HAUSDORFF MEASURES AND THE MORSE–SARD THEOREM

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Abstract

Let $F\colon U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function and $p < m$ an integer. If $k \geq 1$ is an integer, $\alpha \in [0,1]$ and $F \in C^{k+(\alpha)}$, if we set $C_p(F) = \{x \in U \mid \operatorname{rank}(Df(x)) \leq p\}$ then the Hausdorff measure of dimension $(p + \frac{n-p}{k+\alpha})$ of $F(C_p(F))$ is zero.

1 Introduction

The Morse-Sard theorem is a fundamental theorem in analysis that is in the basis of transversality theory and differential topology. The classical Morse-Sard theorem states that the image of the set of critical points of a function $F\colon \mathbb{R}^n \to \mathbb{R}^m$ of class C^{n-m+1} has zero Lebesgue measure in $\mathbb{R}^m$. It was proved by Morse ([M]) in the case $m = 1$ and by Sard ([S1]) in the general case.

Due to its theoretical importance, the Morse-Sard theorem was generalized in many directions. Many of these generalizations are related with Hausdorff measures and Hausdorff dimensions.

Given a metric space X and a positive real number α, we define the Hausdorff measure of dimension α associated to a covering $\mathcal{U} = (U_\lambda)_{\lambda \in L}$ of X by bounded sets U_λ by $m_\alpha(\mathcal{U}) = \sum_{\lambda \in L} (\text{diam } U_\lambda)^\alpha$, where diam U_λ denotes the diameter of U_λ, and, if we define the norm of a covering $\mathcal{U}$ by $||\mathcal{U}|| = \sup_{U \in \mathcal{U}} (\text{diam } U)$, then the Hausdorff measure of dimension α of X is $m_\alpha(X) = \liminf\limits_{\substack{\mathcal{U} \text{ covering of } X \\ ||\mathcal{U}|| \to 0}} m_\alpha(\mathcal{U})$.

It is not difficult to see that there is a unique $d \in [0, +\infty]$ such that if $\alpha > d$ then $m_\alpha(X) = 0$ and if $\alpha < d$ then $m_\alpha(X) = +\infty$. This number d is called the *Hausdorff dimension* of X. It is easy to see that if $X \subset \mathbb{R}^n$ then its Hausdorff dimension $d =: HD(X)$ belongs to $[0, n]$.

Sard himself proved that if $C_p(F) = \{x \in \mathbb{R}^n \mid \operatorname{rank}(DF(x)) \leq p\}$ then for any $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that if F is C^k then $F(C_p(F))$ has zero Hausdorff measure of dimension $p+\varepsilon$ ([S2]). This result was made more precise by Federer ([F]), who proved that if $k \in \mathbb{N}$ then the Hausdorff measure of dimension $p + \frac{n-p}{k}$ of $F(C_p(F))$ is zero. We should also mention the works of Church ([Ch1], [Ch2]), which gave more results about the structure of the set of critical values of differentiable maps. Later, Yomdin ([Y]) proved that the Hausdorff dimension of $F(C_p(F))$ is at most $p + \frac{n-p}{k+\alpha}$, provided that $F \in C^{k+\alpha}$, where $k \in \mathbb{N}$ and $0 \leq \alpha < 1$. More recently, Bates ([B2]) proved that if $F \in C^{k+\alpha}$ with $k \in \mathbb{N}$, $0 < \alpha \leq 1$ and $p + \frac{n-p}{k+\alpha} = m$ then $F(C_p(F))$ has zero Lebesgue measure in $\mathbb{R}^m$ (this in particular improves the hypothesis of the classical Morse-Sard theorem from $F \in C^{n-m+1}$ to $F \in C^{n-m+\text{Lips.}}$, i.e., $F \in C^{n-m}$ and $D^{n-m}F$ Lipschitz.

