Homework 2

Due 29/03/2024

Try as many exercises as you like, no one is expecting you to turn in a complete set. Felipe will grade only 3 or 4 exercises from what you turn in.

1 Exercise. Prove that two quadratic polynomials f and g in two variables have at most four common zeroes, unless they have a nonconstant factor in common.

2 Exercise. Let R be a ring containing $\mathbb{C}$ as a subring and suppose R is finite dimensional as a complex vector space. Show that R contains exactly one maximal ideal M and that M consists of all nilpotent elements of R.

3 Exercise. Let $a \in R$ and let $R' = R[x]/(ax - 1)$. Find the kernel of the map $R \to R'$.

4 Exercise. The nilradical of a ring is the set of nilpotent elements. Prove that it is an ideal and determine it for the ring $\mathbb{Z}/(12)$.

5 Exercise. An irreducible algebraic curve $C \in \mathbb{C}^2$ is the set of zeroes of an irreducible polynomial $f(x, y)$. A point $p \in C$ is called singular if $\partial f/\partial x = \partial f/\partial y = 0$ at p. Prove that an irreducible curve has only finitely many singular points

6 Exercise. Show that an irreducible cubic (that is the set of zeroes of an irreducible cubic polynomial in 2 variables) has at most one singular point.

7 Exercise. Prove that if $n = pq$ where p, q are relatively prime, then every fraction m/n can be written in the form $m/n = a/p + b/q$.

8 Exercise. Let $F \subset \mathbb{C}$ be a subfield and let $f \in F[x]$ be an irreducible polynomial. Prove that f has no multiple root in $\mathbb{C}$.

9 Exercise. Let a, b be coprime integers. Prove that there are integers m, n such that $a^m + b^n = 1 \mod ab$.