ON THE TOPOLOGY OF ARITHMETIC SUMS OF REGULAR CANTOR SETS

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Abstract

The study of geometrical and topological properties of arithmetic sums and differences of regular Cantor sets appears naturally in distinct fields as dynamical systems (particularly, in the study of homoclinic bifurcations related to non-trivial hyperbolic sets) and number theory (particularly, in the study of geometrical properties of the Markov and Lagrange spectra, related to diophantine approximations). We study the topological structure of the sum of two regular Cantor sets and we obtain some local results related to this problem; more precisely, we give persistent examples of several different topological types of these sums or differences.

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1 Introduction

The study of geometrical and topological properties of arithmetic sums and differences of regular Cantor sets appears naturally in distinct fields as dynamical systems (e.g. in the study of homoclinic bifurcations associated to non trivial hyperbolic sets) and number theory (e.g. in the study of the Markov and Lagrange spectra associated to Diophantine approximations).

Regular Cantor sets are, roughly speaking, defined by an iterate system of expanding differentiable functions on intervals. It is not difficult to see that if the sum of the Hausdorff dimensions of two regular Cantor sets K_1 and K_2 is smaller than one then the arithmetic sum $K_1 + K_2 = \{x + y \;|x \in K_1, \;\; y \in K_2\}$ and also the arithmetic difference $K_1 - K_2 = \{x - y\,|x \in K_1, \;\; y \in K_2\}$ both have zero Lebesgue measure. Recently, Moreira and Yoccoz proved a conjecture of J. Palis, according to which, generically, if the sum of the Hausdorff dimensions of K_1 and K_2 is bigger than one then $K_1 + K_2$ (and $K_1 - K_2$) contains intervals. More precisely, it was shown that there is an open, dense and in some sense a total measure subset $\mathcal{U}$ of

$$\Omega = \{(K_1, K_2) \mid K_1 \text{ and } K_2 \text{ regular Cantor sets, } HD\,(K_1) + HD\,(K_2) > 1\}$$

(here HD denotes Hausdorff dimension) which is dense in the C^∞ topology such that if $(K_1, K_2) \in \Omega$ then $K_1 + K_2$ and $K_1 - K_2$ have stably dense interior (see $[MY]$).

Mendes and Oliveira studied in $[MO]$ the topological structure of the sum of some class of affine Cantor sets in the interval $[0, 1]$. In that paper they pose the question of what can be said about the topological structure of the sum of two regular Cantor sets. We will give here some local results about this problem. Since $K_1 - K_2 = K_1 + (-K_2)$ we will restrict our attention to sums instead of differences.

If $HD\,(K_1) + HD\,(K_2) < 1$ then $K_1 + K_2$ has zero Lebesgue measure, and so is a Cantor set (recall that any two Cantor sets on the real line are homeomorphic). So we will be concerned about sums of regular Cantor sets such that the sum of their Hausdorff dimensions is bigger than one. As we mentioned before, these sums have generically dense interior. This fact is of course not enough to characterize its topological structure. However, there are some regions in Ω where we can characterize or give partial descriptions of the topological structure of $K_1 + K_2$ for generic pairs (K_1, K_2) in the considered region. The simplest example is the region of pairs (K_1, K_2) such that $\tau(K_1)\tau(K_2) > 1$, where τ denotes the thickness defined by Newhouse, see for instance $[N]$ or $[PT]$. In this open region $K_1 + K_2$ is stably a finite union of closed intervals (see example 1 in section 4). The same happens in a larger open set $\mathcal{U}^0$ defined in $[M]$.

2 Background and Global Facts

In this section we introduce some preliminary concepts and definitions. For $X \subseteq \mathbb{R}$, $\widehat{X}$ denotes the *convex hull* of X and $|X| = \left|\widehat{X}\right|$ the *length* of the interval $\widehat{X}$, which can be eventually infinite. When X is compact, a connected component of $\mathbb{R}\backslash X$ is called a *gap* of X, and a non trivial connected component of X is called an *interval* of X. A *Cantor set* $K \subseteq \mathbb{R}$ is a compact perfect set with empty interior. A *regular Cantor set* is defined whenever:

(i) *there exist pairwise disjoint compact intervals* $I_1, I_2, \ldots, I_r \subseteq I_0 = \widehat{K}$ *with* $K \subseteq \bigcup_{i=1}^{r} I_i$,

(ii) *there exist expanding functions* $\psi_i : I_i \longrightarrow I_0$ *of class* $C^{1+\varepsilon}$ *whose images are convex hull of disjoint unions of the* I_i *such that if* $\psi : \bigcup_{i=1}^{r} I_i \longrightarrow I_0$ *is defined by* $\psi|_{I_i} = \psi_i$ *then for each* i *there exists* n_i *with* $\psi^{n_i}(K \cap I_i) = K \cap I_0$ *and besides,* $K = \bigcap_{n \geqslant 0} \psi^{-n}\left(\bigcup_{i=1}^{r} I_i\right)$.

A map ψ is $C^{1+\varepsilon}$ if ψ is C^1 and there exists $k > 0$ such that $|\psi'(x) - \psi'(y)| \leqslant k|x-y|^{\varepsilon}$ for all x, y. C^{1+} denotes the union of the spaces $C^{1+\varepsilon}, \varepsilon > 0$, with the induced topology. In particular, when the ψ_i are affine functions we say that the Cantor set is affine. Note that for a regular Cantor set K, $f(K)$ is a regular Cantor set for every C^{1+} diffeomorphism f of $\mathbb{R}$.