The aim of this work is to generalize the mentioned results by proving a general version of the Morse-Sard Theorem involving Hausdorff measures. Let $k \geq 1$ be an integer and $\alpha \in [0,1]$. We say that a function $F\colon U \subset \mathbb{R}^n \to \mathbb{R}^m$ is of class $C^{k+(\alpha)}$ at a subset A of U if F is C^k in U and for each $x \in A$ there are $\varepsilon_x > 0$, $K_x > 0$ such that $|y - x| < \varepsilon_x \Rightarrow |D^k F(y) - D^k F(x)| \leq K_x |y - x|^\alpha$ (this is less restrictive than supposing $F \in C^{k+\alpha}$). Our main result is the following

Theorem: Let $F\colon U \subset \mathbb{R}^n \xrightarrow{C^k} \mathbb{R}^m$ and let $p < m$ be an integer. If $C_p(F) := \{x \in U \mid \operatorname{rank}(DF(x)) \leq p\}$ and if F is of class $C^{k+(\alpha)}$ at $C_p(F)$ then the Hausdorff $\left(p + \frac{n-p}{k+\alpha}\right)$-measure of $F(C_p(F))$ is zero.

In particular, if $k + \alpha = \frac{n-p}{m-p}$, we recover the result of [B2], with a weaker hypothesis. We remark that if $p + \frac{n-p}{k+\alpha} < m$, the Hausdorff $\left(p + \frac{n-p}{k+\alpha}\right)$-measure is not the Lebesgue measure or a product measure in $\mathbb{R}^m$, and so we can not use Fubini's Theorem. This difficulty is solved in the present paper by replacing the use of Fubini's theorem by a careful decomposition of the critical set, combined with a parametrized strong version of the main lemma of Morse's paper([M]) (Theorem 2.1).

We shall also give examples that show that our result is quite sharp, by giving counterexamples to slight changes of the hypothesis or of the conclusion.

2 Functions whose zeros include a given set

We shall prove here a version of Theorem 3.6 of [M] and Lemma 3.4.2 of [F], which will be fundamental for the later results.

THEOREM 2.1. *Let $k \geq 1$, $\alpha \in [0,1]$, $n > p$ and $A \subset U \subset \mathbb{R}^n$, where U is an open set. Then there are sets $A_1, A_2 \ldots, \subset A$ such that $A = \bigcup_{i=1}^{\infty} A_i$, where for each $i = 1, 2, \ldots$ there is a function $\psi_i \colon B_i \times V_i \xrightarrow{C^1} U$ where B_i is a ball in some $\mathbb{R}^{r_i}$, $r_i \geq 0$ and V_i is a ball in $\mathbb{R}^p$ such that $\psi_i(x,y) = (\widetilde{\psi}_i(x,y), y)$, and $|\psi_i(x_1,y_1) - \psi_i(x_2,y_2)| \geq |(x_1,y_1) - (x_2,y_2)|$, $\forall\, (x_1,y_1), (x_2,y_2) \in B_i \times V_i$ and $A_i \subset \psi_i(B_i \times V_i)$, with the following property: We can write $A_i = A_i' \cup A_i''$ so that $\psi_i^{-1}(A_i'')$ has measure zero in $B_i \times V_i$, and if $f \colon U \to \mathbb{R}$ vanishes in A and f is $C^{k+(\alpha)}$ at A we have:*

- $\displaystyle\limsup_{(x,y_0)\to(x_0,y_0)} \frac{f(\psi_i(x,y_0))}{|x-x_0|^{k+\alpha}} < +\infty$, $\forall\, (x_0,y_0) \in B_i \times V_i$ *such that* $\psi_i(x_0,y_0) \in A_i$,
- $\displaystyle\lim_{(x,y_0)\to(x_0,y_0)} \frac{f(\psi_i(x,y_0))}{|x-x_0|^{k+\alpha}} = 0$, $\forall\, (x_0,y_0) \in B_i \times V_i$ *such that* $\psi_i(x_0,y_0)) \in A_i'$.

