Lista 2

Entregar 09/05/2023

1 Exercise. Let R be a commutative ring and $A, A', B, B' \in R-mod$. Show that there is a natural external tensor product

$$\operatorname{Tor}_p(A,B) \otimes_R \operatorname{Tor}_q(A',B') \to \operatorname{Tor}_{p+q}\left(A \otimes_R A', B \otimes_R B'\right),$$

which is associative and commutes with the connecting homomorphism of the long exact sequences associated to Tor (Weibel 2.7.5).

2 Exercise. Let k be a field and R be the ring of polynomial differential operators on the line, that is the k-algebra generated by x and ∂_x with the commutation relation

$$\partial_x \cdot x - x \cdot \partial_x = 1.$$

Let $\mathcal{O} := k[x] \in R-mod$ and $j_* j^* \mathcal{O} := k[x, x^{-1}] \in R-mod$ be given by the standard action where x acts by multiplication by x and ∂_x acts by derivation. Let $\iota_! k$ be the quotient module so that we have a short exact sequence of R-modules

$$0 \to \mathcal{O} \to j_* j^* \mathcal{O} \to \iota_! k \to 0.$$

a) Is $\mathcal{O}$ a projective R-module?

b) Is $\iota_!$ a projective R-module?

c) Is any of the three modules flat?

d) Compute $\operatorname{Tor}_*(\iota_! k, \iota_! k)$.

3 Exercise. Let R be a ring and P be its module, show that P is projective if and only if $\operatorname{Ext}^1_R(P, M) = 0$ for all $M \in R-mod$.

4 Exercise. Let R be a ring, $A \in mod-R$ and $B \in R-mod$. Let $P_\bullet \to A$ and $Q_\bullet \to B$ be projective resolutions. We know that

$$H_\bullet(P_\bullet \otimes B) \simeq H_\bullet(A \otimes Q_\bullet).$$

Find an example of $A, B, P_\bullet, Q_\bullet$ where there is no map of complexes

$$P_\bullet \otimes B \to A \otimes Q_\bullet,$$

nor map in the other direction.

5 Exercise. Let $p, p' \in \mathbb{Z}$ be distinct primes. Find

$$\operatorname{Tor}_\bullet(\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}/p'\mathbb{Z}).$$