $R_c d = d{\cdot}c$) for any $d \in A$. Since $A$ is associative we have $L_b R_c = R_c L_b$. It follows that

$$\left(L_b - R_c\right)^n = \sum_{j=0}^{n} \binom{n}{j} (-1)^j L_b^{n-j} R_c^j = \sum_{j=0}^{n} \binom{n}{j} (-1)^j L_{b^{n-j}} R_{c^j}. \tag{3.5.1}$$

Applying (3.5.1) with $b = c = a$ and noting that $\mathrm{ad}_a = L_a - R_a$ we obtain

$$\mathrm{ad}_a^n = \sum_{j=0}^{n} \binom{n}{j} (-1)^j L_{a^{n-j}} R_{a^j}. \tag{3.5.2}$$

Suppose $a^m = 0$, then choosing $n = 2m$ we see that each summand in the RHS of (3.5.2) vanishes as either $n - j \geq m$ or $j \geq m$. □

**3.6 Theorem** Engel's Theorem. *Let $V$ be a finite dimensional $k$-vector space and let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie subalgebra consisting of nilpotent operators. Then there exists $0 \neq v \in V$ such that $a \cdot v = 0$ for every $a \in \mathfrak{g}$.*

*Proof.* We proceed by induction in $m = \dim \mathfrak{g}$. If $m = 1$ then $\mathfrak{g} = k \cdot a$ for a non-zero nilpotent operator $a$ and the Theorem in this case reduces to Lemma 3.4.

Assume the Theorem holds for $m \geq 1$, we will prove that it holds for $m + 1$. So let $\mathfrak{g}$ be of dimension $m+1$ and let $\mathfrak{h} \subsetneq \mathfrak{g}$ be a proper subalgebra of maximal dimension. Notice $\dim \mathfrak{h} \geq 1$ since every non-zero element of $\mathfrak{g}$ generates a proper subalgebra. Consider the adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}$. Since $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra, it is also a subrepresentation. It follows from 2.13 b) $\mathfrak{g}/\mathfrak{h}$ is an $\mathfrak{h}$-module, that is, we have a map $\mathrm{ad} : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$. By Lemma 3.5, the image $\mathrm{ad}\,\mathfrak{h} \subset \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$ consists of nilpotent operators and by the inductive hypothesis we have a vector $a \in \mathfrak{g} \setminus \mathfrak{h}$ such that $\mathrm{ad}_h a \in \mathfrak{h}$ for all $h \in \mathfrak{h}$. It follows that $\mathfrak{h} \oplus ka \subset \mathfrak{g}$ is a subalgebra, and by the maximality assumption on $\mathfrak{h}$ we must have $\dim \mathfrak{h} = m$ and $\mathfrak{h} \oplus ka = \mathfrak{g}$.

By the inductive hypothesis there exists a non-zero vector $v_0 \in V$ such that $\mathfrak{h} \cdot v_0 = 0$. Let $U \subset V$ be the subspace of all such vectors. $U$ is invariant by $a$, indeed for $u \in U$ we have

$$hau = [h, a]u + ahu = 0,$$

since $[h, a] \in \mathfrak{h}$ and $hu = 0$. By Lemma 3.4 applied to the nilpotent operator $a \in \mathrm{End}_k(U)$ we see that there exists a non-zero vector $v \in U$ such that $av = 0$. This vector therefore satisfies the conditions for the Theorem. □

**3.7 Corollary**. *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie subalgebra as in Theorem 3.6 then there exists a complete flag $F_\bullet \subset V$ and $\mathfrak{g} \hookrightarrow \mathfrak{n} \hookrightarrow \mathfrak{b}_F$.*

*Proof.* By the theorem there exists a vector $0 \neq v_1 \in V$ such that $\mathfrak{g}v_1 = 0$. We let $F_1 = kv_1$. Consider the representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V/F_1)$. The image $\rho\mathfrak{g}$ consists of nilpotent operators, hence by the theorem there exists a non-zero vector $\overline{v_2} \in V/F_1$ such that $\mathfrak{g}\overline{v_2} = 0$. Equivalently, there exists a vector $v_2 \in V \setminus F_1$ such that $\mathfrak{g}v_2 \in F_1$. We let $F_2 = F_1 \oplus kv_2$. Repeating this procedure we obtain the required flag. □