To define the *thickness* of a Cantor set K, denoted by $\tau(K)$, we consider a gap $\mathcal{U}$ of K, and let $I_{\mathcal{U}}$ be the interval bounded by $\mathcal{U}$ and the closest gap $\mathcal{V}$ that is larger than $\mathcal{U}$. Then the thickness of K is

$$\tau(K) = \inf \left\{ \frac{|I_{\mathcal{U}}|}{|\mathcal{U}|} \;;\mathcal{U} \text{ is a bounded gap of } K \right\}.$$

Note that $\tau(K) = \tau(-K)$. The importance of the thickness is the following lemma:

Gap-lemma: *For given K_1 and K_2 Cantor sets with $\tau(K_1)\tau(K_2) > 1$ then exactly one of the following three possibilities holds:*

- $\widehat{K}_1$ *is contained in a gap of* K_2.
- $\widehat{K}_2$ *is contained in a gap of* K_1.
- $K_1 \cap K_2$ *is non empty.*

Thus, if $\tau(K_1)\tau(K_2) > 1$, then $K_1 + K_2$ is a finite union of intervals. Finally, we recall that the thickness, as well as the Hausdorff dimension of a regular Cantor set K depends continuously on K (see $[PT]$).

Given a Cantor set $K \subseteq \mathbb{R}$ and a finite covering of K by open intervals $\mathcal{U} = \{U_i\}_{i\in\mathcal{A}}$, put $|\mathcal{U}| = max\{|U_i| : i \in \mathcal{A}\}$. Given $\alpha \in \mathbb{R}$, we set $H_\alpha(\mathcal{U}) = \sum_{i\in\mathcal{A}} |U_i|^\alpha$, and we define the Hausdorff $\alpha-$measure of K,

$$m_\alpha(K) = \lim_{\epsilon\longrightarrow 0}(\inf\{H_\alpha(\mathcal{U}) \mid \mathcal{U} \text{ is a covering of K with } |\mathcal{U}| < \epsilon\}).$$

The *Hausdorff dimension of* K is defined as being the unique real number $HD(K)$ such that

$$m_\alpha(K) = \begin{cases} +\infty, & \text{for } \alpha < HD(K). \\ 0, & \text{for } \alpha > HD(K). \end{cases}$$

3 Local Results

In this section some local results are presented. To start with, it is considered a simple case where the arithmetic sum is stably an interval.

Example 0: For $\alpha \in \left(0, \dfrac{1}{2}\right)$ let K_α be the affine Cantor set which is the maximal invariant set contained in $[0,1]$ of ψ_1 and ψ_2 which are the orientation preserving affine functions such that $\psi_1([0,\alpha]) = \psi_2([1-\alpha,1]) = [0,1]$. Then, for $\alpha > \frac{1}{3}$ and K_1, K_2 close to K_α, $K_1 + K_2$ is an interval.

Note that $\tau(K_\alpha) = \dfrac{\alpha}{1-2\alpha}$ so, if $\alpha > \dfrac{1}{3}$, $\tau(K_\alpha) > 1$, and since τ is continuous, there is a neighbourhood $\mathcal{U}$ of K_α such that $\tau(K) > 1$ for all $K \in \mathcal{U}$. Moreover, by reducing $\mathcal{U}$ if necessary, assume that for all K_1, $K_2 \in \mathcal{U}$, every gap of K_2 is shorter than $|K_1|$. Then, by the Gap-lemma, $K_1 + K_2$ is an interval, for all K_1, $K_2 \in \mathcal{U}$.

Remark: This example can be generalized, so that the arithmetic sum is stably the union of $n > 1$ disjoint intervals. In fact, let $n > 1$, $\epsilon \in (0,1)$ and K_ε be the Cantor set generated by the orientation preserving affine maps $\psi_0, \dots, \psi_{n-1}$ such that $\psi_i([4i, 4i+2+\epsilon]) = [0, 4n-2+\epsilon]$. Thus, $\tau(K_\varepsilon) = \dfrac{2+\epsilon}{2-\epsilon} > 1$ and there is a neighbourhood $\mathcal{V}$ of K_ε, such that for all $K_2 \in \mathcal{V}$, $\tau(K_2) > 1$. Moreover, by reducing $\mathcal{V}$ and $\mathcal{U}$ neighbourhoods as in the previous example, if necessary, assume that for all $K_1 \in \mathcal{U}$, $K_2 \in \mathcal{V}$, $|K_1| < |K_2|$, and K_2 has exactly $n-1$ gaps longer than $|K_1|$. Then, by the Gap-lemma, $K_1 + K_2$ is a disjoint union of n intervals, for all $K_1 \in \mathcal{U}$ and $K_2 \in \mathcal{V}$.

In general, the arithmetic sum of pairs of regular Cantor sets on the line is more complex, and it can be classified according to the nature of its different types of gaps. It can be distinguished four kind of bounded gaps :

- $LR-gap$ is a gap, between intervals,
- $R-gap$ is a gap with an interval only at its left extreme,
- $L-gap$ is a gap with an interval only at its right extreme,
- $C-gap$ is a Cantor-like gap, i.e. it does not contain intervals at its extremes.

Remark: In this description an interval means a non-trivial interval contained in the sum of the Cantor sets.

$[MO]$ introduced the concept of an M-Cantorval. It is a compact perfect set on the real line with dense interior, whose all bounded gaps are C-gaps. We will give here an example of an open set $\mathcal{U}_1 \subset \Omega$ such that, for generic pairs $(K_1, K_2) \in \mathcal{U}_1$, $K_1 + K_2$ is an M-Cantorval. It is proved in $[MO]$ that all M-Cantorvals are homeomorphic, so it characterizes the topology of $K_1 + K_2$ for generic (K_1, K_2) in $\mathcal{U}_1$. Suppose that K_1 and K_2 are regular Cantor sets defined by increasing surjective differentiable expanding functions $\psi_j^{(i)} : I_j^{(i)} \longrightarrow I_j$ such that if
$\psi_j : I_j^{(1)} \cup I_j^{(2)} \cup ... \cup I_j^{(n_j)} \longrightarrow I_j$ is defined by $\psi_j \Big|_{I_j^{(i)}} = \psi_j^{(i)}$ then

$$\begin{aligned} K_j &= \bigcap_{n \geqslant 0} \psi_j{}^{-n} \left(I_j^{(1)} \cup I_j^{(2)} \cup ... \cup I_j^{(n_j)} \right) \\ &= \left\{ x \in I_j^{(1)} \cup ... \cup I_j^{(n_j)} \Big| \psi_j^n(x) \in I_j^{(1)} \cup ... \cup I_j^{(n_j)}, \quad \forall\, n \geqslant 0 \right\}. \end{aligned}$$

We can order the intervals $I_j^{(i)}$ such that the left extreme of I_j belongs to $I_j^{(1)}$ and the right extreme of I_j belongs to $I_j^{(n_j)}$.

