Representações de Grupos

G - grupo. $V \in \Bbbk\text{-Vect}$

Def. um $\rho : G \longrightarrow GL(V)$ homomorfismo. é dito uma representação de G.

$V \in G\text{-mod.}/\Bbbk$ V é um G-modulo.

$$\rho : G \longrightarrow GL_{\Bbbk}(V)$$

$$\psi : G \longrightarrow GL_{\Bbbk}(W)$$

Um <u>morfismo</u> de representações é $T \in \mathrm{Hom}_{\Bbbk}(V, W)$. tq.

$$T\rho(a) = \psi(a) \cdot T \quad \in \mathrm{Hom}(V, W)$$

$\forall a \in G$.

Observação: $\mathrm{Hom}_G(V, W) = \{\text{homom de repr.}\}$

$\in$

$\Bbbk$-Vect.

$\alpha, \beta \in \Bbbk$ $T, S \in \mathrm{Hom}_G(V, W)$

$$\underline{(\alpha S + \beta T)} \cdot \rho(a) = \alpha \cdot S\rho(a) + \beta \cdot T\rho(a)$$
$$= \alpha \cdot \underline{\psi(a)} S + \beta \underline{\psi(a)} \cdot T$$

$$= \psi(g)\,\alpha S + \psi(g)\,\beta T$$

$$= \psi(g)\,(\alpha S + \beta T)$$

$$\Rightarrow \alpha S + \beta T \in \mathrm{Hom}_G(V, W)$$

Corollario a categoría de Representaçõ
de G (corpo k é fixo).
é enriquecida en k-Vect.

$$\mathrm{Hom}_G(V, W) \times \mathrm{Hom}_G(W, Z) \xrightarrow{\circ} \mathrm{Hom}(V, Z)$$

$$S \qquad T \longrightarrow T \circ S$$

$\circ$ é k-bilinear.

$$\Updownarrow$$

$$\mathrm{Hom}_G(V, W) \otimes_k \mathrm{Hom}_G(W, Z) \to \mathrm{Hom}(V, Z)$$

——— ∘ ———

Porquê estudar representações?

$V \simeq k^{\oplus n}$ (escolhe una base de V)

$GL(V) \simeq GL_n(k)$ matrizes $n \times n$
con entradas en k $\det \neq 0$

$$\rho : G \longrightarrow \rho(g)_{ij} \in k \quad 1 \leq i, j \leq n$$

$$(*) \quad \rho(a\cdot b)_{ij} = \sum_{k=1}^{n} \rho(a)_{ik}\,\rho(b)_{kj}$$

multiplicação em G.

Qué acontece se eu escolho uma outra base. $\underline{S} \in \mathrm{Mat}_{n\times n}(k)$ $\det \neq 0$

$$\underline{S\,\rho(a)\,S^{-1}} = \underline{\bar\rho}(a)$$

$$\bar\rho(a\cdot b)_{ij} = \left(S\,\rho(a\cdot b)\,S^{-1}\right)_{ij}$$

$$= \left(S\,\rho(a)\cdot\rho(b)\,S^{-1}\right)_{ij} \qquad (*)$$

$$= S\rho(a)S^{-1}\,S\rho(b)S^{-1}$$

$$S\,\underline{\rho(a)} = \bar\rho(a)\,S \quad e \quad S \in \mathrm{Hom}_G(\overset{\rho}{V},\overset{\bar\rho}{V})$$

Lemma/Def Um Endomorfismo que é invertível como elemento de End(V) é dito um isomorfismo.

$$S \in \mathrm{End}_G(V,V) \ni S^{-1}$$

$$S\,\rho(a) = \rho(a).S$$

$$\rho(a) = S^{-1}\rho(a)\,S \iff \rho(a)S^{-1} = S^{-1}\rho(a)$$

Prop. $\rho : G \to GL(V)$ $\quad \psi : G \to GL(W)$

(representações são isomorfas s.s.

V e W dim $< \infty$.

1) $\dim V = \dim W = n$

2) se escollo una base de V e W

$$\rho(a)_{ij} \qquad \psi(a)_{ij} \qquad \forall\, a \in G.$$

$$\exists\; S \in Mat_{n\times n}(k) \qquad \det S \neq 0.$$

$$\text{tq} \quad S^{-1}\rho(a)S = \psi(a) \qquad \forall\, a \in G$$

—— . ——

Obtenho un conxunto: representações de G / isomorfismo.

—— o ——

Exemplos de representações

1)
$$\begin{aligned} G &\longrightarrow Gl(k) = GL_1(k) = k^* \\ a &\longmapsto 1 \end{aligned}$$

representação trivial

2) Funções de $G \to k$

$$F(G,k) \qquad k^G \qquad \mathcal{O}_G \qquad k[G]$$

$$f \in F(G,k) \qquad a \mapsto f(a) \in k$$

$F(G,k) \in k\text{-Vect}.$

$f^{*} \qquad a \in G.$

$\downarrow$

$$\underline{\rho(a)}(f)(b) = \begin{cases} f(a\cdot b) \\ f(b\cdot a) \\ f(a^{-1}b) \longleftarrow \\ f(b\cdot a^{-1}) \end{cases}$$

__Def__ seja S um conjunto. G um grupo. G age em S na __direita__

$$S \times G \longrightarrow S$$

$$s, a \longmapsto s a$$

$$(s\cdot a)\cdot b = s\cdot\underset{\downarrow}{(a\cdot b)}$$

multiplicação em G

$$\left.\begin{array}{l} G \times S \longrightarrow S \\ a, s \longrightarrow a\cdot s \\ \underset{\text{multiplicação em } G}{(a\cdot b)}\cdot s = a\cdot(b\cdot s) \end{array}\right\} \text{ação a esquerda}$$

$S \circlearrowright G$ (a direita).

$\text{Func.}(S,k) \in G\text{-mod}$

$f \in V$ $a \in G$. $s \in S$

$b \in G$

$(\rho(a)(f))(s) = f(s\cdot a)$ é uma rep

$(\rho(a)\cdot(\rho(b)\cdot f))(s)$

$\shortparallel$

$(\rho(b)\cdot f)(s\cdot a) = f(s\cdot a\cdot b)$ ✓ G age na direita

$= f(s\cdot(a\cdot b))$

$(\rho(a\cdot b)(f))(s) =$

$\forall\, a, b, s$

$\rho(a)\cdot\rho(b) = \rho(a\cdot b)$ ✓

Lemma. $S \supset G$ age pela direita

$a \bullet s := s\cdot a^{-1}$ é uma ação na esquerda.

$(a\cdot b)\cdot s = s\cdot(ab)^{-1} = s\,(b^{-1}a^{-1})$

$= s\cdot b^{-1}\cdot a^{-1}$

$= a\cdot(s\cdot b^{-1}) = a\cdot bs$

S tem uma ação a esquerda de G então.

$$\mathrm{Func}(S, k) \in G\text{-mod}.$$

$$(\rho(a) \cdot f)(s) = f(a^{-1} \cdot s)$$

$S = G$. admite ações a esquerda e a direita.

$$(\rho(a) f)(b) = \begin{cases} f(a^{-1} \cdot b) & \overset{\psi}{\text{—}} \text{ Thiago} \\ f(\underline{\underline{b \cdot a}}) & \rho \end{cases}$$

Será que essas duas representações são isomorfos ?

$$F(G, k) \circlearrowleft \underline{S} \in \mathrm{Aut}(F(G, k))$$

$$f \longrightarrow \underline{g(b)} = f(b^{-1})$$

$$(\rho(a)(\underline{Sf}))(b) = (Sf)(ba) = f(a^{-1} b^{-1})$$

$$\shortparallel$$

$$S(\psi(a) f)(b) = (\psi(a) f)(b^{-1}) = f(a^{-1} b^{-1})$$

$\underline{\rho(a)}S = S\underline{\psi(a)} \quad \forall a \in G$

$S' = S$

ou seja. $\mathrm{Func}(G, k) \in G\text{-mod}$

representação $\underline{\text{regular}}$ de G.

3) $H \subset G$ subgrupo

$G \xrightarrow[\text{esquerd.}]{} G/H \qquad \underline{H\backslash G} \xleftarrow{\text{direita.}} G$

$$Hb \cdot a$$

$$\shortparallel$$

$$H\cdot(b\cdot a)$$

$\mathrm{Func}(H\backslash G, k) \in G\text{-modulo.}$

$\cong$

$\mathrm{Func}(G/H, k) \in G\text{-modulo.}$

$H = 1.$ dá a representação regular.

4). $V \in G\text{-mod.}$

⚠ $\rho(a)v =: a \cdot v \qquad a \in G \qquad v \in V$

$$\begin{tikzcd} G \arrow[r] & \mathrm{Aut}_G(V) \\ a \arrow[r, mapsto] & \mathrm{Conj}_a \end{tikzcd}$$

$$\mathrm{conj}_a(S) = \underbrace{\rho(a) S \rho(a^{-1})}$$

$\rho(a^{-1}) = \rho(a)^{-1} \qquad \rho(aa^{-1}) = \rho(id_G) = id_V$

$= \rho(a)\rho(a^{-1}) =$

$b \in G. \qquad \rho(b).\,\rho(a)S\rho(a^{-1}) = \rho(a)S\rho(a^{-1})\rho(b)$

$\rho(ba).S\,\rho(a^{-1}) = \rho(a)S\rho(a^{-1}b)$

$\rho(a)S\rho(a^{-1}) = \rho(b^{-1}a)S\rho(a^{-1}b)$

$= \rho(b^{-1}a)S(\rho(b^{-1}a)^{-1})$

$$\mathrm{conj}_{a^{-1}} = (\mathrm{conj}_a)^{-1}$$

$S \in \mathrm{Aut}_G(V) \Rightarrow \mathrm{conj}_a S$

$b \in G. \qquad \rho(b).(\mathrm{conj}_a S) = (\mathrm{conj}_a S)\,\rho(b)$

$$\underbrace{\rho(b).\rho(a)S\rho(a^{-1})}_{S} = \underbrace{\rho(a)S\rho(a^{-1})}_{S}\rho(b)$$

$$\underline{\rho(a)S = S\rho(a)} \qquad S \in \mathrm{Aut}_G(V)$$

$$\rho(a)S\rho(a^{-1}) = S$$

$$\mathrm{Conj}_a(S) = S \quad \forall\ \mathrm{Aut}_G(V).$$

$$G \longrightarrow \mathrm{Aut}_G(V)$$

$$a \longrightarrow \mathrm{Conj}_a = \mathrm{id}$$

* * *

$$V \in G\text{-mod}. \qquad GL_k(V)$$

$$G \longrightarrow \mathrm{Aut}\big(GL_k(V)\big)$$

$$a \longmapsto \mathrm{conj}_a$$

não é um k-vect

$$\mathrm{Conj}_{a\cdot b} S = a\cdot b\, S\, (ab)^{-1} = a(bSb^{-1})a$$

$$= \mathrm{conj}_a\, \mathrm{conj}_b\,(S)$$

$$\mathrm{End}_k(V) \in V\text{-mod}.$$

$(\rho, V), (\psi, W)$ são representações de G.