PROOF. Let us consider first the case $k = 1$ and $df(x) \cdot v = 0\ \forall\, x \in A$, $v \in \mathbb{R}^{n-p} \times \{0\}$. In this case we take $A = (A' \cap A) \cup A''$ where A' is the set of density points of $\overline{A}$ in the direction of $\mathbb{R}^{n-p} \times \{0\}$ ($(x,y) \in A' \Rightarrow \displaystyle\lim_{\varepsilon\to 0} \frac{m((B_\varepsilon(x) \times \{y\}) \cap \overline{A})}{m(B_\varepsilon(x))} = 1$, where m is the $(n-p)$-dimensional measure). The measure of $A'' = A - A'$ is zero, since it is zero in each plane $\mathbb{R}^{n-p} \times \{y\}$.

For $(x_0,y_0) \in A$ take $B((x_0,y_0), \varepsilon(x_0,y_0))$ a ball contained in U and $\psi = Id|_{B((x_0,y_0),\varepsilon(x_0,y_0))}$. We have $\displaystyle\limsup_{(x,y_0)\to(x_0,y_0)} \frac{f(x,y_0)}{|x-x_0|^{1+\alpha}} < +\infty$, since $f(x,y_0) = f(x,y_0) - f(x_0,y_0) = df(tx_0 + (1-t)x)(x-x_0)$, $t \in (0,1) \Rightarrow |f(x,y_0)| \leq K_{x_0}|x-x_0|^{1+\alpha}$. For $(x_0,y_0) \in A'$,

$$\lim_{\delta\to 0} \frac{1}{\mathrm{vol}(S^{n-p-1})} \int_{S^{n-p-1}} \left(\frac{1}{\delta} \int_0^\delta \chi_{\overline{A}}(x_0 + tv, y_0)\,dt \right) dv = 1,$$

so $\forall\, \varepsilon > 0\ \exists \delta_0 > 0$ s.t. $|x - x_0| < \delta_0 \Rightarrow \exists\, v \in S^{n-p-1}$ with $\left|v - \dfrac{x-x_0}{|x-x_0|}\right| < \varepsilon$ and $\left|\dfrac{1}{|x-x_0|} \int_0^{|x-x_0|} \chi_{\overline{A}}(x_0 + tv, y_0)\,dt - 1\right| < \varepsilon$, so, if $\tilde{x} = x_0 + |x-x_0|v$, $|f(x,y_0) - f(x_0,y_0)| \leq |f(x,y_0) - f(\tilde{x},y_0)| + |f(\tilde{x},y_0) - f(x_0,y_0)|$, but $f(x,y_0) - f(\tilde{x},y_0) = df(\theta t + (1-\theta)\tilde{x}, y_0) \cdot (x - \tilde{x}), \theta \in (0,1) \Rightarrow |f(x,y_0) - f(\tilde{x},y_0)| \leq K_{x_0}|x-x_0|^\alpha \cdot \varepsilon|x-x_0| = \varepsilon K_{x_0}|x-x_0|^{1+\alpha}$ and $f(\tilde{x},y_0) - f(x_0,y_0) = \int_0^{|\tilde{x}-x_0|} df(x_0 + tv, y_0) \cdot v\,dt \leq K_{x_0}|x-x_0|^\alpha \cdot m\{t \in [0, |\tilde{x}-x_0|] \mid \dfrac{\partial f}{\partial x}(x_0 + tv, y_0) \neq 0\} \leq K_{x_0}|x-x_0|^\alpha \cdot \varepsilon|x-x_0| = \varepsilon K_{x_0}|x-x_0|^{1+\alpha}$.

So $|f(x,y_0)| = |f(x,y_0) - f(x_0,y_0)| \leq 2\varepsilon K_{x_0}|x-x_0|^{1+\alpha} \Rightarrow \displaystyle\lim_{(x,y_0)\to(x_0,y_0)} \frac{f(x,y_0)}{|x-x_0|^{1+\alpha}} = 0$.