**3.8.** Let $\mathfrak{g}$ be a Lie algebra. Define the following descending sequence of ideals of $\mathfrak{g}$:

$$\text{(central series)} \qquad \mathfrak{g} \supset \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}] \supset \cdots \supset \mathfrak{g}_{k+1} = [\mathfrak{g}_k, \mathfrak{g}] \supset \cdots,$$
$$\text{(derived series)} \qquad \mathfrak{g} =\supset \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] \supset \cdots \supset \mathfrak{g}^{(k+1)} = [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}] \supset \cdots.$$

**3.9 Definition**. A Lie algebra $\mathfrak{g}$ is called *nilpotent* (resp. *solvable*) if $\mathfrak{g}_i = 0$ (resp. $\mathfrak{g}^{(i)} = 0$) for some $i$.

**3.10 Remark**. It follows from Exercise 3.6 that if $\mathfrak{g}$ is nilpotent it is also solvable.

**3.11 Proposition**. *A non-trivial nilpotent Lie algebra $\mathfrak{g}$ has nontrivial center.*

*Proof.* Let $k$ be such that $0 \neq \mathfrak{g}_k \supset \mathfrak{g}_{k+1} = [\mathfrak{g}_k, \mathfrak{g}] = 0$. Then we have $0 \neq \mathfrak{g}_k \subset Z(\mathfrak{g})$. □

**3.12 Definition**. An *extension* of Lie algebras is a short exact sequence of Lie algebras

$$0 \hookrightarrow \mathfrak{h} \xrightarrow{\iota} \tilde{\mathfrak{g}} \overset{\pi}{\twoheadrightarrow} \mathfrak{g} \to 0. \tag{3.12.1}$$

That is, $\iota$ is an injective morphism of Lie algebras, $\pi$ a surjective morphism of Lie algebras and $\ker \pi = \mathrm{Im}\,\iota$. We say that $\tilde{\mathfrak{g}}$ is an extension of $\mathfrak{g}$ by $\mathfrak{h}$. The extension is called *central* if $\mathfrak{h} \subset Z(\tilde{\mathfrak{g}})$.

**3.13 Proposition**. *A central extension of a nilpotent Lie algebra is nilpotent.*

*Proof.* Consider an extension as in (3.12.1) and assume it is central with $\mathfrak{g}$ nilpotent. Notice that $\pi\tilde{\mathfrak{g}}_k \subset \mathfrak{g}_k$ hence there exist $k$ such that $\pi\tilde{\mathfrak{g}}_k = 0$. It follows that $\tilde{\mathfrak{g}}_k \subset \mathfrak{h}$ and therefore $\tilde{\mathfrak{g}}_{k+1} = 0$ since $\mathfrak{h}$ is central. □

**3.14 Corollary**. *If $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent then $\mathfrak{g}$ is nilpotent.*

**3.15 Theorem** Engel's characterization of nilpotent Lie algebras. *A finite dimensional Lie algebra $\mathfrak{g}$ is nilpotent if and only if $\mathrm{ad}_a$ is nilpotent for every $a \in \mathfrak{g}$.*

*Proof.* If $\mathfrak{g}$ is nilpotent and $a \in \mathfrak{g}$ we see that $\mathrm{ad}_a^k \mathfrak{g} \subset \mathfrak{g}_{k+1}$, so for $k \gg 0$ we have $\mathrm{ad}_a^k = 0$.

Conversely, suppose every $\mathrm{ad}_a$ is nilpotent. It follows that $\mathrm{ad}\,\mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$ is a nilpotent Lie algebra by Theorem 3.6. Therefore $\mathfrak{g}$ is nilpotent by Corollary 3.14. □

# Exercises

**3.1.** Let $V$ be a finite dimensional vector space and $F_\bullet \subset V$ a flag. Show that $\mathfrak{p} \subset \mathfrak{gl}(V)$ is a Lie subalgebra and that $\mathfrak{n} \subset \mathfrak{p}$ is a nilpotent ideal.

**3.2.** Prove that a subalgebra $\mathfrak{g} \subset \mathfrak{gl}_n$ consists of nilpotent matrices if and only if there exists a matrix $S \in GL_n$ such that $S\mathfrak{g}S^{-1}$ is a subalgebra of strictly upper triangular matrices.