$\mathrm{Hom}_k(V, W) \in G\text{-mod}.$

$(\; S^{\psi}$

$a \in G$. $\quad (a \cdot S) \longmapsto \psi(a) S \rho(a)^{-1}$

$b \in G$. $\quad (a \cdot b)(S) = \psi(ab) S \rho(ab)^{-1}$

$$= \psi(a) \underbrace{\psi(b) S \rho(b^{-1})} \rho(a^{-1})$$

$$= a \cdot b(S)$$

$\square$

Caso particular. $\quad$

$$\begin{tikzcd} \underline{V} \arrow[r, squiggly] & \underline{\mathrm{Hom}(V, k)} \arrow[d, equal] \\ & \underline{V^*} \end{tikzcd}$$

é uma representação.

$\varphi \in V^* \qquad (a \cdot \varphi)(v) = \varphi(a^{-1} \cdot v)$

$k^* \simeq k \qquad$ "

$$\begin{tikzcd} \mathrm{Fun}(G, k) \arrow[r, phantom, "\simeq"] \arrow[r, bend right] & \mathrm{Fun}(G, k)^* = \end{tikzcd}$$

base $\mathrm{Func}(G,k)$

$\{f_i\}_{i\in I} \longrightarrow \varphi_i \in \mathrm{Fun}(G,k)^*$

$$\mathrm{Func}(G,k) \overset{\sim}{\hookrightarrow} \mathrm{Func}(G,k)^*$$

$\boxed{|G| < \infty}$

$$\mathrm{Func}(G,k)^* \in G\text{-mod}$$

$$\neq$$

$$\mathrm{Func}(G,k)$$

G - é infinito.
não são isomorfos.

$V, W \in G\text{-mod}.$ $V \oplus W \in G\text{-mod}$

$$a(v+w) = a\cdot v + a w$$

$$\mathrm{Fun}(G,k) \oplus \mathrm{Fun}(G,k) \simeq \mathrm{Fun}(G,k^2)$$

$V, W \in G\text{-mod}.$ $\underline{V \otimes W} \in G\text{-mod}$

$$a\cdot(v \otimes w) = a\cdot v \otimes a\cdot w$$

$$\begin{tikzcd} V^* \otimes W \arrow[r, phantom, "\overset{\dim<\infty.}{\cong}"] \arrow[r, bend right] & \operatorname{Hom}(V, W) \end{tikzcd}$$

isomorfismo de
Reresentações.

__Ex__: $\operatorname{Hom}_G(V \otimes W, Z) \cong \operatorname{Hom}_G(V, \operatorname{Hom}(W, Z))$

Uma subrepresentação. $U \subset V$

$G \cdot U \subset U$

$i\ U \hookrightarrow V$ morfismo injetivo.

__Def__. Uma representação é dita irredutível. se $U \subset V$
é ou $U = 0$ ou $U = V$.

Uma representação V é indecomponível.

$V \cong U \oplus W$

ou $U = 0$ ou $U \cong V$.

$$U \subseteq U \oplus W$$

$$u \longmapsto u + 0_w$$

irredutível $\Longrightarrow$ indecomponível.

Teorema Se G grupo finito.
Toda representação de dimensão finita é soma de irredutíveis.

Compacto $SU(\underline{3}) \longleftrightarrow SL(\underline{3},\mathbb{R})$

— . —

Recap. $\rho : G \to GL(V)$ rep. de um grupo finito. $\mathbb{k} = \mathbb{R}, \mathbb{C}$.

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[dr] & GL(V) \\ & O(V) \arrow[u, hook] \end{tikzcd}$$

$\langle , \rangle_0 : V \otimes \overline{V} \longrightarrow \mathbb{k}$. $\rightsquigarrow$ $\langle , \rangle$ invariante.

— . —

$$\rho(a)\underbrace{\rho(a)^*} = Id_V$$

Existe uma base de V. $\underline{\underline{\{e_i\}}}$ t.q. $\langle , \rangle$.

$v = \sum v_i e_i$ $\qquad w = \sum w_i e_i$

$$\langle v, w \rangle \underset{\uparrow}{=} \sum v_i \overline{w_i}$$

$\rho(a)_{ij}$ matriz de $\rho(a)$ na base e_i

$$(\rho(a)^*)_{ij} = \overline{\rho(a)_{ji}} \longrightarrow \text{Definição.}$$

$$\sum_{j=1}^{\dim V} \rho(a)_{ij}(\rho(a)^*)_{jk} = \delta_{ik} = (\mathrm{Id}_V)_{ik}$$

$$= \sum_{j=1}^{\dim V} \rho(a)_{ij}\,\overline{\rho(a)_{kj}} = \delta_{ik} = \begin{cases} 1 & i=k \\ 0 & i\neq k \end{cases}$$

— o —

Se $V \otimes_{\mathbb{k}} \overline{V} \longrightarrow \mathbb{k}$. $\langle\, , \rangle$

$$\langle \rho(a).v, w\rangle \simeq: \langle v, \underline{\underline{\rho(a)^* w}} \rangle \quad \longrightarrow \text{Definição.}$$

$\langle\, , \rangle$ invariante. $\Rightarrow$ $\rho(a)^* = \rho(a)^{-1}$

$\Rightarrow$ $\rho(a)\rho(a)^* = \mathrm{Id}_V$.

$\Rightarrow$ $\rho(a)$ é diagonalizável.
autovalores λ_i satisfazem
$\langle \lambda_i, \lambda_i \rangle = 1 \quad \forall\, i = 1, \ldots, \dim V.$ $\Big\}$

$G \ni a \quad \exists\, n$ tq $a^n = 1 \in G$.

$GL(V) \ni \rho(a)^n = \mathrm{Id}_V = \rho(1)$

→ autovalores de $\rho(a)$. $\{\lambda_i\}$

$\lambda_i^n = 1$. e raízes n-esima da unidade em $\mathbb{C}$ tem norma 1.

* * *

$V, W \in G$-mod. $\rho : G \to GL(V)$

$v \quad w$ $\eta : G \to GL(W)$.

$\rho \underline{\otimes \eta} : G \longrightarrow GL(V \otimes W)$

$$(\rho \otimes \eta)(a) . (v \otimes w) = \underset{\overset{\uparrow}{V}}{\rho(a)(v)} \otimes \underset{\overset{\uparrow}{W}}{\eta(a)(w)}$$

$$a \cdot (v \otimes w) = \underline{a} \cdot v \otimes \underline{a} \cdot w$$

base $\{v_i\}_{i \in I}$ em V $\{w_j\}_{j \in J}$ em W.

$\rho(a)_{i i'}$ $\eta(a)_{j j'}$ $v_i \otimes w_j$ formam uma base de $V \otimes W$.

$$\underline{((\rho \otimes \eta)(a))_{(i,j)(i',j')}}$$

$I \times J$

$\rho(a)_{ij}$ ⇝ definição de uma matriz associada a um endomorfismo

$$\rho(a)\, v_j = \sum_{i=1}^{\dim V} \underset{=}{\rho(a)_{ij}}\, v_i$$

$$(\rho\otimes\eta)(a)\,(v_{i'}\otimes w_{j'}) = \rho(a)(v_{i'}) \otimes \eta(a)(w_{j'})$$

$$= \left(\sum_i \rho(a)_{ii'}\, v_i\right) \otimes \left(\sum_j \eta(a)_{jj'} \cdot w_j\right)$$

$$= \sum_{i,j} \rho(a)_{ii'}\, v_i \otimes \eta(a)_{jj'}\, w_j$$

$$= \sum_{i,j} \left(\rho(a)_{ii'} \cdot \eta(a)_{jj'}\right) \cdot (v_i \otimes w_j)$$

$$\shortparallel$$

$$(\rho\otimes\eta)(a)_{(i,j),(i',j')}$$

$$\dim(V\otimes W) = \dim(V) \cdot \dim(W).$$

$W = V$. a representação $V\otimes V$ não é irredutível. $\dim V > 1$.

$$\theta(v\otimes v') = v'\otimes v$$

$$\in \mathrm{Aut}_{\underline{\underline{G}}}(V\otimes V)$$

$\forall a \in G.$ $\quad \Theta(a \cdot v \otimes a v') = \Theta(a \cdot (v \otimes v'))$

$$= a \cdot v' \otimes a \cdot v = a(v' \otimes v)$$

$$= a\,\Theta(v \otimes v')$$

$\Rightarrow \Theta a = a \Theta \quad \forall a \in G.$

se $\underline{\dim V > 1}$. $\underline{\Theta \neq \lambda \, Id}_{(V \otimes V)}$

$\Theta^2 = id. \Rightarrow$ autovalores são ± 1.

$$\begin{array}{ccl} \boxed{Sym^2 V} & \subset \underline{\underline{V \otimes V}} & \text{autovalor } 1. \\ \oplus & & \\ \boxed{\Lambda^2 V} & \subset V \otimes V & \text{autovalor } -1. \\ \wr\!\wr & & \\ V \otimes V & & \end{array}$$

$\dim V = n \qquad \dim(V \otimes V) = n^2$

$\dim \Lambda^2 V = \frac{n(n-1)}{2}$

$+$

$\dim Sym^2 V = \frac{n(n+1)}{2}$

Exemplo. $G = S_n \curvearrowright \{e_1, \dots, e_n\}$

$$S_n \longrightarrow GL_n(k)$$

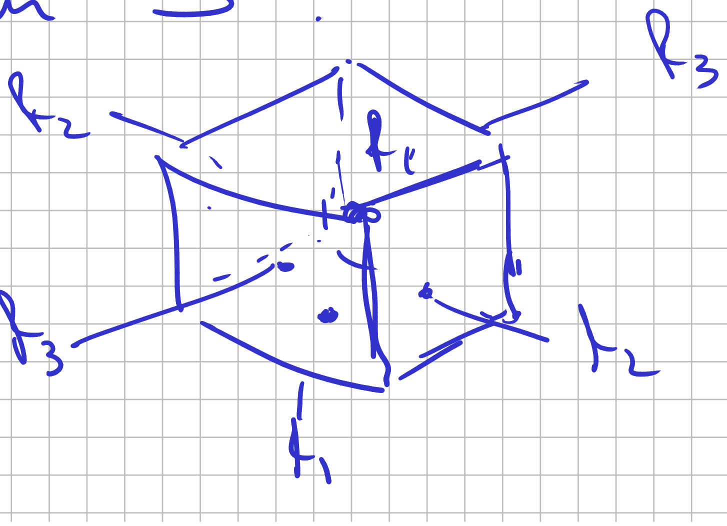
$$\sigma \longrightarrow \text{permuta a base } \{e_1, \dots, e_n\}$$
$$\text{pela ação de } \sigma.$$

$e = \sum_{i=1}^{n} e_i$ é invariante.

$$\begin{tikzcd} 0 \arrow[r] & k\cdot e \arrow[r] & k^{\oplus n} \arrow[r] & V \arrow[r] & 0 \\ & G \arrow[u, phantom, "\circlearrowleft"] & G \arrow[u] & & \end{tikzcd}$$

<u>Teorema.</u> V é irredutível. $\dim V = n-1$

S_4 = Automorfismo do cubo.
Tem uma representação irredutível de dim 3.



S_4 tem uma rep. de dim 6. permutando as faces do cubo.

8 _ vértices do cubo.

4 _ Diagonais principais do cubo.

12 _ permutando. Arestas do cubo.