We can take a countable subcovering of A by the $B((x_0,y_0), \varepsilon(x_0,y_0))$ to finish the proof in this case.

Consider now the case $k \geq 1$, n arbitrary. We have $A = A^* \cup A^{**}$ where $A^* = \{x \in A \mid \exists\, g \colon U \xrightarrow{C^k} \mathbb{R},\ g|_A \equiv 0, \exists\, v \in \mathbb{R}^{n-p} \times \{0\},\ dg(x) \cdot v \neq 0\}$. $A^{**} = A \backslash A^*$. If $(x_0,y_0) \in A^*$ there is g as above, so there is $\varepsilon > 0$ such that $g^{-1}(0) \cap B_\varepsilon(x_0, y_0)$ is contained in the image of $\psi\colon B \times V \xrightarrow{C^k} U$ where B is a ball in $\mathbb{R}^{n-p-1}$, as in the statement, and $A \subset g^{-1}(0)$. Taking a countable subcovering of A^* by these balls we reduce the proof in this case to a case with smaller n. If $k = 1$, the result was yet proved for A^{**}. If $k > 1$, and assuming by induction the result for $k - 1$, we have

$$A^{**} = \bigcup_{i=1}^{\infty} A_i^{**}, A_i^{**} = (A_i^{**})' \cup (A_i^{**})'', A_i^{**} \subset \psi_i(B_i \times V_i), \psi_i \in C^1,$$

$$\psi_i(x_0, y_0) \in A_i^{**} \Rightarrow \limsup_{x \to x_0} \frac{||df(\psi_i(x, y_0))|_{\mathbb{R}^{n-p} \times \{0\}}||}{|x - x_0|^{k-1+\alpha}} < +\infty \Rightarrow \limsup_{x \to x_0} \frac{|f(\psi_i(x, y_0))|}{|x - x_0|^{k+\alpha}} < +\infty$$

and

$$\psi_i(x_0, y_0) \in (A_i^{**}) \Rightarrow \lim_{x \to x_0} \frac{||df(\psi_i(x, y_0))|_{\mathbb{R}^{n-p} \times \{0\}}||}{|x - x_0|^{k-1+\alpha}} = 0 \Rightarrow \lim_{x \to x_0} \frac{f(\psi_i(x, y_0))}{|x - x_0|^{k+\alpha}} = 0,$$

both by the mean value theorem, and the proof is finished by induction. □

Corollary 2.2. *Let $k \geq 1$, $\alpha \in [0, 1]$, $n > p$ and $A \subset U \subset \mathbb{R}^n$, where U is an open set. Then there are sets $A_1, A_2 \ldots, \subset A$ such that $A = \bigcup_{i=1}^{\infty} A_i$, where for each $i = 1, 2, \ldots$ there is a function $\psi_i\colon B_i \times V_i \xrightarrow{C^1} U$ where B_i is a ball in some $\mathbb{R}^{r_i}$, $r_i \geq 0$ and V_i is a ball in $\mathbb{R}^p$ such that $\psi_i(x, y) = (\widetilde{\psi}_i(x, y), y)$, and $|\psi_i(x_1, y_1) - \psi_i(x_2, y_2)| \geq |(x_1, y_1) - (x_2, y_2)|$, $\forall\, (x_1, y_1), (x_2, y_2) \in B_i \times V_i$ and $A_i \subset \psi_i(B_i \times V_i)$, with the following property: We can write $A_i = A_i' \cup A_i''$ so that $\psi_i^{-1}(A_i'')$ has measure zero in $B_i \times V_i$, and if $f\colon U \to \mathbb{R}$ is $C^{k+(\alpha)}$ at A and $D_x f \equiv 0$ in A we have:*

