Homework 3

Due 26/04/2024

Try as many exercises as you like, no one is expecting you to turn in a complete set.

1 Exercise. Let a, b, $p \in \mathbb{Z}_+$ with p prime. Show that $a + b = p \Rightarrow (a, b) = 1$.

2 Exercise. Factor $x^2 - 3x - 3 \in \mathbb{F}_5[x]$ into irreducible factors.

3 Exercise. Let $F \subset \mathbb{C}$ be a subfield and $f \in F[x]$ be an irreducible polynomial. Prove that f has no multiple roots in $\mathbb{C}$ [Hint: think about the derivative of f]

4 Exercise. Find the gcd of $11 + 7i$ and $18 - i$ in $\mathbb{Z}[i]$.

5 Exercise. Show that the kernel of the map $\mathbb{C}[x, y] \to \mathbb{C}[t]$, $f(x, y) \mapsto f(t^2, t^3)$ is a principal ideal and that its image is the set of polynomials $p(t)$ such that $f'(0) = 0$.

6 Exercise. Factor the following polynomials in $\mathbb{Q}[x]$

$$x^2 + 2345x + 125, \qquad x^4 + 2x^3 + 3x^2 + 2x + 1.$$

7 Exercise. Show that an integer prime p is prime in $\mathbb{Z}[\sqrt{3}]$ if and only if the polynomial $x^2 - 3$ is irreducible in $\mathbb{F}_p[x]$.

8 Exercise. Factor $1 - 3i$ and $6 + 9i$ into Gauss primes.

9 Exercise. Show that in $\mathbb{F}_p^\times$, $p > 2$, there are no elements of order 2 besides -1.