$f = f_1 + f_2 + f_3$

$$\underbrace{kf_1 \oplus kf_2 \oplus kf_3}_{W} \curvearrowleft S_4$$

$$0 \to k \cdot f \longrightarrow W \longrightarrow K \to 0$$

$K \searrow$ $\underline{\underline{\text{irred.}}}$?

$\dim K = 2.$

$$V \rightsquigarrow \underline{\underline{\Lambda^2 V}} \qquad \operatorname{Sym}^2 V$$

$$\underset{3.}{} \qquad \underset{k}{} \qquad \underset{k}{}$$

$$3. \overset{\sim}{\underset{?}{\longrightarrow}} 3 \qquad 6$$

Teoria de Charateres

$$\begin{tikzcd}
\rho : G \arrow[r] \arrow[rr, bend right] \arrow[d] & GL(V) \arrow[r, "det"] & k^* = GL_1(k) \\
\underline{U(V)} \arrow[r, "det"] & \begin{cases} U(1) \subset \mathbb{C}^* \\ \pm 1 \subset \mathbb{R}^* \end{cases} &
\end{tikzcd}$$

representação dim 1.

$$det \circ \rho : G \longrightarrow \mathbb{Z}_2 = (\pm 1, \cdot)$$

$$det \circ \rho . \; G \longrightarrow U(1) = S^1 \subset \mathbb{C}^*$$

— o —

Charater de $\rho : G \to GL(V)$ $\quad \chi_V$

$$\chi_V : G \to k$$

$$\begin{tikzcd}
\underline{a} \arrow[r] & \rho(\underline{a}) \in GL(V) \arrow[r, "Tr"] \arrow[d, phantom, "\cap"] & k . \\
& End V \simeq V^* \otimes V \arrow[d] & \psi \otimes v \arrow[d, mapsto] \\
& k & \psi(v)
\end{tikzcd}$$

$a = 1 \in G$. $\rho(a) \in$ ~~Id~~ $GL(V)$

$$\chi(1) = \dim V$$

$$\chi(a^{-1}) = \overline{\chi(a)}$$

$$\left.\begin{array}{l} \operatorname{Tr}\rho(a)^{*} = \operatorname{Tr}\rho(a)^{-1} \quad \text{(unitariedade).} \\ \parallel \qquad\qquad = \operatorname{tr}\rho(a^{-1}) \\ \overline{\chi(a)} \qquad = \chi(a^{-1}) \end{array}\right]$$

a é diagonalizável. $\{\lambda_i\}$

$$\chi(a) = \sum \lambda_i \qquad \chi(a^{-1}) = \sum \lambda_i^{-1}$$

$$\chi(a^{-1}) = \sum \overline{\lambda_i}$$

$\bar{\lambda}_i$ são os auto valores de $\rho(a)^{*}$

$$\Rightarrow \overline{\chi(a)} = \chi(a^{-1}).$$

$$\operatorname{Tr}\rho(a)\rho(b) = \operatorname{tr}\rho(b)\cdot\rho(a)$$

$$\parallel$$

$$\operatorname{Tr}(\rho(ab)) = \operatorname{tr}\rho(ba).$$

$$\chi(ab) = \chi(ba).$$

$$\chi(aba^{-1}) = \chi(b)$$

$$\chi : G/\sim \longrightarrow \mathbb{R}.$$

$$a \sim b \quad \exists\ c \in G \quad \underline{ca\underline{c^{-1}}} = b.$$

* * *

Caracter determina. V - completamente.

$$\chi_{V\oplus W} = \chi_V + \chi_W$$

$\{v_i\} \cup \{w_j\}$ base de $V\oplus W$

$$Tr\begin{bmatrix}[P_V] & 0\\ 0 & [P_W]\end{bmatrix} = tr[P_V] + Tr[P_W].$$

$$\chi_{V\otimes W} = \chi_{\underset{\rho}{V}} \cdot \chi_W \;\text{—}\; \eta$$

$v_i,\ w_j \qquad v_i \otimes w_j$

$$Tr(P\otimes\eta)(a) = \sum_{i,j} (P\otimes\eta(a))_{(i,j)(i,j)}$$

$$= \sum_{i,j} P(a)_{i,i} \cdot \eta(a)_{j,j} = \left(\sum_i P(a)_{ii}\right)\left(\sum_j \eta(a)_{jj}\right)$$

$$= \chi_V(g) \cdot \chi_W(g) = \chi_{V\otimes W}(g).$$

$$\begin{tikzcd}
G\text{-mod} \arrow[r, squiggly] & \text{Anel} \ (\text{ex. Fun } G \to \mathbb{C}) \\
\underline{V} \arrow[r, mapsto] & \overset{\psi}{\chi_V} \\
\boxed{\begin{array}{c} \underline{V \oplus W} \\ \underline{V \otimes W} \end{array}} & \\
\underline{V \oplus W} \arrow[r] & \chi_V + \chi_W \\
\underline{V \otimes W} \arrow[r] & \underline{\chi_V \cdot \chi_W}
\end{tikzcd}$$

Cat. conjunto.

$\cong$

[Grothendieck Ring
of G-mod.]

$\chi_V - \chi_W$ não é necessariamente
o caracter de uma repre.

$V - W$?

Como aplicação.

$$V \otimes V. \qquad \chi_{\underline{V\otimes V}} = (\chi_V)^2$$

$$\shortparallel \qquad\qquad\qquad \shortparallel$$

$$\Lambda^2 V \oplus \underline{\mathrm{Sym}^2 V} \qquad \chi_{\underline{\Lambda^2} V} + \chi_{\mathrm{Sym}\underline{^2} V}.$$

$a \in G$. $\rho(a) \in GL(V)$. λ_i

$$a^2 \rightsquigarrow \underline{\lambda_i^2} \qquad \chi(a^2) = \sum \lambda_i^2$$

$$a \cdot (e_i \otimes e_j + e_j \otimes e_i) = \underline{(\lambda_i \lambda_j)} (e_i \otimes e_j + e_j \otimes e_i)$$

$$a \cdot (e_i \otimes e_j - e_j \otimes e_i) = \underline{(\lambda_i \lambda_j)} (e_i \otimes e_j - e_j \otimes e_i)$$

$$i < j$$

$$i \leq j$$

$$\frac{n(n+1)}{2} \qquad \frac{n(n-1)}{2}$$

$$\chi_{\Lambda^2 V}(a) = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j =$$

$$\chi_{\mathrm{Sym}^2}(a) = \sum_{1 \leq i \leq j \leq n} \lambda_i \lambda_j = \sum_{i=1}^{n} \lambda_i^2 + \sum_{i<j} \lambda_i \lambda_j$$

$$\left(\sum \lambda_i\right)^2 = \underline{\sum \lambda_i^2} + 2 \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j$$

$$\boxed{2\cdot\sum_{i<j}\lambda_i\lambda_j + \sum_i\lambda_i^2 = \left(\sum_i\lambda_i\right)^2 = \chi(a)^2}$$

$$= \chi_{V\otimes V}(a).$$

$$\chi(a^2) = \sum\lambda_i^2$$

$$\chi_{Sym^2(V)}(a) = \chi_V(a^2) + \chi_{\Lambda^2(V)}(a)$$

$$\Rightarrow \chi_{Sym^2(V)}(a) = \frac{1}{2}\left(\chi_V(a)^2 + \chi(a^2)\right)$$

$$\chi_{\Lambda^2(V)}(a) = \frac{1}{2}\left(\chi_V(a)^2 - \chi(a^2)\right)$$

* * *

Temos $V \rightsquigarrow \chi_V$:

Quantas repres. irred. o grupo pode ter.

$$\chi_V : G \to k.$$

$$\underline{\chi(aba^{-1}) = \chi(b)}.$$

$$\boxed{\chi_V : \text{Classes de Conj.} \longrightarrow k.}$$

χ_V determina V.

$$\Downarrow$$

Para cada função. Classes $\to$ $\underline{k}$.
conj.

Voltar na representação V.

com esse caráter.

$$\underline{\underline{\chi(1)}} = \dim V \in \mathbb{Z}_+ .$$

Rep. G (determina G)

$$\underline{\mathcal{C}}, \underline{\oplus}, \underline{\otimes} \xrightarrow{\Gamma} \underline{\mathrm{Vect}}, \oplus, \otimes$$

$$X \oplus Y \longmapsto \Gamma(X \oplus Y) \simeq \Gamma(X) \oplus \Gamma(Y)$$

$$G := \underline{\mathrm{Aut.}}(\Gamma) = \underset{\text{natural}}{\mathrm{Transf.}}(\Gamma, \Gamma)$$

$$\underline{\mathcal{C} \simeq \mathrm{Rep}\, G} .$$

$\mathcal{C}$ - categoria .

- Kernels - cokernels. ← A~~neis~~ ~~Grupos~~

Reps de Aneis
Reps. de Grp

- $\text{Hom}_{\mathcal{C}}(X,Y) \in k\text{-Vect}$

  ↳ k-linear linear.

$$\text{Hom}(X,Y) \otimes_k \text{Hom}(Z,X) \longrightarrow \text{Hom}(Z,Y)$$

$$T \otimes S \longmapsto T \circ S$$

A~~neis~~ k-algebras. ✓

- $\times$ produtos $\sqcup$ coprodutos

$$X \times Y \simeq X \sqcup Y$$

$$X \oplus Y$$

Grupos Abelianos ✓

Representações sim.

- $\underline{\underline{\otimes}}$

$X, Y \in \mathcal{C} \rightsquigarrow X \otimes Y$

$$X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$$

$$\begin{tikzcd}
X \otimes (Y \otimes (Z \otimes T)) \arrow[rr] \arrow[d] & & X \otimes ((Y \otimes Z) \otimes T) \arrow[d] \\
(X \otimes Y) \otimes (Z \otimes T) \arrow[rrd] & = & (X \otimes (Y \otimes Z)) \otimes T \arrow[d] \\
& & ((X \otimes Y) \otimes Z) \otimes T
\end{tikzcd}$$

$\mathbb{1} \in \mathcal{C}$. $\quad \mathbb{1} \otimes X \simeq X \otimes \mathbb{1} \simeq X$

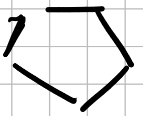
$\mathcal{C}$ é dita monoidal.

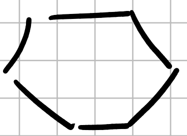
$$X \otimes Y \overset{\tau}{\simeq} Y \otimes X$$

$$\begin{tikzcd}
X\otimes(Y\otimes(Z\otimes T)) \arrow[rr] \arrow[d] \arrow[dd, bend left] \arrow[ddrr, bend left] & & (Z\otimes T)\otimes(X\otimes Y) \arrow[d] \\
(Y\otimes(Z\otimes T))\otimes X \arrow[d] & = & (T\otimes Z)\otimes X\otimes Y \arrow[d] \\
(Z\otimes T)\otimes Y\otimes X \arrow[rr] & & T\otimes Z\otimes Y\otimes X
\end{tikzcd}$$

+ Diagrama similar 6 - entradas.

Categoria que satisfaz.





é dita braided.

+ $$\begin{tikzcd} X\otimes Y \arrow[r, "\sigma", "\simeq"'] \arrow[rr, bend right, "\sigma^2 = \mathrm{id}_{X\otimes Y}"'] & Y\otimes X \arrow[r, "\sigma", "\simeq"'] & X\otimes Y \end{tikzcd}$$

Simétrica ou tensorial.