- $\displaystyle\limsup_{(x, y_0) \to (x_0, y_0)} \frac{|f(\psi_i(x, y_0)) - f(\psi_i(x_0, y_0))|}{|x - x_0|^{k+\alpha}} < +\infty$, *$\forall\, (x_0, y_0) \in B_i \times V_i$ such that $\psi_i(x_0, y_0) \in A_i$,*
- $\displaystyle\lim_{(x, y_0) \to (x_0, y_0)} \frac{|f(\psi_i(x, y_0)) - f(\psi_i(x_0, y_0))|}{|x - x_0|^{k+\alpha}} = 0$, *$\forall\, (x_0, y_0) \in B_i \times V_i$ such that $\psi_i(x_0, y_0)) \in A_i'$.*

Proof. If $k \geq 2$ this is an immediate consequence of Theorem 2.1 applied to $D_x f$ and of the mean value theorem. If $k = 1$ this can be proved exactly as the case $k = 1$ of the Theorem 2.1. □

Corollary 2.3. *In the statements of Theorem 2.1 and Corollary 2.2, for any $x \in B_i$ s.t. $\psi_i(x) \in A_i$ there are $\varepsilon_x > 0$, $K_x > 0$ such that $|y - x| < \varepsilon_x \Rightarrow |f(\psi_i(y)) - f(\psi_i(x))| \leq K_x |y - x|^{k+\alpha}$, and for any $\varepsilon > 0$ there is a $\delta > 0$ so that $\frac{\lambda(\psi_i^{-1}(A_i) \cap B_r(x))}{\lambda(B_r(x))} > 1 - \delta \Rightarrow |f(\psi_i(y)) - f(\psi_i(x))| \leq \varepsilon K_x r^{k+\alpha}$, if $r \leq \varepsilon_x$ and $|y - x| \leq r$ (δ depends only on ε and n, but not on f or on x).*

Proof. This is only a more precise formulation of the results proved in the demonstration of the theorem. □

REMARK 2.1. For $k = 0$ we have the same results, except the statement $\lim\limits_{y\to x} \frac{f(\psi_i(y))}{|y-x|^{k+\alpha}} = 0$, for each $x \in B_i$ such that $\psi_i(x) \in A_i'$.

3 The main results

LEMMA 3.1. *Let $A \subset \mathbb{R}^m$ with $\lambda(A) < \infty$ and let $\mathcal{U}$ be a family of balls $B_r(x)$, $x \in A$ such that for each $x \in A$ there is an $\varepsilon_x > 0$ such that $r \leq \varepsilon_x \Rightarrow B_r(x) \in \mathcal{U}$. Then for each $\varepsilon > 0$ there are $x_n \in A$, $r_n > 0$ with $B_{r_n}(x_n) \in \mathcal{U}$ and $A \subset \bigcup_{n=1}^{\infty} B_{r_n}(x_n)$ such that $\sum_{n=1}^{\infty} \lambda(B_{r_n}(x_n)) < \lambda(A) + \varepsilon$.*

PROOF. This lemma is essentially the Vitali covering theorem from measure theory. Take $U \supset A$ an open set with $\lambda(U) < \lambda(A) + \frac{\varepsilon}{2}$. If we choosed $B_{\tilde{r}_1}(x_1), \ldots, B_{\tilde{r}_n}(x_n)$, define $s_n = \sup\{r > 0 \mid \exists x \in A \text{ s.t. } r < \frac{\varepsilon_x}{5}, B_r(x) \subset U \text{ and } B_r(x) \cap (B_{\tilde{r}_1}(x_1) \cup \cdots \cup B_{\tilde{r}_n}(x_n)) = \emptyset\}$. Choose $B_{\tilde{r}_{n+1}}(x_{n+1})$ such that $\tilde{r}_{n+1} > \frac{s_n}{2}$, $\tilde{r}_{n+1} < \frac{\varepsilon_{x_{n+1}}}{5}$, $B_{\tilde{r}_{n+1}}(x_{n+1}) \subset U$ and $B_{\tilde{r}_{n+1}}(x_{n+1}) \cap (B_{\tilde{r}_1}(x_1) \cup \cdots \cup B_{\tilde{r}_n}(x_n)) = \emptyset$. Since the $B_{\tilde{r}_i}(x_i)$ are disjoint and contained in U we have $\sum_{i=1}^{\infty} \lambda(B_{\tilde{r}_i}(x_i)) < \lambda(A) + \frac{\varepsilon}{2}$, and so there is a $n_o \in \mathbb{N}$ such that $\sum_{i=n_o}^{\infty} \lambda(B_{5\tilde{r}_i}(x_i)) < \frac{\varepsilon}{2}$. We take $B_{r_i}(x_i) = B_{\tilde{r}_i}(x_i)$, $i < n_o$ and $B_{r_i}(x_i) = B_{5\tilde{r}_i}(x_i)$, $i \geq n_o$.