Given two such Cantor sets K_1 and K_2 there are diffeomorphisms φ_j of I_j, $j = 1, 2$ such that $\varphi_1 \circ \psi_1^{(n_1)} \circ \varphi_1^{-1}$ and $\varphi_2 \circ \psi_2^{(1)} \circ \varphi_2^{-1}$ are affine, with slope λ_1 and λ_2 respectively. We define the *right linearized Cantor set* of K_1 as $\widetilde{K}_1^R := \varphi_1(K_1)$, and the *left linearized Cantor set* of K_2 as $\widetilde{K}_2^L := \varphi_2(K_2)$, so that $\widetilde{K}_1^R$ and $\widetilde{K}_2^L$ are defined by the functions $\widetilde{\psi}_1^R := \varphi_1 \circ \psi_1 \circ \varphi_1^{-1}$ and $\widetilde{\psi}_2^L := \varphi_2 \circ \psi_2 \circ \varphi_2^{-1}$.

An interval $J_1 = [a, b]$ is *left linked* with $J_2 = [c, d]$ if $a < c$, $b < d$ and $c < b$. We say that a pair of Cantor sets (K_1, K_2) as before satisfies the **L property* if there are $\lambda > 0$, $t \in \mathbb{R}$ such that the support interval of $\widetilde{K}_1^R$ is left-linked with the support interval of $\lambda \widetilde{K}_2^L + t$ and $\widetilde{K}_1^R \cap \left(\lambda \widetilde{K}_2^L + t \right) = \varnothing$. The **R property* is defined in the symmetric way: (K_1, K_2) satisfies the *R property if and only if (K_2, K_1) satisfies the *L property .

It was proved in $[M]$ (Theorem IV. 1) that if $\dfrac{\ln \lambda_1}{\ln \lambda_2} \notin Q$, then for $\lambda > 0$, $K_1 + (-\lambda K_2)$ does not contains any interval containing the left extreme of $K_1 + (-\lambda K_2)$. As a consequence we can prove the proposition below.

Remark: The results of the all following propositions hold with total measure in generic 1-parameter families.

Proposition 1 *In the open set* $\mathcal{U}_1 = \{(K_1, K_2)$ *pairs of regular Cantor sets defined by increasing functions such that* $(K_1, -K_2)$ *and* $(K_2, -K_1)$ *satisfy the* $^*L$ *property and* $HD\,(K_1) + HD\,(K_2) > 1\}$. *Then for pairs* (K_1, K_2) *in a residual subset* $\mathcal{U}_1$, $K_1 + K_2$ *is an* M*-Cantorval.*

Proof. Suppose that $(K_1,\ -K_2) \in \mathcal{U}$, the open and dense subset of $\Omega = \{(K_1, K_2)\,|K_1, K_2 \,\text{regular Cantor sets},\, HD\,(K_1) + HD\,(K_2) > 1\}$ defined in the introduction. Then $K_1 + K_2$ have dense interior, i.e. $K_1 + K_2 = \overline{int(K_1 + K_2)}$. Suppose moreover that for any a, b extremes of gaps of K_1 and c, d extremes of gaps of K_2 we have $a + c \neq b + d$. This last property is generic, since for any (K_1, K_2) there is only a countable subset of $\lambda \in \mathbb{R}$ such that the property fails for $(K_1, \lambda K_2)$. Suppose moreover that the ratio of the logarithms of the slopes of $\widetilde{\psi}_1^L\Big|_{\widetilde{I}_1^{(1)L}}$ and $\widetilde{\psi}_2^L\Big|_{\widetilde{I}_2^{(1)L}}$ is irrational and the same for $\widetilde{\psi}_1^R\Big|_{\widetilde{I}_1^{(n_1)L}}$ and $\widetilde{\psi}_2^R\Big|_{\widetilde{I}_2^{(n_2)L}}$ $\Big($in our previous notation $\widetilde{I}_1^{(n_1)R} := \varphi_1\left(I_1^{(n_1)}\right)$ and $\widetilde{I}_2^{(1)L} := \varphi_2\left(I_2^{(1)}\right)\Big)$.

We claim that under these conditions $K_1 + K_2$ is an M-Cantorval.

Indeed, an extreme of a gap of $K_1 + K_2$ is always of the form $a + b$ where $a \in K_1$ and $b \in K_2$ are extremes of gaps. If we consider a left extreme, then a and b are left extremes of gaps in K_1 and K_2 respectively, so there are intervals of the construction of K_1 and K_2, say J_1 and J_2 of which a and b are its left extremes, and if $\overline{K}_1 = K_1 \cap J_1$ and $\overline{K}_2 = K_2 \cap J_2$ then $\overline{K}_1$ and $\overline{K}_2$ are regular Cantor sets conjugated to K_1 and K_2, and so their linearized Cantor sets are similar to the linearized of K_1 and K_2.