Vector spaces / k.

Representações. ✓

Fibrados vetoriais ✓

Feixes em variedades ✓
algébricas

$$X_1 \otimes X_2 \otimes X_3 \otimes \cdots \otimes X_n$$

$$\wr\wr \qquad\qquad \sigma \in S_n$$

$$X_{\sigma(1)} \otimes X_{\sigma(2)} \otimes \cdots \otimes X_{\sigma(n)}$$

Exemplos de Categorias que
são braided não são simétricas

- grupos Quânticos. q
- Algebras dimensão infinita. x

$$\begin{tikzcd} V \otimes W \arrow[r, phantom, "\simeq"] \arrow[rr, bend right, "\textcircled{q}"'] & W \otimes V \arrow[r, phantom, "\simeq"] & V \otimes W \end{tikzcd}$$

* * *

Rigidez $\otimes$, $\mathbb{1}$

Hom - internos. $X, Y \in \mathcal{C}$

$\mathrm{Hom}(X,Y) \in k\text{-Vect}.$

$\underline{\mathrm{Hom}}(X,Y) \in \mathcal{C}$

$$\underline{\mathrm{Hom}}(X,Y) \otimes_{\mathcal{C}} \underline{\mathrm{Hom}}(Z,X) \xrightarrow{\circ} \underline{\mathrm{Hom}}(Z,Y)$$

$\otimes$ $\mathbb{1}$ $\underline{\mathrm{Hom}}(X,Y).$

$X \rightsquigarrow X^* = \underline{\mathrm{Hom}}(X, \mathbb{1})$

Duais existem. Categorias onde.

$\otimes$, $\mathbb{1}$, $\underline{\mathrm{Hom}}$

k, Vect ✓   Rep. de grupos ✓.

Reps em geral ✓

Rigidez. $(X^*)^* \simeq X$.

↗ k-~~Vect~~ k-Vect dim $<\infty$ ✓

$$\underline{\mathrm{Hom}}(X,Y) \simeq X^* \otimes_{\mathcal{C}} Y$$

Categoría rígida tensorial.

k-Vect. dim $< \infty$

dim $< \infty$.

Representações grupos, Aneis

Fibrados vetoriais. ✓

—— . ——

$F : \mathcal{C} \to \mathcal{D}$ $\qquad X \longrightarrow F(X)$

$\varphi \in X \to Y \rightsquigarrow F(\varphi) : F(X) \to F(x)$

$$F(X \oplus Y) \simeq F(X) \oplus F(Y)$$

$$F(X \otimes Y) \simeq F(X) \otimes F(Y)$$

$$F(\mathbb{1}_{\mathcal{C}}) \simeq \mathbb{1}_{\mathcal{D}}$$

$$F(\underline{\mathrm{Hom}}(X,Y)) \simeq \underline{\mathrm{Hom}}(F(X), F(Y))$$

—— . ——

<u>Def</u>. Uma categoria é dita <u>Tannakiana</u> se: $\mathcal{C}$, rígida tensorial.

$$\Gamma : \mathcal{C} \longrightarrow k\text{-Vect dim} < \infty.$$

Funt $(\mathcal{C}, \mathcal{D})$ é uma categoria.

F, G morfismos são transf. naturais α

$X \xrightarrow{\varphi} Y$

$$F \xrightarrow{\alpha} G \qquad \forall X \in \mathcal{C}.$$

$$\begin{tikzcd} F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(\varphi)"'] \arrow[dr, phantom, "\circlearrowright"] & G(X) \arrow[d, "G(\varphi)"] \\ F(Y) \arrow[r, "\alpha_Y"] & G(Y) \end{tikzcd}$$

$\alpha_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$

Aut objetos. é um grupo.
composição.

$\mathcal{C}$ Tannakiana. $\Gamma: \mathcal{C} \to \mathbb{Q}\mathrm{Vect}$.

$G := \mathrm{Aut}_{\Gamma}$

Teorema. Tannaka-Krein duality.

$\mathcal{C} \simeq G\text{-rep}.$

Programa Geométrico de Langlands.

Começa com aquele teorema.

$\underline{G}$ - grp de Lie $(SO(n))$

Construimos uma categoria de $\underline{\mathcal{C}_G}$ G-fibrados num espaço.

Fibrados vetoriais com uma <u>métrica</u>

$\mathcal{C}_G$ é rígida tensorial.

$\mathcal{C}$ é tannakiana

$$\begin{tikzcd} F \arrow[d] \arrow[r, "\Gamma"] & H^*(X, F) \in k\text{-Vect} \\ X & \end{tikzcd}$$

$\Gamma : \mathcal{C}_G \longrightarrow k\text{Vect}$ $\qquad \Gamma = H^*(X, \cdot)$

Teorema. $\boxed{\mathcal{C}_G} \cong : \text{rep}\, \underline{G}^L$ — $SP(n)$

$$\begin{tikzcd} & \mathcal{C}_G \arrow[d, dash] \\ \underline{G} \arrow[ur] \arrow[d, dash] & \circ \arrow[r] & \underline{G}^L \\ SO(n) & & \end{tikzcd}$$

Teorema Tannaka Milne - Deligne

Tannakian Categories

MacLane Categories for the working mathematician.

* * *

Caracteres de representações de Grupos finitos.

$V \in G\text{-mod}.$ $\chi_V(a) = Tr_V\, \rho(a)$

$\rho : G \to GL(V)$

Func. en G. tem um produto interno natural.

$V \otimes \overline{V} \longrightarrow \mathbb{C}$ $(\overline{V} \otimes V \longrightarrow \mathbb{C})$

$(\alpha, \lambda \cdot \beta) = \overline{\lambda}(\alpha, \beta)$ $(\alpha, \lambda\beta) = \lambda(\alpha, \beta)$

$\lambda \in \mathbb{C}$ $\alpha, \beta \in V$

* * *

$f : G \to k$ $g \in G \to k.$

$$\langle f, g \rangle = \frac{1}{|G|} \sum_{a \in G} f(a)\, \overline{g(a)}$$

$$\langle 1, 1 \rangle = \frac{1}{|G|} \sum_{a \in G} 1 \cdot 1 = 1$$

$$\langle f, f \rangle = \frac{1}{|G|} \sum_{a \in G} f(a) \overline{f(a)} > 0 \qquad \checkmark$$

Teorema $V, W \in G$-mod irredutíveis. $\chi_V = \chi_W$ se são isomorfos.

$$\langle \chi_V, \chi_W \rangle = 0 \text{ se não}$$

forem isomorfos.

$V \in G$-mod. $v_0 \in G$.

$$v = \frac{1}{|G|} \sum_{a \in G} a \cdot v_0 \in V^G.$$

$$V, W \in G\text{-mod} \Rightarrow \begin{array}{c} \mathrm{Hom}(V, W). \\ \rotatebox{90}{$\in$} \\ S_0 \end{array}$$

$$S = \frac{1}{|G|} \cdot \sum_{a \in G} a \cdot S_0 \in \begin{array}{c} \mathrm{Hom}(V, W)^G \\ \rotatebox{90}{$=$} \\ \mathrm{Hom}_G(V, W) \end{array}$$

$$(a\cdot S_0)\cdot v = a S_0 a^{-1}\cdot v \qquad \forall\, v\in V$$

$$a\cdot S_0 = a S_0 a^{-1}$$

$$\parallel$$

$$\rho_W(a)\, S_0\, \rho_V(a)^{-1}$$

$$S = \frac{1}{|G|}\cdot \sum a S_0 a^{-1} \in \mathrm{Hom}_{\underline{\underline{G}}}(V,W)$$

se V, W irred. não isomorfos.

$S = 0$!

V, W são isomorfos. $\Rightarrow$

$$\boxed{S = \underline{\underline{\lambda}}\, \mathrm{Id}_V} \qquad \lambda\in\mathbb{C}.$$

Schur.

$V = V.$

$$\mathrm{Tr}_V S = \lambda \cdot \mathrm{Tr}_V \mathrm{Id}_V = \lambda \dim V.$$

$$\Rightarrow \lambda = \frac{\mathrm{Tr}_V S}{\dim V}.$$

$$\mathrm{Tr}_V S = \frac{1}{|G|}\sum_a \mathrm{Tr}\,(a S_0 a^{-1})$$

$$= \frac{1}{|G|}\sum_a \mathrm{tr}\, S_0 = \mathrm{Tr}\, S_0 .$$

$S_0 \in \mathrm{End}(V)$ $\quad$ V irred.

*). $$S = \frac{1}{|G|}\sum a S_0 a^{-1} = \frac{(\mathrm{Tr} S_0)}{\dim V}\,\mathrm{Id}_V .$$

seja $V \in G$-mod. escolha uma base.

$\rho(a) = \rho_{ij}(a)$.

$S^0 \in \mathrm{End}\, V$ $\quad$ S^0_{ij} Matriz.

$$S_{ij} = \frac{1}{|G|}\sum \rho_{ik}(a)\, S^0_{kl}\, \rho_{lj}(a^{-1})$$

$$\frac{1}{|G|}\sum_{a\in G}\ \sum_{k,l=1}^{\dim V} \rho_{ik}(a)\, S^0_{kl}\, \rho_{lj}(a^{-1}) =$$

$\forall S^0$.

$$\begin{cases} i \neq j & 0 \\ i = j & \dfrac{\operatorname{Tr} S^0}{\dim V} \end{cases} .$$

se $V \ncong W$. então $S = 0$.

$$0 = \frac{1}{|G|} \sum_{a \in G} \sum_{k=1}^{\dim W} \sum_{l=1}^{\dim V} \rho^W_{\underline{\underline{i}}k}(a) S^0_{kl} \rho^V_{l\underline{\underline{j}}}(a^{-1})$$

$\forall$ i, j $\quad 1 \leq i \leq \dim W \quad 1 \leq j \leq \dim V$.

Qual é a relevância com os caracteres de V, W ?

Observação

$$(\chi_V, \chi_W) = \frac{1}{|G|} \sum_{a \in G} \chi_V(a) \overline{\chi_W(a)}$$

$$= \frac{1}{|G|} \sum_{a \in G} \chi_V(a) \chi_W(a^{-1})$$

$$\frac{1}{|G|}\sum_{a\in G}\sum_{k=1}^{\dim W}\sum_{l=1}^{\dim V}\rho^{W}_{\underset{\nearrow}{\underset{=}{i}}k}(a)\,S^{0}_{kl}\,\rho^{V}_{l\underset{\nearrow}{\underset{=}{j}}}(a^{-1})$$

$$S^{0}_{kl} \in Mat_{n\times m}(k).$$

$$n=\dim W \qquad m=\dim V.$$

$$S^{0} = e_{kl} = (e_{kl})_{ij} = \begin{cases} 1 & i=k\ l=j \\ 0 & \text{caso contrário} \end{cases}$$

$$\frac{1}{|G|}\sum_{a\in G}\rho^{W}_{ik}(a)\,\rho^{V}_{lj}(a^{-1}).$$

Definição. f, g Funções en G.

$$\langle f,g\rangle = \frac{1}{|G|}\sum_{a G} f(a)\,g(a^{-1})$$

$$\rho^{W}_{ik},\ \rho^{V}_{ll} \in Func(G)$$

$$(*)\ \langle \rho^{W}_{ik}, \rho^{V}_{lj}\rangle = \begin{cases} 0 & i\neq j \\ 0 & k\neq l \\ 0 & V\neq W \\ 1/\dim V & i=j\ k=l\ V=W \end{cases}$$

(Peter-Weyl) Funções em G. definidas entradas matriciais de irreps. são orthogonais!