Clearly we have $\sum_{i=1}^{\infty} \lambda(B_{r_i}(x_i)) < \lambda(A) + \varepsilon$. To prove that $A \subset \bigcup_{n=1}^{\infty} \overline{B_{r_n}(x_n)}$, take $x \in A$ and $r = \min\{\tilde{r}_{n_o}, \varepsilon_x/5, d(x, U^c \cup \bigcup_{i<n_o} B_{r_i}(x_i))\}$. If $r > 0$, take $n \geq n_o$ such that $s_n < r \leq s_{n-1}$ (we have $r \leq \tilde{r}_{n_o} \leq s_{n_o-1}$), and note that $s_n < r \Rightarrow B_r(x) \cap (B_{\tilde{r}_1}(x_1) \cup \cdots \cup B_{\tilde{r}_n}(x_n)) \neq \emptyset \Rightarrow \exists\, i \leq n$ such that $B_r(x) \cap B_{\tilde{r}_i}(x_i) \neq \emptyset$. We have $n \geq n_o$ since $r \leq d(x, B_{\tilde{r}_i}(x_i))$, and $\tilde{r}_i > \frac{s_{n-1}}{2} \geq \frac{r}{2}$, since $i \leq n$. Therefore, we have $x \in B_{5\tilde{r}_i}(x_i)$. If $r = 0$ then $x \in \overline{B_{r_i}(x_i)}$ for some $i < n_o$. This proves that $A \subset \bigcup_{n=1}^{\infty} \overline{B_{r_n}(x_n)}$. Taking $\tilde{r}_n = \left(\frac{\lambda(A)+\varepsilon}{\sum_{i=1}^{\infty} \lambda(B_{r_i}(x_i))}\right)^{1/2m} \cdot r_n$, we have $A \subset \bigcup_{n=1}^{\infty} B_{\tilde{r}_n}(x_n)$, with $\sum_{n=1}^{\infty} \lambda(B_{\tilde{r}_n}(x_n)) = (\lambda(A)+\varepsilon)^{1/2} (\sum_{i=1}^{\infty} \lambda(B_{r_i}(x_i)))^{1/2} < \lambda(A) + \varepsilon$. $\square$

REMARK 3.1. In the Lemma 3.1 we can replace a family of balls $B_r(x)$ by a family of cubes $C_r(x) = \prod_{i=1}^{m} [x_i - r, x_i + r]$, where $x = (x_1, \ldots, x_m)$, using the same proof.

LEMMA 3.2. *Let $F\colon U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function, $A \subset U$ and $d > 0$ such that for any $x \in A$ there are $\varepsilon_x > 0$, $K_x > 0$ such that $m_d(F(B_\varepsilon(x) \cap A)) \leq K_x.\lambda(B_\varepsilon(x))$, $\forall\, \varepsilon < \varepsilon_x$, where m_d is the Hausdorff measure of dimension d, and there is $A' \subset A$ such that $\lambda(A \backslash A') = 0$ and $\lim\limits_{\varepsilon\to 0} \frac{m_d(F(B_\varepsilon(x)\cap A))}{\lambda(B_\varepsilon(x))} = 0$, $\forall\, x \in A'$. Then $m_d(F(A)) = 0$.*

REMARK 3.2. The same result is true if we replace $B_\varepsilon(x)$ by $C_\varepsilon(x)$.