Thus, by the hypothesis, $\overline{K}_1 + \overline{K}_2$ does not contain any interval adjacent to $a + b$, and so $a + b$ is accumulated by gaps and intervals. Analogously the same is true for right extremes. Since $\overline{K}_1 + \overline{K}_2$ contains a neighbourhood of $a + b$ in $K_1 + K_2$ (by uniqueness of a and b), $K_1 + K_2$ is an M-Cantorval, which completes the proof.

Example: Consider the Cantor set K which is the maximal invariant set contained in $[0, 23]$ of the orientation preserving affine functions ψ_i, $1 \leqslant i \leqslant 4$, such that $\psi_i(I_i) = [0, 23]$ where

$$I_1 = [0, 1]\ , I_2 = [3, 11]\,, \, I_3 = [12, 20] \text{ and } I_4 = [22, 23]\,.$$

Then, there is a neighborhood $\mathcal{U}$ of K in the C^{1+} topology and a residual set $\mathcal{R} \subseteq \mathcal{U} \times \mathcal{U}$ such that for all $(K_1, K_2) \in \mathcal{R}$, $K_1 + K_2$ is an M-Cantorval.

In fact, the maximal invariant set $\widetilde{K} \subseteq K$ of ψ_2 and ψ_3 has thickness $\frac{8}{7} > 1$, so there is a neighbourhood $\mathcal{U}$ of K, such that for each $K_1 \in \mathcal{U}$ there exists $\widetilde{K}_1 \subseteq K_1$ a Cantor set with $\tau\left(\widetilde{K}_1\right) > 1$. Therefore, by the Gap- lemma, for all $K_1,\ K_2 \in \mathcal{U},\ K_1 + K_2$ has dense interior. Moreover, since (K, K) satisfies the $*L$ and the $*R$ properties defined above, there is a residual $\mathcal{R} \subseteq \mathcal{U} \times \mathcal{U}$ such that every bounded gap of $K_1 + K_2$ is a $C-$gap. It follows that for such (K_1, K_2), $K_1 + K_2$ is an $M-$Cantorval.

Similarly, as in $[MO]$, an R-(resp. L-) Cantorval is a perfect compact subset of the real line, with dense interior, whose all its bounded gaps are R-(resp. L-) gaps.

As in $[M]$, we define *left extremal stable intersection* as follows: (K_1, K_2) has left extremal stable intersection if there are $\delta > 0$ and a neighbourhood $\mathcal{V}$ of (K_1, K_2) such that $\left(\widetilde{K}_1,\ \widetilde{K}_2\right) \in \mathcal{V} \Longrightarrow \widetilde{K}_1 + \left(-\widetilde{K}_2\right)$ contains an interval of lenght δ to which the left extreme of $\widetilde{K}_1 + \left(-\widetilde{K}_2\right)$ belongs. Then we have the following proposition, whose proof is analogous to the previous one.

Proposition 2 *In the open set* $\mathcal{U}_2^{R(resp.\ L)} = \{(K_1, K_2)$ *pairs of regular Cantor sets defined by increasing functions such that* $(K_1,\ -K_2)$ *satisfy the* $*L$ *property and* $(K_1,\ -K_2)$ *have extremal stable intersection (resp.* $(-K_2,\ K_1)$ *satisfy the* $*L$ *property and* $(-K_2,\ K_1)$ *have extremal stable intersection) and* $HD(K_1) + HD(K_2) > 1\}$. *Then for generic pairs* $(K_1,\ K_2) \in \mathcal{U}_2^{R(resp.\ L)}$, $K_1 + K_2$ *is an* $R(resp.\ L)$*-Cantorval .*

Example: If K is defined by the increasing affine functions $\psi_i,\ i = 1, 2, 3$ such that

$$\psi_1(I_i) = [0, 14],\ \text{where } I_1 = [0, 5],\ I_2 = [6, 11] \text{ and } I_3 = [13, 14].$$

Then there is a neighbourhood $\mathcal{U}$ of K in the C^{1+} topology such that for generic $(K_1, K_2) \in \mathcal{U} \times \mathcal{U}$ then $K_1 + K_2$ is an $R-$Cantorval and for generic $(-K_1, -K_2) \in \mathcal{U} \times \mathcal{U}$ then $K_1 + K_2$ is an $R-$Cantorval.

Proposition 3 *Let* K *be the affine Cantor set whose convex hull is* $[0, 24]$ *defined by the affine functions* ψ_i, $i = 1, 2, 3$*, such that*

$$\psi_i(I_i) = [0, 24],\ \textit{where } I_1 = [0, 10],\ I_2 = [11, 21] \textit{ and } I_3 = [23, 24],$$

and only ψ_2 *is decreasing. Then there is a neighbourhood* $\mathcal{U}_3$ *of* K *in the* C^{1+} *topology such that if* $K_1, K_2 \in \mathcal{U}_3$ *then* $K_1 + K_2$ *is an infinite disjoint union of closed intervals converging to a point.*

Proof. The maximal invariant set of ψ_1 and ψ_2 in $[0,21]$ is an affine Cantor set $\widetilde{K}$ whose convex hull is $[0,21]$ with thickness $\tau\left(\widetilde{K}\right) = \frac{5}{2}$, so $\tau(K)\,\tau\left(\widetilde{K}\right) = \frac{5}{4} > 1$. This implies that K has stable intersection with $\lambda\widetilde{K} + t$ if K and $\lambda\widetilde{K} + t$ are linked.

If $I = [0,21]$ then $K \subset \{24\} \cup \bigcup_{n=0}^{\infty} \psi_3^{-n}(I)$, and $\psi_3^{-n}(I) \supset \psi_3^{-n}\left(\widetilde{K}\right)$, a Cantor set whose thickness is $\frac{5}{2}$ and whose convex hull is $\psi_3^{-n}(I)$.