— · —

Teorema. V, W. rep irred.

$V \not\cong W$.

1). $(\chi_V, \chi_V) = 1$

2) $(\chi_V, \chi_W) = 0$.

$$(\chi_V, \chi_V) = \frac{1}{G} \sum_{a \in G} \chi_V(a) \chi_V(a^{-1})$$

$$= \frac{1}{G} \sum_a \left(\sum_i \rho_{ii}(a) \right) \left(\sum_j \rho_{jj}(a^{-1}) \right)$$

$$= \frac{1}{G} \sum_{a \in G} \sum_{i,j} \rho_{ii}(a) \rho_{jj}(a^{-1})$$

$$= \frac{1}{G} \sum_{a \in G} \sum_i \rho_{ii}(a) \rho_{ii}(a^{-1})$$

$$= \sum_i \left(\frac{1}{|G|} \sum_e \rho_{\underline{ii}}(a)\, \rho_{\underline{ii}}(a^{-1}) \right)$$

$$\sum_i \frac{1}{\dim V} = 1 . \quad \checkmark$$

2) $$(\chi_V, \chi_W) = \frac{1}{|G|} \sum_{a \in G} \chi_V(a)\, \chi_W(a^{-1})$$

$$= \frac{1}{|G|} \sum_{a \in G} \sum_{ij} \underbrace{\rho^V_{ii}(a)\, \rho^W_{jj}(a^{-1})}_{\overset{\shortparallel}{0} \quad V \neq W .}$$

$$(\chi_V, \chi_W) = 0$$

$$\left\{ \begin{matrix} \text{reps. irreds.} \\ \text{de } G . \end{matrix} \Big/ \sim \right\} \longrightarrow \{ \text{Func. } G \} \quad (\, , \,)$$

$$V \longmapsto \chi_V \quad \text{orthonormal.}$$

Corollario. este conjunto é finito!

máximo eu tenho $|G|$ reps. irreds. diferentes.

$$\chi_V(aba^{-1}) = \chi_V(b). \quad \checkmark$$

Def Seja $C(G) \subset \mathrm{Fun}(G)$ as funções $f(aba^{-1}) = f(b)$.

Lemma. $\langle , \rangle|_{C(G)}$ continua sendo um produto interno não deg.

No máximo tenho $|C(G)|$ irred. representações.

$$\parallel$$

número de conjugações de G.

$$(*)\, S_{ij} = \frac{1}{|G|} \sum_{a \in G} \rho_{ik}(a) S^{0}_{kl} \rho_{lj}(a^{-1})$$

$$\frac{1}{|G|} \sum_{a \in G} \sum_{k,l=1}^{\dim V} \rho_{ik}(a) S^{0}_{kl} \rho_{lj}(a^{-1}) =$$

$\forall S^{0}$.

V, W duas reps. irredutíveis dim $< \infty$ de G. $\{v_i\}_{i=1}^{\dim V}$ $\{w_j\}_{j=1}^{\dim W}$ bases de V e W.

$S^{0} \in \mathrm{Hom}_{k}(V, W)$ $\quad S^{0}$

$\rightsquigarrow S^{0}_{ij} \in \mathrm{Mat}_{\dim V \times \dim W}(k)$

$S \in \mathrm{Mat}(k)$ dada pela fórmula. $(*)$

$$(*)\, S_{ij} = \frac{1}{|G|} \sum_{a \in G} \rho_{ik}(a) S^{0}_{kl} \rho_{lj}(a^{-1})$$

$S_{ij} = 0 \ \forall i,j$ se $V \not\cong W$.

$$V = W \Rightarrow S_{ij} = \begin{cases} 0 & i \neq j \\ \mathrm{Tr}\, S^{0} / \dim V. & i = j \end{cases}$$

Teorema/ Def. No espaço de fuções de $G \to \mathbb{C}$. $\underline{(}f,g) = \frac{1}{|G|}\sum_{a\in G} f(a)\overline{g(a)}$

Seja $\chi_V(a) = \mathrm{Tr}_V\, \rho_V(a)$ o carater de V.

Para duas fucões defino.

$$\langle f, g\rangle = \frac{1}{|G|}\sum_{a\in G} f(a)\, g(a^{-1})$$

$$(\chi_V, \chi_W) = \langle \chi_V, \chi_W\rangle.$$

$$\hookrightarrow \quad \overline{\chi_V(a)} = \chi_V(a^{-1})$$

autovalores de $\rho_V(a)$ são raízes da unidade $\Leftarrow$ G é finita

Teorema. V, W irredutíveis.

$$(\chi_V, \chi_W) = \begin{cases} 0 & V \not\cong W \\ 1 & V \simeq W \end{cases}$$

$$\langle \chi_V, \chi_W\rangle = \frac{1}{|G|}\sum_{a\in G} \chi_V(a)\, \chi_W(a^{-1}).$$

$$= \frac{1}{|G|}\sum_{a\in G}\left(\sum_{i=1}^{\dim V} \rho^V_{ii}(a)\right)\left(\sum_{j=1}^{\dim W} \rho^W_{jj}(a^{-1})\right)$$

$$= \frac{1}{|G|}\sum_{a\in G}\sum_{i,j} \rho^V_{ii}(a)\, \rho^W_{jj}\underset{*}{(a^{-1})}$$

$$(*)\ S_{ij} = \frac{1}{|G|} \sum_{a\in G} \rho_{ik}(a)\, S^{0}_{kl}\, \rho_{lj}(a^{-1})$$

$$S^{0}_{kl} = \begin{cases} 1 & k=i \quad l=j \\ 0 & \text{caso contrário.} \end{cases}$$

esse produto é uma das entradas dos S_{ii} $\forall i$.

Teorema. Seja $V \in G$-mod χ_V.

$V \simeq V_1 \oplus \dots \oplus V_n$ V_i irredutíveis.

$\chi_i = \chi_{V_i}$ o caráter de V_i

W uma rep irredutível de G.

o número de reps. V_i são isom a W. é igual. (χ_V, χ_W)

⚠ $\underset{\|}{(\chi_V, \chi_W)} \in \underset{\cup}{\mathbb{Z}_{\geq 0}}$

$$\frac{1}{|G|} \sum_{a\in G} \operatorname{Tr}\underset{\uparrow}{\rho_V(a)}\ \overline{\operatorname{Tr}\rho_W(a)}$$

Corollario o número de V_i's não depende da decomposição.

Corollario. V, V' são isomorfos $\iff \chi_V = \chi_{V'}$

$$V = V_1 \oplus \cdots \oplus V_n$$

$$\chi_V = \chi_{V_1} + \chi_{V_2} + \cdots + \chi_{V_n}$$

$$(\chi_V, \chi_W) = \sum_{i=1}^{n} (\chi_{V_i}, \chi_W) \quad \begin{cases} 0 & V_i \cong W \\ 1 & V_i \cong W \end{cases}$$

$$= (\# \, V_i \cong W) \quad \checkmark$$

$$V \simeq V_1^{\oplus n_1} \oplus \cdots \oplus V_k^{\oplus n_k}.$$

$$V_i \not\simeq V_j \qquad n_i \geq 0$$

$$n_i = (\chi_V, \chi_{V_i})$$

$$\chi(V \otimes W) = \chi_V \cdot \chi_W$$

$$(\chi_V \cdot \chi_W, \chi_{V_i}) = ?$$

para $V = V_i$ $W = V_j$ irred.

$$V_i \otimes V_j = \bigoplus_k V_k^{c_{ij}^k}$$

$$c_{ij}^k = (\chi_{V_i} \cdot \chi_{V_j}, \chi_{V_k})$$

↳ coeffs. de Clebsh-Gordon.

$$V = \bigoplus V_i^{\oplus n_i} \qquad n_i \geqslant 0$$

$$(\chi_V, \chi_V) = \left(\sum n_i \chi_{V_i}, \sum_j n_j \chi_{V_j}\right)$$

$$= \sum_{ij} n_i n_j (\chi_{V_i}, \chi_{V_j})$$

$V \neq 0$ $$= \sum_i n_i^2 (\chi_{V_i}, \chi_{V_i}) = \sum n_i^2$$

$(\chi_V, \chi_V) \in \mathbb{Z}_{>0}$ é 1 $\Leftrightarrow$ V é irredutível.

$G \rightsquigarrow \quad \mathrm{Fun}(G, \mathbb{C}) = R_G$

$\mathcal{O}_G = \mathrm{Fun}(G) \quad \mathbb{C}^G$

$\mathbb{C}[G]$

$\chi_R(a) = \mathrm{Tr}_R\, P(a).$

R tem uma base $\{e_g\}_{g \in G}$.

$a \cdot e_g = e_{a \cdot g}$. por permutações.

$G \longrightarrow \mathrm{Aut}(S) =$ permutações.

$G \times S \longrightarrow S$

$a \times s \longrightarrow a \cdot s$

$\mathrm{Fun}(S)$

$\{e_s\}_{s \in S}$

$P(a) \cdot e_s = e_{a \cdot s}$.

$\mathrm{Tr}_{\mathrm{Fun} S}\, P(a) = \{\# s \in S \mid as = s\}$

$\chi_p(a) =$

Cor: Se a ação é simples.

$G \hookrightarrow Aut(S)$

Não tem pontos fixos.

$$\chi_\rho(a) = \begin{cases} 0 & a \neq id \\ |S| & a = id. \end{cases}$$

$S = G \qquad G \times G \to G$

$\qquad\qquad a \quad b \to ab.$

$$\chi_{R_G}(a) = \begin{cases} 0 & a \neq id_G \\ |G| & a = id_G \end{cases}$$

Teorema: Toda representação irred de G aparece como somando da representação regular.

<u>Pr</u>. V irrep G.

$$(\chi_{R_G}, \chi_V) = \frac{1}{|G|} \sum \chi_{R_G}(a)\, \chi_V(a^{-1})$$

$$= \frac{1}{\cancel{|G|}} \cancel{|G|}\, \chi_V(id) = \dim V$$

$$R_G \simeq \bigoplus_{\substack{V \text{ irred} \\ \text{de } G/\sim}} \underline{V^{\oplus \dim V}}$$

$$\dim\left(V^{\oplus \dim V}\right) = (\dim V)^2.$$

Corollario sejam $V_1, \dots, V_k$ reps. dos reps. irredutíveis de G $\dim V_i = n_i$

$$\sum_{i=1}^{k} n_i^2 = |G| \quad ✓$$

$k \leq$ classes de conjugação de G.

se $a \in G$ $a \neq id$.

$$\Rightarrow \sum n_i \cdot \chi_i(a) = 0$$

$$\chi_i = \chi_{V_i}$$

$$0 = \chi_R(a) = \left(\sum_{i=1}^{k} n_i \chi_i\right)(a) = \sum_{i=1}^{k} n_i \chi_i(a)$$

$a \neq id_G$.