**3.3.** Find a subspace of $\mathfrak{gl}(V)$ consisting of nilpotent operators such that there is no common eigenvector.

**3.4.** If $a$ and $b$ are commuting nilpotent operators on a vector space over a field of characteristic zero. Show that $e^{a+b} = e^a e^b$ where

$$e^a = \sum_{n\geq 0} \frac{a^n}{n!}.$$

Notice that this sum is finite. What can you say if the characteristic of the base field is positive?

**3.5.** Compute the center of $\mathfrak{gl}_n$, $\mathfrak{sl}_n$ and $\mathfrak{so}_n$.

**3.6.** Show that $\mathfrak{g}_i$ and $\mathfrak{g}^{(i)}$ as defined in 3.8 are ideals of $\mathfrak{g}$ and that $\mathfrak{g}^{(i)} \subset \mathfrak{g}_i$.

**3.7.** Let $V$ be a finite dimensional vector space and $F_\bullet \subset V$ a full flag. Consider the corresponding Lie subalgebras $\mathfrak{n} \subsetneq \mathfrak{b} \subsetneq \mathfrak{gl}(V)$. Show that $\mathfrak{b}$ is solvable and $\mathfrak{n}$ is nilpotent. What happens if the flag is not full?

# Lecture 4: Lie's Theorem

**4.1 Lemma**. *Consider an extension of Lie algebras*

$$0 \to \mathfrak{h} \xrightarrow{\iota} \tilde{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \to 0.$$

*Then $\tilde{\mathfrak{g}}$ is solvable if and only if $\mathfrak{h}$ and $\mathfrak{g}$ are solvable.*

*Proof.* We have $\iota\,\mathfrak{h}^{(k)} \subset \tilde{\mathfrak{g}}^{(k)}$ and $\pi, \tilde{\mathfrak{g}}^{(k)} = \mathfrak{g}^{(k)}$, hence if $\tilde{\mathfrak{g}}$ is solvable we have $\tilde{\mathfrak{g}}^{(k)} = 0$ and therefore $\iota\,\mathfrak{h}^{(k)} = \mathfrak{g}^{(k)} = 0$. Therefore $\mathfrak{g}$ is solvable and since $\iota$ is injective this implies $\mathfrak{h}$ is solvable.

Conversely. Suppose $\mathfrak{g}^{(k)} = 0$, therefore by exactness of the extension we have $\tilde{\mathfrak{g}}^{(n)} \subset \iota\,\mathfrak{h}^{(n)}$ for every $n \geq k$. In particular for $n \gg 0$ we have $\tilde{\mathfrak{g}}^{(n)} \subset \mathfrak{h}^{(n)} = 0$ and $\tilde{\mathfrak{g}}$ is solvable. ☐

**4.2 Definition**. Let $\mathfrak{h}$ a Lie algebra and $V$ its representation. Let $\lambda \in \mathfrak{h}^*$ and define the subspace

$$V_\lambda = V_\lambda^{\mathfrak{h}} = \left\{ v \in V \,\middle|\, av = \lambda(a)v \,\forall\, a \in \mathfrak{g} \right\}.$$

Then provided $V_\lambda \neq 0$ we call $\lambda$ a *weight* of $V$ and $V_\lambda$ its *weight space.*

**4.3 Lemma** Lie's Lemma. *Let $\mathfrak{g}$ be a Lie algebra over a field $k$ of characteristic zero. Let $V$ be its finite dimensional representation. Let $\mathfrak{h} \subset \mathfrak{g}$ be an ideal and $\lambda \in \mathfrak{h}^*$. Then the weight spaces of $V$ (viewed as $\mathfrak{h}$-module) are $\mathfrak{g}$-invariant. That is*

$$\mathfrak{g} \cdot V_\lambda^{\mathfrak{h}} \subset V_\lambda^{\mathfrak{h}}.$$