Let K_1, K_2 be Cantor sets close to K defined by $\left(\psi_1^{(1)}, \psi_2^{(1)}, \psi_3^{(1)}\right)$ and $\left(\psi_1^{(2)}, \psi_2^{(2)}, \psi_3^{(2)}\right)$ respectively. Let $\widetilde{K}_1$ and $\widetilde{K}_2$ be constructed as $\widetilde{K}$. We may suppose $\tau\left(\widetilde{K}_1\right)\tau(K_2) > 1$ and $\tau\left(\widetilde{K}_2\right)\tau(K_1) > 1$, since they are close to $\frac{5}{4}$. It is not difficult to check (by self similarity, as in the Theorem IV.1 of $[M]$) that the rightmost point of $K_1 + K_2$ is accumulated by a sequence of gaps of $K_1 + K_2$. Let b_1 and b_2 be the rightmost points of K_1 and K_2, respectively. The results follows from

$$\begin{aligned}
K_1 + K_2 \;\subset\; & \{b_1 + b_2\} \cup \bigcup_{n=0}^{\infty} \left(\left(\psi_3^{(1)}\right)^{-n} \left(I^{(1)}\right) + \left(\psi_3^{(2)}\right)^{-[\frac{n}{2}]} (K_2) \right) \\
& \cup \bigcup_{n=0}^{\infty} \left(\left(\psi_3^{(1)}\right)^{-[\frac{n}{2}]} (K_1) + \left(\psi_3^{(2)}\right)^{-n} \left(I^{(2)}\right) \right) \\
= \; & \{b_1 + b_2\} \cup \bigcup_{n=0}^{\infty} \left(\left(\psi_3^{(1)}\right)^{-n} \left(\widetilde{K}_1\right) + \left(\psi_3^{(2)}\right)^{-[\frac{n}{2}]} (K_2) \right) \\
& \cup \bigcup_{n=0}^{\infty} \left(\left(\psi_3^{(1)}\right)^{-[\frac{n}{2}]} (K_1) + \left(\psi_3^{(2)}\right)^{-n} \left(\widetilde{K}_2\right) \right) \subset K_1 + K_2,
\end{aligned}$$

where $I^{(1)}$ and $I^{(2)}$ are the convex hull of $\widetilde{K}_1$ and $\widetilde{K}_2$. The equality holds by the Gap-lemma since the sums in the right hand side are sums of Cantor sets whose product of thicknesses is close to $\frac{5}{4}$, and $\left(\psi_3^{(i)}\right)^{-[\frac{n}{2}]}(K_i)$ is never contained in a gap of $\left(\psi_3^{(j)}\right)^{-n}\left(\widetilde{K}_j\right)$, $i,j = 0,1$. Moreover, these sums are finite unions of intervals that converge to $\{b_1 + b_2\}$, which shows that $b_1 + b_2$ is the only accumulation point of the boundary of $K_1 + K_2$. Thus, the proof has been completed.

Remark: If K_1 and K_2 are near K and $\lambda > 0$ is small (for instance, if $\lambda < \frac{1}{30}$) then the topological type of $K_1 + \lambda K_2$ is a finite union of (2 or more) copies homeomorphic to $K + K$, and the number of copies increases as λ tends to 0.

Proposition 4 *Given $\varepsilon > 0$ consider the affine Cantor set K_ε which is the maximal invariant set, contained in $[0,11]$ of the affine functions ψ_i, such that $\psi_i(I_i) = [0,11]$, $1 \leqslant i \leqslant 4$, where*

$$I_1 = [0,4]\,,\ I_2 = [6,10]\,,\ I_3 = [11-2\varepsilon,\ 11-\varepsilon]\ ,\ I_4 = \left[11-\varepsilon^2,\ 11\right],$$

and only ψ_1 preserves orientation. If $\varepsilon > 0$ is small enough (for example $\varepsilon \in \left(0, \frac{1}{300}\right)$) there is a neighbourhood $\mathcal{U}_4$ of K_ε in the C^{1+} topology such that for all $K_1, K_2 \in \mathcal{U}_4$, every gap of $K_1 + K_2$ is an RL-gap and the accumulation points of the boundary of $K_1 + K_2$ are not countable. (In particular $K_1 + K_2$ is not a finite union of intervals).

Proof. 1.- Note that the maximal invariant set $\widetilde{K}_\varepsilon$ of ψ_i, $i = 1,2$ is contained in K_ε and has thickness $\frac{4}{3}$. Moreover, since ψ_4 reverses orientation, K_ε contains another subset at the right extreme which has thickness $\frac{4}{3}$. Therefore, by the Gap-lemma all K_1 and K_2 close to K_ε in the C^{1+} topology are such that every gap of $K_1 + K_2$ is an RL-gap for any $\lambda > 0$.

2.- Given a neighbourhood $\mathcal{V}$ of K_ε there is a neighbourhood $\mathcal{U}_4$ of K_ε such that for $K \in \mathcal{U}_4$ and for each interval I of the construction of $K \cap I$ is the image by an affine map of an element of $\mathcal{V}$.

We shall prove that if $\mathcal{V}$ is small enough and $\mathcal{U}_4$ is as above, then, for any $\lambda > 0$ and $K_1, K_2 \in \mathcal{U}_4$, we can find two disjoint open intervals I', I'' such that $(K_1 + \lambda K_2) \cap I' = \mu' K_1' + \lambda' K_2'$ and $(K_1 + \lambda K_2) \cap I'' = \mu'' K_1'' + \lambda'' K_2''$ where K_1', K_2', K_1'' and K_2'' belong to $\mathcal{U}_4$, and both $\mu'\lambda'$ and $\mu''\lambda''$ are positive.

Repeating the process for the sums $\mu' K_1' + \lambda' K_2'$ and $\mu'' K_1'' + \lambda'' K_2''$ and successively, we get our result.

