Homework 5

due by year's end

1 Exercise. Let V be the standard irreducible representation of A_5 of dimension 4. What are the irreducible components of $V \otimes V$? Is $V \wedge V$ irreducible?

2 Exercise. Let $H = Heis(\mathbb{Z}/n\mathbb{Z})$ be the Heisenberg group on $3n$ elements, this is the group of 3×3 upper triangular matrices with entries in $\mathbb{Z}/n\mathbb{Z}$ and 1 in the diagonal. Find an irreducible representation of dimension n. Is there another irreducible representation of this dimension?

3 Exercise. Let $G = GL(3, \mathbb{F}_2)$ be the group of invertible 3×3 matrices with entries in the field with 2 elements. Find a representation of this group on $\mathbb{C}^3$. This exercise is not simple. I'll be happy if you do the following subgroup, let H be the group of $4x4$ matrices with entries $0, 1$ and such that each line and column has exactly one entry 1. Show that H is a subgroup of G and that H has an irreducible representation of dimension 3. How can we extend the action of this H to G on the same space?

4 Exercise. Seja G um grupo cíclico com n elementos. Descreva o conjunto de representações irredutíveis de G módulo isomorfismos.

5 Exercise. Seja $H \subset G$ um subgrupo e $V \in H - mod$. Prove que

$$\left(Ind_H^G V\right)^* \simeq Ind_H^G V^*$$

como G-módulos.

6 Exercise*.Seja $G = PSL_2(\mathbb{F}_7)$ o grupo de matrizes $2x2$ com entradas em $\mathbb{F}_7$ módulo o subgrupo central $\pm Id$. Seja H o grupo $GL_3(\mathbb{F}_2)$.

a) Prove que $G \simeq H$

b) Encontre todas as representações irredutíveis de G.

[Hint. Keywords Klein, Elkies]

7 Exercise. Seja G um grupo finito e V uma representação fiel (ou seja $G \hookrightarrow GL(V)$). Mostre que qualquer representação irredutível de G acontece como submódulo de $\operatorname{Sym}^n V$ para algum n (e por tanto acontece como submódulo de $V^{\otimes n}$).

8 Exercise. Let G be a finite group and let H be the algebra of class functions. Let $\{\chi_i\}_{i=1}^k \in H$ be the irreducible characters of G and let $Z = \mathbb{Z}[\chi_1, \dots, \chi_k] \subset H$ be the sub-ring generated by all the χ_i.

a) Let $\psi = \chi_1 - \chi_2 \in Z$, show that ψ is not the character of an irreducible representation of G.

b) Show that ψ above is not the character of any representation of G.

c) Let $\psi = \sum_{i=1}^k a_i \chi_i \in Z$ and suppose that $\langle \psi, \psi \rangle = 1$ and moreover $\psi(1) > 0$. Show that ψ is the character of an irreducible representation.