*Proof.* Let $a \in \mathfrak{g}$ and $v \in V_\lambda^{\mathfrak{h}}$. We want to prove that for every $h \in \mathfrak{h}$ we have

$$hav = \lambda(h)av.$$

Note that the LHS equals

$$[h,a]v + ahv = [h,a]v + \lambda(h)av = \lambda([h,a])v + \lambda(h)av,$$

where we have used that $\mathfrak{h}$ is an ideal. It is therefore equivalent to prove that $\lambda([h,a]) = 0$ when $V_\lambda^{\mathfrak{h}} \neq 0$. Let $0 \neq v \in V_\lambda^{\mathfrak{h}}$ and $0 \neq a \in \mathfrak{g}$. Consider the vector subspaces $W_n$ of $V$ generated by vectors of the form $a^k v$, $0 \leq k \geq n$. We have $k \cdot v = W_0 \subset W_1 \subset \cdots \subset V$. As $V$ is finite dimensional there exists a minimal $n$ such that $W_n = W_m$ for all $m \geq n$. Let $W = W_n$ and consider the full flag $W_\bullet \subset W$. I claim that a) $\mathfrak{h}$ preserves $W$, b) $\mathfrak{h}$ preserves the flag $W_\bullet$, that is $\mathfrak{h}|_W \subset \mathfrak{b}$ and c) $h \in \mathfrak{h}$ acts by $\lambda(h)$ on $W_i/W_{i-1}$. We prove this by induction on $n$. For $n = 0$ we have $h|_{W_0} = \lambda(h)\,\mathrm{Id}$ for every $h \in \mathfrak{h}$. Suppose that the claim is true for $n$ then we have

$$ha^{n+1}v = [h,a]a^n v + aha^n v.$$

By induction we have $ha^n v = \lambda(h)a^n v \mod (W_{n-1})$, therefore $aha^n v = \lambda(h)a^{n+1}v \mod (W_n)$. Also since $\mathfrak{h}$ is an ideal, by induction we have $[h,a]a^n v \in W_n$. It follows that $ha^{n+1}v = \lambda(h)a^{n+1}v \mod (W_n)$ proving the induction step.

Now consider the action of $h$, $a$ and $[h,a]$ on $W$ and we consider the trace of the corresponding operators. By the claim in the previous paragraph and the fact that $\dim W = n+1$ we have

$$\mathrm{tr}[h,a] = (n+1)\lambda([h,a])$$

On the other hand, the trace is a morphism of Lie algebras, hence $\mathrm{tr}[h,a] = 0$. It follows that $\lambda([h,a]) = 0$ as wanted. ☐

**4.4 Theorem** Lie's Theorem. *Let $\mathfrak{g}$ be a solvable Lie algebra over an algebraically closed field of characteristic zero. Let $V$ be its finite dimensional representation. Then there exists a $\lambda \in \mathfrak{g}^*$ with non-trivial weight space $0 \neq V_\lambda^{\mathfrak{g}} \subset V$.*

*Proof.* It is enough to prove the Theorem for the image of $\mathfrak{g}$ in $\mathfrak{gl}(V)$ hence we may assume $\mathfrak{g}$ is finite dimensional and use induction in $\dim \mathfrak{g}$. If $\dim \mathfrak{g} = 1$, that is $\mathfrak{g} = k \cdot a$ for $a \in \mathfrak{gl}(V)$, there exists an eigenvector for $a$ since $k$ is algebraically closed. Assume by induction that the theorem is known for all solvable Lie algebras of dimension less than $\dim \mathfrak{g}$. Since $\mathfrak{g}$ is solvable we have $\mathfrak{g} \supsetneq \mathfrak{g}^{(1)} = [\mathfrak{g},\mathfrak{g}]$. It follows that there exists a codimension 1 ideal $\mathfrak{h}$ of $\mathfrak{g}$, indeed any subspace of codimension 1 containing $[\mathfrak{g},\mathfrak{g}]$ would do. We have therefore $\mathfrak{g} = \mathfrak{h} \oplus k \cdot a$ for some $0 \neq a \in \mathfrak{g}$. By induction there exists a $\lambda' \in \mathfrak{h}^*$ such that $V_{\lambda'}^{\mathfrak{h}} \neq 0$, and by Lie's Lemma we have $aV_{\lambda'}^{\mathfrak{h}} \subset V_{\lambda'}^{\mathfrak{h}}$. Since $k$ is algebraically closed the operator $a$ acting on $V_{\lambda'}^{\mathfrak{h}}$ has an eigenvector, say $v \in V_{\lambda'}^{\mathfrak{h}}$, with eigenvalue $\mu$. Then defining $\lambda \in \mathfrak{g}$ by $\lambda(h + \alpha a) = \lambda'(h) + \alpha\mu$ for $h \in \mathfrak{h}$ and $\alpha \in k$, we obtain that $0 \neq v \in V_\lambda^{\mathfrak{g}}$ as we wanted. ☐