Our previous statement follows from the following analysis:

(i) If $\frac{1}{100} \leq \lambda \leq 100$, $K_1 + \lambda K_2 \supset \left(\widetilde{\psi}_4^{-1}(K_1) + \lambda \overline{\psi}_4^{-1}(K_2)\right) \cup \left(\widetilde{\psi}_3^{-1} \widetilde{\psi}_4^{-1}(K_1) + \lambda \overline{\psi}_3^{-1} \overline{\psi}_4^{-1}(K_2)\right)$ we denote the functions that define K_1 and K_2 by $\widetilde{\psi}$ and $\overline{\psi}$ respectively). Since support intervals of the two previous sums are disjoint, we get the statement because $\widetilde{\psi}_j, \overline{\psi}_j$, $j = 3, 4$ reverse orientation.

(ii) If $\lambda \leq \frac{1}{100}$, $K_1 + \lambda K_2 \supset \left(\widetilde{\psi}_1^{-1}(K_1) + \lambda K_2\right) \cup \left(\widetilde{\psi}_2^{-2}(K_1) + \lambda K_2\right)$. As in the previous case, the support intervals of the sums of the right side are disjoint, and both $\widetilde{\psi}_1$ and $\widetilde{\psi}_2^2$ preserve orientation, so again we get the desired result. The case $\lambda \geq 100$ is proved in the same way, changing the role of K_1 and K_2. Therefore, the proof has been completed.

Proposition 5 *Given $\varepsilon > 0$ small, let K_ε be the Cantor set defined by the affine increasing functions ψ_i such that ψ_i, $1 \leqslant i \leqslant 6$, $\psi_i(I_i) = [0, 25]$, where*

$$\begin{aligned} I_1 &= [0, 10], \; I_2 = \left[11, 11 + \frac{\varepsilon^2}{2}\right], \; I_3 = \left[12, 12 + 2\varepsilon^2\right], \\ I_4 &= \left[13, 13 + \frac{\varepsilon^2}{2}\right], \; I_5 = \left[14, 14 + 2\varepsilon^2\right] \; \textit{and} \; I_6 = [15, 25]. \end{aligned}$$

If ε small enough (for instance if $0 < \varepsilon < \dfrac{1}{100}$) then there is a neighbourhood $\mathcal{U}_5$ of K_ε in the C^{1+} topology such that if $(K_1, K_2) \in \mathcal{U}_5 \times \mathcal{U}_5$ and $\lambda \in \left(\dfrac{\varepsilon}{3}, 3\varepsilon\right)$ then $K_1 + \lambda K_2$ has non-countably many connected components and all of its gaps are of the type RL.

Proof. Let $\mathcal{U}_5$ be a small neighbourhood of K_ε and $\widetilde{K}$ the maximal invariant set of ψ_1 and ψ_6, then $\tau\left(\widetilde{K}\right) = 2$ and the convex hull of $\widetilde{K}$ is $[0, 25]$. In a similar way, given $K_1 \in \mathcal{U}_5$ one defines $\widetilde{K}_1$, which has thickness close to 2, and whose convex hull is equal to that of K_1.Thus if

$(K_1, K_2) \in \mathcal{U}_5 \times \mathcal{U}_5\}$ then $\tau\left(\widetilde{K}_1\right)\tau\left(\widetilde{K}_2\right) > 1$ since it is close to 4, and therefore all gaps of $K_1 + K_2$ are of type RL.

Let $\widetilde{\mathcal{U}}$ be a small neighbourhood of K_ε, thus if $\mathcal{U}_5$ is small enough, then every sub-Cantor set of $K_1 \in \mathcal{U}_5$ belongs to $\widetilde{\mathcal{U}}$ after been properly reescaled.

Let $(K_1, K_2) \in \mathcal{U}_5 \times U_5$, and $\psi_i^{(j)}$, $1 \leqslant i \leqslant 6$,$1 \leqslant j \leqslant 2$ be the functions that define K_1 and K_2. For $\lambda \in \left(\frac{\varepsilon}{3}, 3\varepsilon\right)$, we have

$$K_1 + \lambda K_2 = \bigcup_{i=1}^{6}\left(\left(\psi_i^{(1)}\right)^{-1}(K_1) + \lambda K_2\right).$$

Since $\lambda \in \left(\frac{\varepsilon}{3}, 3\varepsilon\right)$, the diameter of λK_2 is smaller than the lengh of the gaps of K_1 in the first step of its construction, so the convex hulls of $\left(\psi_i^{(1)}\right)^{-1}(K_1) + \lambda K_2$, $1 \leqslant i \leqslant 6$ are disjoint.

Since λ belongs to $\left(\frac{\varepsilon}{3}, 3\varepsilon\right)$ and if $\mathcal{U}_5$ is small enough, the ratio of the diameters of the Cantor sets $\left(\psi_i^{(1)}\right)^{-1}(K_1)$ and λK_2 belongs to $\left(\frac{2\varepsilon}{5}, \frac{5\varepsilon}{2}\right) \subset \left(\frac{\varepsilon}{3}, 3\varepsilon\right)$ for at least two values of i between 2 and 5. For these values of j, the sets $\left(\psi_j^{(1)}\right)^{-1}(K_1) + \lambda K_2$ are open and closed in $K_1 + \lambda K_2$, and $\left(\frac{25}{|K_2|} \cdot K_2, \frac{25}{\lambda\,|K_2|} \cdot \psi_j^{(1)}(K_1)\right) \in \mathcal{U}_5 \times \mathcal{U}_5$, so we can repeat the argument to these pairs and conclude that $K_1 + K_2$ has non-countably many connected components. Therefore the proof has been completed.

Remark: If $\mathcal{U}_5$ is small enough, λ is close to 1 and $(K_1, K_2) \in \mathcal{U}_5 \times \mathcal{U}_5$ then $(K_1) + \lambda K_2$ is an interval, so the topological structure of $K_1 + \lambda K_2$ for K_1 and K_2 close to K depends strongly on the size of λ, that is not the case of the Proposition 4.