REMARK 3.3. We can replace the condition "$m_d(F(B_\varepsilon(x) \cap A)) \leq K_x\,\lambda(B_\varepsilon(x)),\ \forall\,\varepsilon < \varepsilon_x$" by "$F(B_\varepsilon(x) \cap A)$ can be covered by balls $B_{\delta_i}(y_i)$, $i \in \mathbb{N}$, with $\sum_{i=1}^\infty \delta_i^d \leq K_x\,\lambda(B_\varepsilon(x)),\ \forall\,\varepsilon < \varepsilon_x$", and the condition "$\lim\limits_{\varepsilon\to 0} \frac{m_d(F(B_\varepsilon(x)\cap A)}{\lambda(B_\varepsilon(x))} = 0,\ \forall x \in A'$" by "$F(B_\varepsilon(x) \cap A)$ can be covered by balls $B_{\delta_i^{(\varepsilon)}}(y_i)$, $i \in \mathbb{N}$ with $\lim\limits_{\varepsilon\to 0} \frac{\sum_{i=1}^\infty (\delta_i^{(\varepsilon)})^d}{\lambda(B_\varepsilon(x))} = 0,\ \forall\,x \in A'$". The proof remains essentially the same, and Remark 3.2 is still valid.

REMARK 3.4. If we replace the conditions of this lemma by "$F(B_\varepsilon(x) \cap A)$ can be covered by balls $B_{\delta_i}(y_i)$, $i \in \mathbb{N}$, with $\sum_{i=1}^\infty \delta_i^d \leq k\,\lambda(B_\varepsilon(x)),\ \forall\,\varepsilon < \varepsilon_x$ (note that here k does not depend on x), and $\lambda(A) < \infty$", then we can conclude, using the same proof, that $m_d(F(A)) \leq k\,\lambda(A)$.

PROOF. We may suppose that A has finite Lebesgue measure, since A is a countable union of sets with finite measure, and a countable union of sets with Hausdorff d-measure zero has Hausdorff d-measure zero. Moreover, since $A = \bigcup_{k=1}^\infty A_k$, where $A_k = \{x \in A \mid K_x \leq k\}$, we may suppose $K_x \leq K,\ \forall\,x \in A$. Let C be the Lebesgue measure of A.

Let $\varepsilon > 0$. For each $x \in A'$ take $\delta_x > 0$ such that $B_{\delta_x}(x) \subset U$ and $r \leq \delta_x \Rightarrow \frac{m_d(F(B_r(x)\cap A))}{\lambda(B_r(x))} \leq \frac{\varepsilon}{2(C+1)}$. By the Lemma 3.1 we can cover A' by $\bigcup_{n=1}^\infty B_{r_n}(x_n)$ with $\sum_{n=1}^\infty \lambda(B_{r_n}(x_n)) < C+1$ and $r_n \leq \delta_{x_n} \Rightarrow \sum_{n=1}^\infty m_d(F(B_{r_n}(x_n) \cap A)) \leq \frac{\varepsilon}{2(C+1)} \cdot (C+1) = \frac{\varepsilon}{2} \Rightarrow m_d(F(A')) \leq \frac{\varepsilon}{2}$. By Lemma 3.1 we can cover $A \backslash A'$ by $\bigcup_{n=1}^\infty B_{\tilde{r}_n}(\widetilde{x}_n)$ such that $B_{\tilde{r}_n}(\widetilde{x}_n) \subset U$ and $\tilde{r}_n < \varepsilon_{\widetilde{x}_n},\ \forall n \in \mathbb{N}$, with $\sum_{n=1}^\infty \lambda(B_{\tilde{r}_n}(\widetilde{x}_n)) < \frac{\varepsilon}{2K} \Rightarrow \sum_{n=1}^\infty m_d(F(B_{\tilde{r}_n}(\widetilde{x}_n))) \leq \frac{\varepsilon}{2K} \cdot K = \frac{\varepsilon}{2} \Rightarrow m_d(F(A \backslash A')) \leq \frac{\varepsilon}{2} \Rightarrow m_d(F(A)) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Since $\varepsilon > 0$ is arbitrary we have $m_d(F(A)) = 0$. $\square$