Def. Uma função $G \xrightarrow{f} \mathbb{C}$
é dita de classe se $f(aba^{-1}) = f(b)$
$\forall a, b \in G$.

$\text{Func}(G, \mathbb{C}) \circlearrowleft \overset{\overset{\downarrow}{a}}{G}$. por conjugação.

$$\big(a \cdot \overset{\psi}{(f)}\big)(b) = f(a^{-1}ba)$$

$$(a \cdot (b \cdot f))(c) = (bf)(a^{-1}ca)$$
$$= f(b^{-1}a^{-1}cab) =$$
$$f((a \cdot b)^{-1}c(a \cdot b)) = ((ab) \cdot (f))(c)$$

$f := \frac{1}{|G|} \sum_{a \in G} (a \cdot f_0)$ $\Leftarrow$ é uma função de Classe.

Teorema. Seja G grupo finito
$V_1, \ldots, V_k$ irreps. de G.
$\chi_1, \ldots, \chi_k$ - carateres. formam uma base ortonormal do espaço de funções de classe de G.

Seja $W = \text{Span}\,\mathbb{C}\,\langle \chi_1, \dots, \chi_k \rangle$

$$\begin{tikzcd} & \cap \\ f \in \arrow[ur, "g"] & \end{tikzcd}$$

$f \in$ Função de classe de G.

$$\sum_{i=1}^{k} (f, \chi_i) \cdot \chi_i = g \in W$$

$$(g, \chi_j) = \sum_{i=1}^{k} \big(\underline{(f, \chi_i)}\, \chi_i, \chi_j\big)$$

$$= (f, \chi_j)$$

$W \subset$ Funç. de Classe. f

$\qquad\qquad \|$

$\qquad W \oplus W^{\perp}$

$\qquad g$

$(f - g) \in W^{\perp}$

Se $W \neq$ Funções de classe

$\exists\, 0 \neq f \in W^{\perp} \Rightarrow (f, \chi_k) = 0$

$$\frac{1}{|G|}\sum_{a\in G} f(a)\cdot\chi_R(a^{-1}) = \frac{1}{|G|} f(id) = 0$$

$$\Rightarrow f(\underline{id}) = 0.$$

Eu quero ver $\underline{f(a) = 0 \quad \forall a}$

$V \in G\text{-mod}$

$$\rho_f = \sum_{a\in G} f(a)\cdot\rho^V_{(a)} \in \mathrm{End}_k V$$

$$\rho_f(b) = \sum_{a\in G} f(a)\cdot\rho^V(b)$$

Claim: $\rho_f(b) = \alpha\,\rho^V(b) \qquad \alpha\in\mathbb{C}$

$\forall b.$

$$\underline{(a^{-1}\rho_f\, a)} = \sum_{b\in G} f(b)\cdot a^{-1}\rho^V_{(b)}\, a$$

$$= \sum_{b\in G} f(b)\cdot\rho^V(a^{-1}ba)$$

$$= \sum_{b\in G} f(aba^{-1})\,\rho^V(b)$$

$$= \sum_{b\in G} f(b)\,\rho^V(b) = \underline{\underline{\rho_f}}$$

$\Rightarrow \rho_f$ é uma função de Classe.

$\uparrow$ End V

$$\Rightarrow \rho_f \in \mathrm{End}_G(V) \qquad V \text{ irred.}$$

$$\rho_f = \lambda \cdot \underline{id}. \quad \checkmark$$

$$\mathrm{Tr.} = \lambda \cdot \dim V = \mathrm{tr} \sum_{a \in G} f(a) \cdot \rho(a)$$

$$= \sum_{a \in G} f(a) \, \mathrm{Tr}_V \, \rho(a) \quad \frac{|G|}{|G|} \sum_{a \in G} f(a) \, \overline{\chi(a)}$$

$$= |G| \, (f, \overline{\chi}) = \lambda \dim V.$$

$\Rightarrow$ Vemos que. $\lambda = \frac{1}{\dim V} \cdot \sum_{a \in G} f(a) \chi(a)$

$$= \boxed{\frac{|G|}{\dim V} \, (f, \overline{\chi}) = \lambda}$$

$$\|$$

$$0$$

of $f \in x/\perp$ $\quad \lambda = 0 \quad \forall \; V$ irred.

$$= P_{\underline{\underline{f}}} = \sum f(a)\,\rho(a) = 0$$

$\forall$ irred V.

$\Rightarrow P_f = 0$ $\quad \rho = \rho_R$ é zero.

$$\underset{\underset{R.}{\cap}}{P_f \cdot e_1} = \sum_{a\in G} f(a)\cdot\rho(a)e_1$$

$$= \sum_{a\in G} f(a)\cdot e_a = 0$$

$\Rightarrow f(a) = 0 \;\forall a$

$\Rightarrow W^{\perp} = 0 \Rightarrow W =$ funções de classe $\square$

$|C_i|$

$G/\sim$ $\leftarrow$ conjugação $\quad C_1, \dots, C_k$ classes de conjugação.

$X_1, \dots, X_k \qquad V_1, \dots, V_k$

Teorema. seja $a \in G$. $\quad a \in C_1 \quad |C_1|$

$$\sum \chi_i(a)\, \chi_1(a)^* = \frac{|G|}{|C_1|} \in \mathbb{Z}_{>0}$$

$$\sum \chi_i(a) \underline{\underline{\cdot}}\, \chi_i(b)^* = 0$$

$b \notin C_1$

seja f a função de classe

$$f(b) = \begin{cases} 1 & b \in C_1 \quad b \sim a \\ 0 & b \notin C_1 \quad b \not\sim a \end{cases}$$

$$f = \sum_{i=1}^{k} \alpha_i \chi_i \qquad \alpha_i = (f, \chi_i)$$

$$(f, \chi_i) = \frac{1}{|G|} \sum_{b \in G} f(b)\, \overline{\chi_i(b)}$$

$$= \frac{|C_1|}{|G|} \cdot \overline{\chi_i(a)}$$

$$\underline{\underline{f(b)}} = \sum_{i=1}^{k} \frac{|C_1|}{|G|}\, \overline{\chi_i(a)} \cdot \chi_i(b)$$

$$f(b) = \sum_{i=1}^{k} \frac{|C_i|}{|G|} \overline{\chi_i(a)}\, \chi_i(b) = \begin{cases} 1 & b \sim a \\ 0 & b \not\sim a \end{cases}$$

$$\left.\begin{aligned} &b = a & \sum \chi_i(a)\overline{\chi_i(a)} &= \frac{|G|}{|C_i|} \\ &b \not\sim a & \sum \chi_i(b)\overline{\chi_i(a)} &= 0 \end{aligned}\right\} \checkmark$$

———— . ————

G - grupo finito. $\mathrm{Irrep}_G = \bigoplus \mathbb{C}\cdot[V]$

Soma é sobre V classe de isomorfismos de representações irred. de G.

Tem a mesma dimensão que

$$C(G) = \text{Funções de } G \to \mathbb{C} \text{ invariantes por conjugação.}$$

São canonicamente isomorfos

$$\mathrm{Irrep}_G \xrightarrow{\sim} C(G)$$

$$(V \xrightarrow{\sim} \overline{\chi_V}$$

Vira um anel via o isomorfismo.

Irrep_G é o anel de Grothendieck de Rep_G.

$\mathcal{C}, \oplus, \underline{\underline{\otimes}}, \mathbb{1}, 0,$ ~~symmetrica~~ monoidal tensorial.

Espaço vetorial $[X]$ $\quad X \in \underline{\mathcal{C}}$

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] & \underline{\underline{Y}} \arrow[r] & Z \arrow[r] \arrow[l, bend right] & 0 \end{tikzcd} \in \mathcal{C}$$

$$[X] + [Z] = [Y].$$

$$K(\mathcal{C}) \quad [X]+[Y] = [X \oplus Y]$$

$$[X] \cdot [Y] = [X \otimes Y]$$

$$1 = [\mathbb{1}]$$

$$0 = [0]$$

Anel associativo. $\mathcal{C}$-symetrica.

$$X \otimes Y \simeq Y \otimes X$$

$$\Downarrow$$

é comutativo.

$$\begin{tikzcd}
\mathrm{Irrep}_G \simeq R(G) \arrow[d] & V \arrow[d] \\
C(G) & \chi_V
\end{tikzcd}$$

isomorfismo de anéis.

$$\chi_{V \oplus W} = \chi_V + \chi_W$$

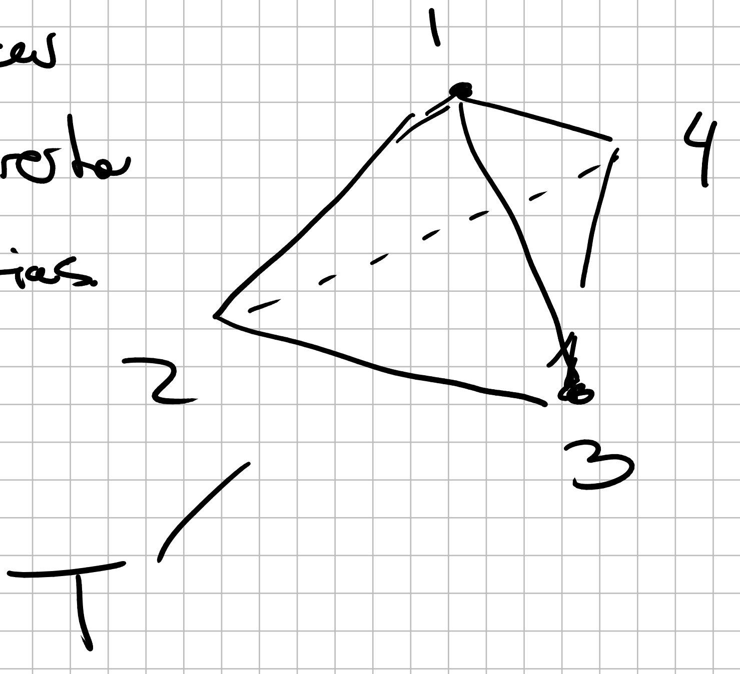
$$\chi_{V \otimes W} = \chi_V \cdot \chi_W$$

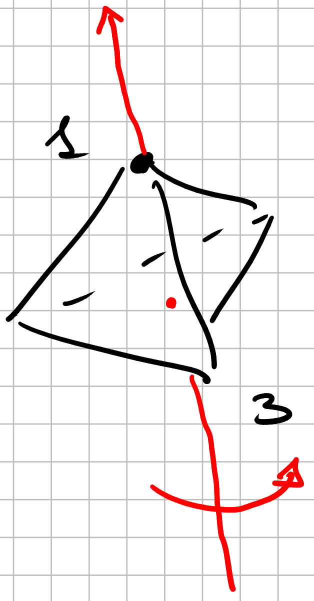
* * *

Alguns exemplos.