Proposition 6 *Let K be the Cantor set which is the maximal invariant set of the maps ψ_i, $1 \leqslant i \leqslant 4$, such that $\psi_i(I_i) = [0, 52]$, where*

$$I_1 = [0, 21],\ I_2 = [22, 43],\ I_3 = [48, 49],\ I_4 = [51, 52],$$

and only ψ_2 reverses orientation. Then, there is a neighbourhood $\mathcal{U}_6$ of K and a residual set $\mathcal{R} \subseteq \mathcal{U}_6 \times \mathcal{U}_6$ such that for all $(K_1, K_2) \in \mathcal{R}$ and $\lambda > 0$, *(i) the bounded gaps of* $K_1 + \lambda\ K_2$ *are of type* RL *or* L .

(ii) $K_1 + \lambda\ K_2$ *has infinitely many gaps of type* RL *and of type* L*.*

Proof. 1.- Fix $\eta > 1$ close to 1 and let $\mathcal{V} = \mathcal{V}(\eta)$ be a neighbourhood of K such that all the ratios in the first step in the construction of a Cantor set in $\mathcal{V}$ are like the ones in K with a factor in (η^{-1}, η).

Then, by bounded distortion, there is a neighbourhood $\mathcal{U}_6 = \mathcal{U}_6(\eta)$ of K such that every sub-Cantor set of $K \in \mathcal{U}_6$ belongs to $\mathcal{V}$ after been properly reescaled. Note that $\tau(K) = \dfrac{1}{2}$ and, if $\widetilde{K}$ denotes the maximal invariant set contained $[0, 52]$ of ψ_1 and ψ_2, then $\tau\left(\widetilde{K}\right) = \dfrac{21}{10}$, so $\tau\left(\widetilde{K}\right) \cdot \tau(K) > 1$.

Therefore, if $\eta > 1$ is close to 1 for all $K_1, K_2 \in \mathcal{U}_6$, $\tau\left(\widetilde{K}_1\right) \cdot \tau(K_2) > 1$, where $\widetilde{K}_1$ is the analogous set of $\widetilde{K}$ for K_1. Thus, by the Gap-lemma, $\widetilde{K}_1 + K_2$ is a finite union of intervals. In particular, the leftmost point of $K_1 + K_2$ belongs to an interval of $K_1 + K_2$, since it belongs to $\widetilde{K}_1 + K_2$.

2.- Let $t \in K_1 + K_2$ be the right boundary point of a gap of $K_1 + K_2$, so $t = t_1 + t_2$ where t_i is the left extreme of a sub-Cantor set K_i' of K_i, $i = 1, 2$. By the observation in 1 and considering that $\tau(K) = \tau(-K)$, it follows that if K_1' or K_2' is an oriented copy of K_1 or K_2 , respectively, then t belongs to an interval of $K_1 + K_2$.

On the other hand, if K_1' and K_2' are reversed copies of K_1 and K_2 let n_i be the least integer such that t_i is an extreme of an interval in the n_i^{th} step of the construction of K_i, $i = 1, 2$. Note that $n_i > 1$ and let I_i be the interval in the $(n_i - 1)^{th}$ step in the construction of K that contains t_i, $i = 1, 2$.

Thus $I_1 \cap K_1$ is an oriented copy of K_1 and $t_1 \in int J \subseteq I_1$, where J is the support of two thick Cantor set, i.e. $J \cap K_1$ is a Cantor set such that $\widehat{J \cap K_1} = J$ and $\tau(J \cap K_1)\, \tau(K_2) > 1$.

Reducing $\mathcal{U}_6$, if necessary, one may suppose that for t' close to t, $-K_2 + t'$ is not contained in a gap of $J \cap K_1$. Since $t_1 = t - t_2 \in int(J) \cap (-K_2 + t)$, for t' close to t, $J \cap K_1$ is not contained in a gap of $-K_2 + t'$. Then, by the Gap-lemma $(J \cap K_1) \cap (-K_2 + t') \neq \phi$ for all t' close to t. Thus, t belongs to the interior of an interval of $K_1 + K_2$, contradicting its definition.

This proves that all gaps of $K_1 + K_2$ are of type RL or L, since they all have an interval to their right, so part (i) has been proved.

3.- If $K_1, K_2 \in \mathcal{V}$ and $\lambda > 0$, there are $m, n \in \mathbb{N}$ (that can be choose n arbitrarily large) such that $\frac{\left|\widetilde{\psi}_4^{-m}(K_1)\right|}{\lambda\left|\overline{\psi}_4^{-n}(K_2)\right|}$ is close to 1 (where $\widetilde{\psi}$, $\overline{\psi}$ denote the functions that define K_1 and K_2), provided that the ratio of the logarithms of the eigenvalues of $\widetilde{\psi}_4$ and $\overline{\psi}_4$ at their fixed points is irrational (which is a generic condition, called irrational condition).

Now, assuming the generic condition that no point can be written in two or more different ways as the sum of extremes points of gaps of K_1 and K_2 (this is called the separation condition), there is a gap between the convex hulls of $\widetilde{\psi}_4^{-m-1}(K_1) + \lambda\overline{\psi}_4^{-n-1}(K_2)$ and $\left(\widetilde{\psi}_4^{-m}\widetilde{\psi}_3^{-1}(K_1) + \lambda\overline{\psi}_4^{-n-1}(K_2)\right) \cup \left(\widetilde{\psi}_4^{-m-1}(K_1) + \lambda\overline{\psi}_4^{-n}\overline{\psi}_3^{-1}(K_2)\right)$ of type L and there is a gap between the convex hulls of $\widetilde{\psi}_4^{-m}\widetilde{\psi}_3^{-1}(K_1) + \lambda\overline{\psi}_4^{-n}\overline{\psi}_3^{-1}(K_2)$ and $\left(\widetilde{\psi}_4^{-m}\widetilde{\psi}_3^{-1}(K_1) + \lambda\overline{\psi}_4^{-n}\overline{\psi}_2^{-1}(K_2)\right) + \left(\widetilde{\psi}_4^{-m}\widetilde{\psi}_2^{-1}(K_1) + \lambda\overline{\psi}_4^{-n}\overline{\psi}_3^{-1}(K_2)\right)$ of type RL. Since m, n can be taken arbitrarily large, this gives our result.