**4.5 Corollary.** *Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite dimensional representation of a solvable Lie algebra $\mathfrak{g}$ over an algebraically closed field of characteristic zero. Then there exists a full flag $F_\bullet \subset V$ such that $\rho(\mathfrak{g}) \subset \mathfrak{b}_F$.*

*Proof.* By Lie's theorem there exists a some common eigenvector $0 \neq v_1 \subset V$. Let $F_1 = k \cdot v_1 \subset V$. Consider the quotient representation of $\mathfrak{g}$ on $V/F_1$. Proceed by induction on $\dim V$. ☐

**4.6 Corollary.** *A subalgebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ of a finite dimensional vector space $V$ over an algebraically closed field of characteristic zero is solvable if and only if there exists a flag $F_\bullet \subset V$ such that $\mathfrak{g} \subset \mathfrak{b}_F$.*

4.7 **Corollary**. *If $\mathfrak{g}$ is a finite dimensional solvable algebra over an algebraically closed field of characteristic zero, then $[\mathfrak{g}, \mathfrak{g}]$ is a nilpotent Lie algebra.*

*Proof.* Consider the adjoint representation of $\mathfrak{g}$. By the previous corollary we have $\operatorname{ad}\mathfrak{g} \subset \mathfrak{b}_F$ for some full flag $F_\bullet \subset \mathfrak{g}$. It follows that $[\operatorname{ad}\mathfrak{g}, \operatorname{ad}\mathfrak{g}] \subset \mathfrak{n}_F$ so that $\operatorname{ad}\mathfrak{g}$ is nilpotent by Exercise 4.3. It follows that $\operatorname{ad}[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, and therefore $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent by Corollary 3.14. ☐

## 4.8 Some other consequences of algebraically closed base field

Let $\mathfrak{g}$ be a Lie algebra and $V, W$ be $\mathfrak{g}$-modules. Suppose $0 \neq \varphi \in \operatorname{Hom}_{\mathfrak{g}}(V, W)$ is a morphism. Then $\ker\varphi \subsetneq V$ is an $\mathfrak{g}$-submodule. If $V$ is irreducible we see that any non-zero morphism from it has to be injective. Similarly, $0 \neq \operatorname{Im}\varphi \subset W$ is an $\mathfrak{g}$-submodule, therefore if $W$ is irreducible any non-zero morphism to it has to be surjective. It follows that any non-trivial morphism between two irreducible modules has to be an isomorphism. When in addition the base field is algebraically closed we have

4.9 **Lemma** Schur's Lemma. *Let $V$ be a finite dimensional irreducible representation of a Lie algebra $\mathfrak{g}$ over an algebraically closed field. Then*

$$\operatorname{End}_{\mathfrak{g}}(V) = k \cdot \operatorname{Id}_V .$$

*Proof.* We have seen that any non-zero endomorphism has to be an isomorphism. Let $0 \neq \varphi \in \operatorname{End}_{\mathfrak{g}}(V)$. For $\lambda \in k$ we have $\lambda \operatorname{Id}_V \in \operatorname{End}_{\mathfrak{g}}(V)$, therefore we have $\varphi_\lambda := \varphi - \lambda \operatorname{Id}_V \in \operatorname{End}_{\mathfrak{g}}(V)$. On the other hand, since $k$ is algebraically closed, $\varphi$ has an eigenvector, say with eigenvalue $\lambda \in k$. It follows that $\ker\varphi_\lambda \neq 0$. Therefore we must have $\varphi_\lambda = 0$ and $\varphi = \lambda \operatorname{Id}_V$. ☐

# Exercises

4.1. Let $\pi : \mathfrak{g} \twoheadrightarrow \mathfrak{h}$ be a surjective morphism of Lie algebras. Show that for any ideal $I \subset \mathfrak{h}$, its preimage $\pi^{-1} I \subset \mathfrak{g}$ is an ideal. Is the converse true?