$$\mathrm{Aut}(T) \cap SO(3, \mathbb{R})$$

4-faces
6-Arestas
4-vertices

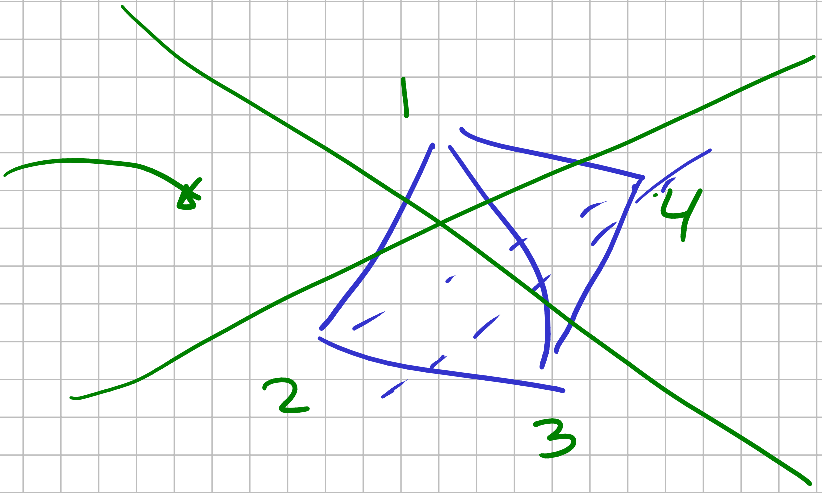




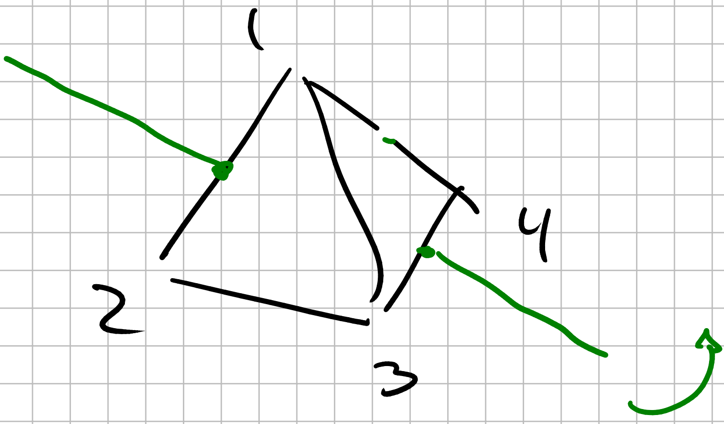
120° ~ $\mathbb{Z}/3$

Permutações dos vertices.

⇝ (1)(2 3 4)

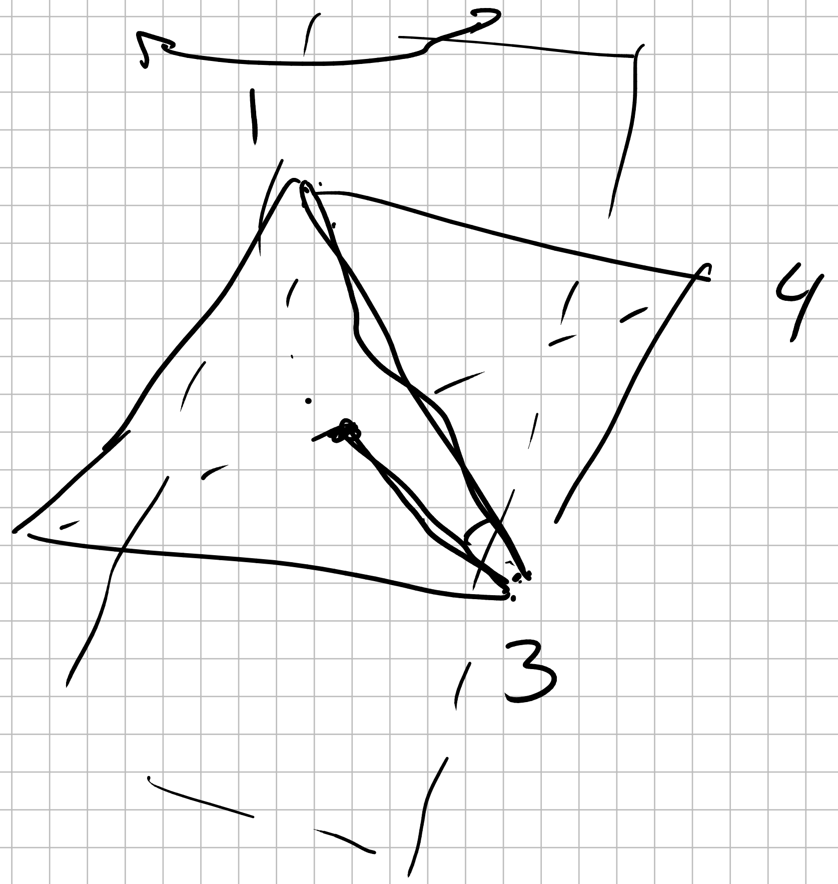


(2 4) (1)(3)



120°

(12)(34)



Reflexão. a travez de plano.

(2,4). está. em. $SO(3,\mathbb{R})$

?

~~∅~~

$$\det \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} = -1$$

$$S_4 \supset \quad (123) \quad (12)(34)$$

$$\begin{tikzcd} S_4 \\ \mathrm{Aut}(T) \arrow[u, hook] \end{tikzcd}$$

A_4

$\mathrm{Aut}(T) = A_4$

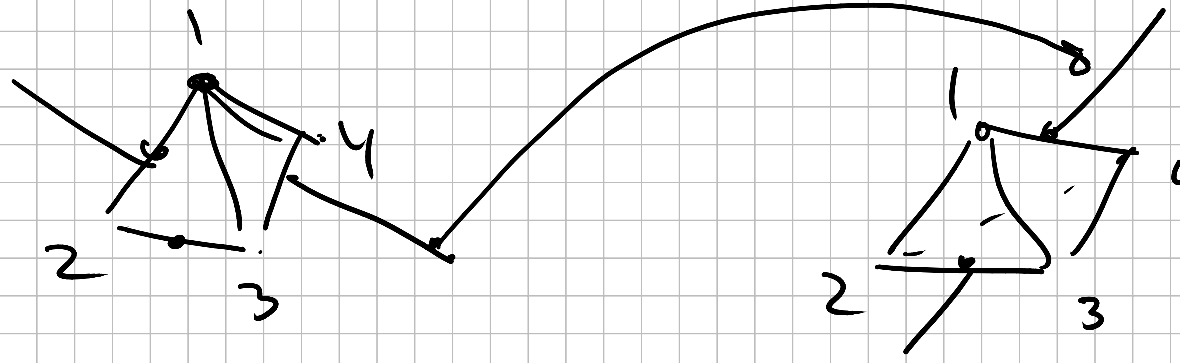
Reps. irredutíveis de A_4 ?

$\mathbb{1}$ _ $\chi(a) = 1 \quad \forall a \in G$

<u>Regular</u> $|A_4| = 12$

Quantas irredutíveis ?

$(123) \not\sim (12)(34)$



$(12)(34) \sim (14)(23)$

Esses dois são conjugados.

(234).

$$\boxed{(123) \sim (234) \; ?}$$

$(1) \leftrightarrow 1$.

Quantos elementos de cada classe de conjugacões?

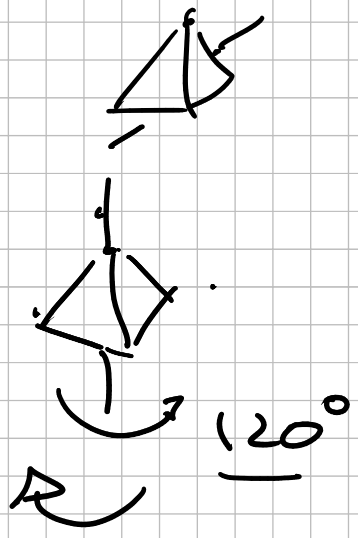
(1) 1 $\left.\right\}$ $\mathbb{Z}_2 \times \mathbb{Z}_2$

$(12)(34)$ 3

(123) - 4

(132) 4

são conjugados em S_4



em S_4 (1) $(12)(34)$. (123) (1234) (12)

$\exists\ \sigma \in A_4$ tq $\sigma(123)\sigma^{-1} = (132)$?

$= (123)^2$

$\downarrow$

$\sigma = (32) = \sigma^{-1}$ $\quad \sigma \in S_4 \setminus A_4$

$\{1234\} \xrightarrow{\sigma} \{1324\} \longrightarrow \{2134\}$

$\longrightarrow \{2314\}$

$\Rightarrow \nexists\,!\ \sigma \in A_4$ tq $\sigma(123)\sigma^{-1} = (132)$

Classes de conjugação em $S_n \longleftrightarrow$ formas dos ciclos.

(1) $\boxed{1}$

$(12)(34)$ $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$

(123) $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}$

(1234) $\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline \end{array}$

(12) $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}$

	(1)	$(12)(34)$	(123)	(132)
	1	3	4	4
$\mathbb{1}$	1	1	1	1
Reg.	1	0	0	0
Vert.	4	0	1	1
$\overline{\text{Vert}}$	3	-1	0	0

$A_4 \hookrightarrow S_4 \curvearrowright \{1,2,3,4\}$

char. de uma reps. por permutações avaliado $a \in G$. $=$ # de elementos fixos por a.

$\sum e_i$ é invariante. !

$$0 \to \mathbb{1} \hookrightarrow \text{Vert}. \longrightarrow \overline{\underline{\text{Vert}}}_3 \longrightarrow 0$$

Pergunta é $\overline{\text{Vert}}$ irred. ?

$$\left(\chi_{\overline{Vert}}, \chi_{\overline{Vert}}\right) = \frac{1}{|G|}\sum_{a\in G} \chi_{Vert}(a)\,\overline{\chi_{Ver}(a)}$$

$$\frac{1}{|G|}\sum_{a\in C} |C|\,\chi_{\overline{Vert}}(a)\,\overline{\chi_{\overline{Ve}}(a)}$$

	(1)	(12)(34)	(123)	(132)	
	1.	3	4	4	
1	1	1	1	1	←
Reg.	12	0	0	0	
~~Vert.~~	~~4~~	~~0~~	~~1~~	~~1~~	
$\overline{\text{Vert}}$	3	-1	0	0	← irred
	1.				
	1				

$$\frac{1}{12}\cdot\left(9+3+0+0\right) = 1$$

↑

$$1^2+3^2+a^2+b^2 = 12$$

$$a^2+b^2 = 2 \qquad a=b=1$$

Reps irred. de $\mathbb{Z}_3$.

G - grupo Abeliano V. rep. irredutível.

$\Rightarrow \dim V = 1$.

Reps. irred corresp classes de conjugação. $C = G$.

$$\sum_i n_i^2 = G \iff n_i = 1 \quad \forall i$$

Rep. de $\mathbb{Z}_3$ Tres. reps. todas de dimen são 1.

$\sigma \in C_n$ gerador.

$$\sigma \longrightarrow \chi(\sigma) \in \mathbb{C}$$

$\sigma^n = 1$ raiz n-esima da unidade.

$$\xi = e^{2\pi i/3}.$$

	(1)	(123)	(132)
$\mathbb{1}$	1	1	1
$\mathbb{C}_\xi$	1	ξ	$\bar{\xi}$
$\mathbb{C}_{\bar{\xi}}$	1	$\bar{\xi}$	ξ

$\longrightarrow \mathbb{Z}_3$

G. grupo. V uma rep. de dim. 1.

$$K \hookrightarrow G \xrightarrow[\chi_V]{} GL(1) = \mathbb{C}^*$$

K = subgrupo normal.