Remark: In fact is possible to adapt this argument to show that there is an uncountable set $\mathcal{C} \subset K_1 + K_2$ such that every neighbourhood of any point of C contains infinitely many gaps of both types R and RL.

Proposition 7 : *Consider the Cantor set K defined by the affine functions ψ_i, $1 \leqslant i \leqslant 4$, such that $\psi_i(I_i) = [0, 52]$, where*

$$I_1 = [0, 21], I_2 = [22, 43], I_3 = [48, 49], I_4 = [51, 52],$$

and only ψ_3 reverses orientation. Then, there is a neighbourhood $\mathcal{U}$ of K and a residual subset $\mathcal{R}$ of $\mathcal{U} \times \mathcal{U}$ such that for all $(K_1, K_2) \in \mathcal{R}$, $K_1 + K_2$ has infinitely many gaps of each four types as described at the beginning of this section.

Proof. Let we first remark that if we denote by $\widetilde{K}$ the maximal invariant set of ψ_1 and ψ_2 contained in $[0, 52]$. Note that $\widetilde{K} \subseteq K$ and $\tau\left(\widetilde{K}\right) = \dfrac{21}{10}$ and so $\tau(K)\,\tau\left(\widetilde{K}\right) = \frac{21}{20} > 1$.

Let K_1, K_2 be regular Cantor sets close to K that satisfy the irrational condition.

As in proposition 6, we have, arbitrarily close to the right extreme of $K_1 + K_2$, sums of regular Cantor sets which are close to K after reescaling, that are open and closed in K_1+K_2 (and indeed are intersections of K_1+K_2 with intervals). The result follows from noting that if K_1 and K_2 are close to K and satisfy the separation condition, then denoting by $\widetilde{\psi}$ and $\overline{\psi}$ the functions that define K_1 and K_2, we have:

(i) The gap between the convex hulls of $\widetilde{\psi}_4^{-1}(K_1)+\overline{\psi}_4^{-1}(K_2)$ and $(\widetilde{\psi}_3^{-1}(K_1) + \overline{\psi}_4^{-1}(K_2)) \cup (\widetilde{\psi}_4^{-1}(K_1) + \overline{\psi}_3^{-1}(K_2))$ is of type RL.

(ii) The gap between the convex hulls of $(\widetilde{\psi}_3^{-1}(K_1)+\overline{\psi}_4^{-1}(K_2))\cup(\widetilde{\psi}_4^{-1}(K_1)+\overline{\psi}_3^{-1}(K_2)$ and $\widetilde{\psi}_3^{-1}(K_1) + \overline{\psi}_3^{-1}(K_2)$ is of type R.

(iii) The gap between the convex hulls of $\widetilde{\psi}_3^{-1}(K_1)+\overline{\psi}_3^{-1}(K_2)$ and $(\widetilde{\psi}_2^{-1}(K_1)+\overline{\psi}_4^{-1}(K_2)) \cup (\widetilde{\psi}_4^{-1}(K_1) + \overline{\psi}_2^{-1}(K_2))$ is of type C.

(iv) We have, arbitrarily close to the right end point of the gap in $\mathcal{C}$ mentioned in last remark, sums of small Cantor sets which are close to $-K$ after rescaling, that are open and closed in K_1+K_2 (this gives, by item (ii) gaps of type L).

Thus, the proof has been completed.

Remark: In fact one can prove, by slightly reducing $\mathcal{R}$, that every end point of a gap is the boundary of an interval, or it is accumulated by infinitely many gaps of each one of the four types.

We can use this fact to show that there is an uncountable set $\mathcal{C} \subset K_1+K_2$ such that every neighbourhood of any point of $\mathcal{C}$ contains infinitely many gaps of each of the four types (C, L, R and RL).

4 Open Questions

The following is a natural question.

Problem 1: Does there exists a generic set $\mathcal{R} \subset \{(K_1, K_2) \mid K_1, K_2$ are C^{1+} regular Cantor sets$\}$ such that each $(K_1, K_2) \in \mathcal{R}$ has a neighbourhood $\mathcal{U}$ such that $(\widetilde{K_1}, \widetilde{K_2}) \in \mathcal{R} \cap \mathcal{U} \implies \widetilde{K_1} + \widetilde{K_2}$ is homeomorphic to $K_1 + K_2$?

In particular, if generic is taken in the Baire sense, it would imply that there are countably many robust topologies of a sum of Cantor sets. The remarks after Example 0 and Proposition 3 show that there are infinitely many robust topologies of sums of Cantor sets.

An interesting particular instance of Problem 1 is the problem of determining the typical topology (or topologies) for sums of pairs of Cantor sets in the open sets $\mathcal{U}_2^{R(resp.\, L)}$ of Proposition 2. For generic pairs $(K_1, K_2) \in \mathcal{U}^L, K_1 + K_2$ is an L-Cantorval, which that not determine its topology. There are two natural possibilities for the topology of $K_1 + K_2$ in this case: the L_0 topology, obtained from a Cantor set by blowing up each right extreme of a gap creating intervals; and the L_∞ topology, obtained by blowing up each right extreme of a gap of an M-Cantorval. We do not know robust examples of sums of Cantor sets with the L_0 topology, (resp. with the L_∞ topology), but it is reasonable to think that generically in $\mathcal{U}^L, K_1 + K_2$ is either L_0 or L_∞.

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