Homework 1

Due 22/03/2024

1 Exercise. Let S be a set with an associative binary map and with an identity element. Prove that the subset of S consisting of invertible elements is a group. Find a counterexample for the set of *left invertible* elements.

2 Exercise. Let Δ be the regular tetrahedron in $\mathbb{R}^3$ so that its four vertices are in the sphere of radius 1 centered at the origin. Let

$$G = \{A \in SO_3(\mathbb{R}) \mid A\Delta = \Delta\}.$$

Show that G is a subgroup of $SO_3(\mathbb{R})$ isomorphic to A_4, the alternating group of permutations of 4 elements.

3 Exercise. Let $H = \{\pm 1, \pm i\}$ be the subgroup of $G = \mathbb{C}^\times$ of fourth root of unity. Describe the cosets of H in G explicitly and prove that G/H is isomorphic to G.

4 Exercise. Let H be the Heisenberg real group of real, upper triangular 3×3 matrices with 1 in the diagonal. Let $\Gamma \subset H$ be the subgroup of such matrices with integer entries. Let T^2 be the group $S^1 \times S^1$. Show that there exists a surjection $\pi : H/\Gamma \twoheadrightarrow T^2$ such that the fibers $\pi^{-1}(x)$ are isomorphic to S^1.

5 Exercise. Compute the group of automorphisms of the quaternion group, this is the group consisting of $\{\pm 1, \pm i, \pm j, \pm k\}$ with the usual quaternionic multiplication.