4.2. Find a counterexample to Lie's Theorem if the characteristic of $k$ is not zero.

4.3. Let $F_\bullet \subset V$ be a full flag of a finite dimensional vector space. Prove that $\mathfrak{b}_F$ is solvable and $\mathfrak{n}_F$ is nilpotent.

4.4. Let $\operatorname{char} k = p > 0$. Consider the Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(p)$ generated by

$$a = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ & & \cdots & \cdots & \\ 0 & 0 & \cdots & p-2 & 0 \\ 0 & 0 & \cdots & 0 & p-1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ & & \cdots & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \qquad c = \operatorname{Id}_{p\times p}$$

Show that $\mathfrak{g}$ is nilpotent therefore solvable but there is no full flag of $k^p$ such that $\mathfrak{g} \subset \mathfrak{b}$.

# Lecture 5: Jordan Form

## Exercises

**5.1.** Let $\mathfrak{g} = \mathfrak{sl}_2$ and let $V$ be it's two dimensional defining representation. Let $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathfrak{sl}_2$.

a) Describe $\mathfrak{g} = \oplus_\alpha \mathfrak{g}_\alpha^h$.

b) Describe $V = \oplus_\lambda V_\lambda^h$.

c) Check $g_\alpha^h V_\lambda^h \subset V_{\lambda+\alpha}^h$.

**5.2.** Let $\mathfrak{g}$ be the three dimensional Lie algebra of Exercise 4.4 over a field $k$ with characteristic $p$. Describe $\mathfrak{g} = \oplus \mathfrak{g}_\alpha^a$.

**5.3.** Let $\mathfrak{g}$ be the Lie algebra of Exercise 4.4 and let $V = k^p$ be its defining representation. Describe $V = \oplus V_\lambda^a$.

# Lecture 6:

# Lecture 7:

# Lecture 8:

# Lecture 9:

## Exercises

**9.1.** Show that the defining representation of $\mathfrak{sl}_n$ and $\mathfrak{so}_n$ is irreducible.

**9.2.** Show that the trace form on $\mathfrak{sl}_n$ associated to the standard representation and the Killing form are non-degenerate if $\operatorname{char} k = 0$

**9.3.** Show that $\mathfrak{g}_0^a$ is a Cartan subalgebra of $\mathfrak{g}$ for every regular element $a \in \mathfrak{g}$ for any field $k$.

**9.4.** Find a four dimensional irreducible representation of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ such that each summand does not act by zero.

**9.5.** Let $\mathfrak{g}$ be a semisimple finite dimensional Lie algebra over $k = \overline{k}$ with $\operatorname{char} k = 0$ and let $\mathfrak{h}$ be a Cartan subalgebra. Show that there exists another Cartan subalgebra $\mathfrak{h}' \subset \mathfrak{g}$ such that $\mathfrak{h} \cap \mathfrak{h}' = 0$. [Hint: Use an inner automorphism similar to the one in the proof of Chevalley's Theorem]

# Lecture 10:

## Exercises

**10.1.** Let $\mathfrak{g}$ be a finite dimensional solvable Lie algebra over $k = \overline{k}$, $\operatorname{char} k = 0$. Show that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. Find counterexamples for $k \neq \overline{k}$ or $\operatorname{char} k \neq 0$.

**10.2.** Let

$$b = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ -3 & -3 & -3 \end{pmatrix}$$

and let $\mathfrak{h}$ be the set of all $3 \times 3$ matrices $a$ with complex numbers and trace 0 such that $[a, b] = 0$.

a) Find $\dim \mathfrak{h}$.

b) Show that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{sl}_3(\mathbb{C})$.

*c) Can you find $\mathfrak{h}$ explicitly?

**10.3.** Let $\mathfrak{g}$ be a finite dimensional Lie algebra and let $\rho : \mathfrak{g} \to \mathfrak{sl}(V) \subset \mathfrak{gl}(V)$ be a representation with $\dim V = n$. Define $\det V = \wedge^n V$. $\det V$ is naturally a one dimensional vector space and is a representation of $\mathfrak{g}$. Show that this representation is trivial. Show also that this may not be the case if $\rho$ does not factor through $\mathfrak{sl}(V)$.

**10.4.** Let $\mathfrak{g} \subset \mathfrak{gl}_n$ be a subspace consisting of matrices with arbitrary first rows and 0 for the rest of the rows. Find $R(\mathfrak{g})$.