A_4 não tem subgrupos normais.

$K = \{1\}$ ou $\boxed{A_4}$ — reps. trivial.

	(1)	(12)(34)	(123)	(132)	
	1	3	4	4	
$\mathbb{1}$	①	1	1	1	–
~~Reg.~~	~~1~~	~~0~~	~~0~~	~~0~~	
~~Vert.~~	~~4~~	~~0~~	~~1~~	~~1~~	
$\overline{\text{Vert}}$	③	-1	0	0	
$\mathbb{C}_\xi$	①	1	ξ	$\bar{\xi}$	←
$\mathbb{C}_{\bar{\xi}}$	①	1	$\bar{\xi}$	ξ	

$$\frac{1}{12}\left(1 + 3\alpha\bar{\alpha} + 4\xi\bar{\xi} + 4\xi\bar{\xi}\right) = 1$$

$$\Rightarrow \alpha\bar{\alpha} = 1 \Rightarrow \underline{(\alpha,\alpha)} = \underline{1}.$$

$$(\chi_{\mathbb{C}_\xi}, \chi_{\overline{Vert}}) = \frac{1}{12}(3 - \alpha \cdot 3) = 0$$

$$\Rightarrow \alpha = 1$$

Deixo vocês ver que é a tabela completa

—————o—————

$$A_4 \hookrightarrow \underline{SO(3)} \hookleftarrow \mathrm{Aut}(T)$$

Reps. que define Aut(T) é uma reps. de dim 3.

Será que é irredutível?

$\nexists$ nenhum vetor $\mathbb{R}^3$ fixo por todo automorfismo do tetraedro. $SO(2)$

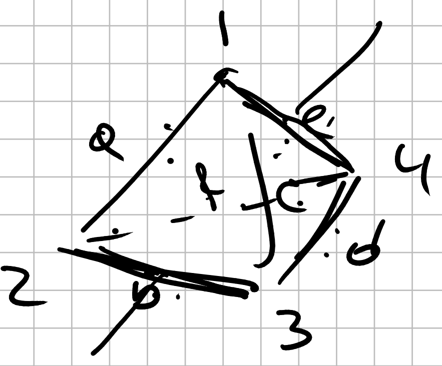
$\nexists$

$\mathbb{1} \hookrightarrow \mathbb{R}^3$ $\quad$ $\mathbb{C}_\xi$? $\quad$ $\mathbb{C}_{\bar{\xi}}$?

$$\overline{Vert} \simeq \mathbb{R}^3$$

<u>Edges</u>. representação dim 6.

$\frac{\shortparallel}{E}$



$\chi_E = ?$

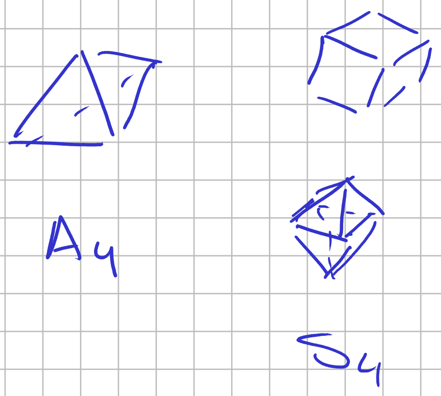
	(1)	$(12)(34)$	(123)	(132)
	1	3	4	4
χ_E	6	2	0	0
$\chi_{\overline{E}}$	5	1	-1	-1 ?
$\chi_{\overline{Vert}}$	3	-1	0	0
$\underline{V}$	2	2.	-1.	-1.
$\mathbb{C}_\zeta$	$\underline{1}$	$\underline{1}$	ζ.	$\bar{\zeta}$.
$\mathbb{C}_{\bar{\zeta}}$	$\underline{1}$	$\underline{1}$	$\bar{\zeta}$	ζ

Como é decomp. de $\overline{E}$ em irred.

$$(\chi_{\underline{\overline{E}}}, \chi_{\underline{\overline{Vert}}}) = \frac{1}{12}(15-3) = \mathbb{1}$$

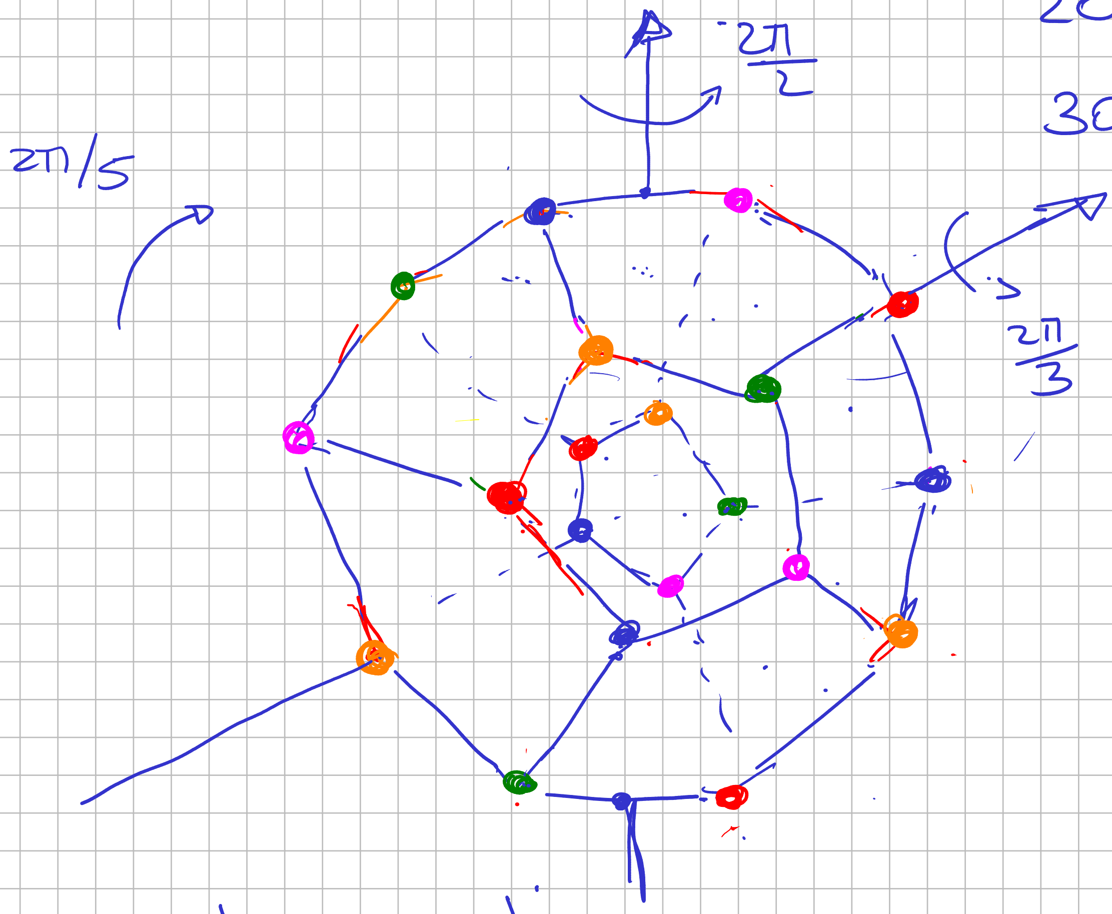
$$E = \mathbb{1} \oplus \overline{Vert} \oplus \mathbb{C}_\zeta \oplus \mathbb{C}_{\bar{\zeta}}$$

Representações do grupo A_5.



isosae.

12 Faces.
20 Vertices.
30 Arestas.



72 cada
rotações
não são.
iso.

20 vertices colorear com 5 cores.
Em cada face (contem 5 vertices)
os 5 cores aparecem.

5 Tetraedros são os que
se permutam.

Quantas colorações diferentes existem

$D \subset SO_3(\mathbb{R})$ rotações do dodec.

$D \to S_5$.

Toda rotação do dodeca. leva vértices de uma cor em vértices de uma cor

$\mathbb{Z}_5 \rightsquigarrow$ Fixar dois faces opostas. rotar $\frac{2\pi k}{5}$ $0 \leq k \leq 4$.

$$\left\{\begin{array}{l}(12345)\\(13524)\\(14253)\\(15432)\end{array}\right.$$

$$\left\{\begin{array}{l}(123)\\(132)\end{array}\right.$$ Fixa dois vértices opostos. $\frac{2\pi k}{3}$ $0 \leq k \leq 2$

$\{(12)(34)$ Fixa dois arestas opostas. $\frac{2\pi}{2}k$ $k=0,1$.

(1)

$$\begin{tikzcd}
D \arrow[rr] \arrow[dr, "\simeq"] & & S_5 \\
& A_5 \arrow[ur] &
\end{tikzcd}$$

Classes de conjugação do S_5

(1)	(12)	(123)	$\underline{(12)(34)}_{2}$	$(123)(45)$	(1234)	(12345)
1	$\binom{5}{2}$	$\binom{5}{3}2$	$\binom{5}{2}3/2$	$\binom{5}{3}\cdot 2$	$5\cdot 3!$	$4!$
1	10	20	~~30~~ 15	20	30	24

$|A_5| = |S_5|/2 = 60$

(1)	(123)	(132)	(12)(34)	(12345)	(13245)
1	10	10	15	12	12

Def. $G \longrightarrow \mathrm{Aut}(G)$ é um grupo composição

$a \longmapsto a \cdot a^{-1}$

$\mathrm{inn}(G) \lhd \mathrm{Aut}(G)$

$\cup$

$\mathrm{Inn}(G)$

Def. $\mathrm{Out}(G) = \mathrm{Aut}(G) / \mathrm{Inn}(G)$.

$S_5 \curvearrowright A_5$ por conjugação.

σ^{ψ} $\quad \tau^{\psi}$ $\quad \sigma \tau \sigma^{-1} \in A_5$

$$\begin{tikzcd} S_5 \arrow[r] & \mathrm{Aut}(A_5) \\ A_5 \arrow[ur] & \end{tikzcd}$$

$S_5 / A_5 \cong \mathbb{Z}_2 = \underline{\mathrm{Out}}(A_5)$

	(1)	(123)	↔	(132)	(12)(34)	(12345)	(13245)
	1	10	20	10	15	12	12
$\mathbb{1}$	①	1		1	1	1	1
$\mathbb{R}^3$	3	0		0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{3}}{2}$
S	4	1		1	0	-1	-1

$$\begin{array}{l|cc|ccc}
\Lambda\ 3: & 0 & 0 & -1 & \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \\
\perp\ 5: & -1 & -1 & 1 & 0 & 0 \\
F\ 11 & -1 & -1 & -1 & 1 & 1
\end{array}$$

$$\frac{1}{60}\left(25+20+15\right)=1 \qquad \underline{\text{irred}}$$

$$\underbrace{(45)(23)}\,(123)\,\underbrace{(23)(45)}=(132)$$

Teorema. As classes de conjugação de A_n são as mesmas que conjugadas em S_n ou união disjunta de duas conjugadas por uma transposição.

$$\frac{1}{60}\left(121+20+15+24\right)=3$$

soma de 3 reps. irredutíveis.

$$\frac{1}{60}\left(11-20-15+24\right)=0$$

$$\frac{1}{60}\left(33+15+12\right)=1$$