We first use Lemma 3.2 to prove the following strong version of Constantin's result ([Co]), that does not suppose continuity of the derivatives. Here we do not suppose differentiability in every point, but only in the set of critical points under consideration.

THEOREM 3.3. *Let $F\colon X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function, and let $A = \{x \in X \mid DF(x)$ exists and is not surjective$\}$. Then $\lambda(F(A)) = 0$.*

PROOF. It is a simple consequence of Lemma 3.2, since if $x \in A$ then $\lim\limits_{r\to 0} \frac{\lambda(F(B_r(x)))}{\lambda(B_r(x))} = 0$. Indeed, $x \in A \Rightarrow F(x+h) = F(x) + DF(x).h + r(h)$, where $\lim\limits_{h\to 0} \frac{r(h)}{|h|} = 0$. Let $K = ||DF(x)||$, and let $\varepsilon \in (0,1)$. Let $\delta > 0$ such that $|h| \leq \delta \Rightarrow \frac{|r(h)|}{|h|} < \frac{\varepsilon}{2(K+1)^{n-1}}$. Then, if $|h| \leq \delta$, $F(x+h) - F(x)$ belongs to an $\frac{\varepsilon.|h|}{2(K+1)^{n-1}}$ neighbourhood of a ball of radius $K|h|$ in a subspace of $\mathbb{R}^n$ of dimension $n-1$ (a fixed subspace of $\mathbb{R}^n$ of dimension $n-1$ which contains the image of DF), and thus belongs to the orthogonal product of a ball of radius $(K+1)|h|$ in this subspace by an interval of radius $\frac{\varepsilon|h|}{2(K+1)^{n-1}}$. Therefore, $\lambda(F(B_r(x)) \leq \frac{\varepsilon.r.r^{n-1}(K+1)^{n-1}}{(K+1)^{n-1}} v_{n-1} = \varepsilon r^n v_{n-1}$, where v_{n-1} is the volume of the unitary ball in $\mathbb{R}^{n-1}$, and, since $\varepsilon > 0$ is arbitrary, $\lim\limits_{r\to 0} \frac{\lambda(F(B_r(x)))}{\lambda(B_r(x))} = 0$. $\square$

THEOREM 3.4. *Let $F\colon U \subset \mathbb{R}^n \xrightarrow{C^k} \mathbb{R}^m$ be a function of class $C^{k+(\alpha)}(\alpha \in (0,1])$ at $C_p(F) := \{x \in U \mid \text{ rank } (DF(x)) \leq p\}$. Then the Hausdorff measure of dimension $d = p + \frac{n-p}{k+\alpha}$ of $F(C_p(F))$ is zero, $\forall\, p < min\{m,n\}$.*

PROOF. Since $C_p(F) = \bigcup_{r=0}^{p} \{x \in U \mid \text{ rank } (DF(x)) = r\}$, and $r + \frac{n-r}{k+\alpha} \leq p + \frac{n-p}{k+\alpha}$ for $0 \leq r \leq p$, we may restrict our attention to $\widetilde{C}_p(F) = \{x \in U \mid \text{ rank } (DF(x)) = p\}$. If $x_o \in C_p(F)$, we have, after a change of coordinates of class C^k, $F(z,y) = (z, G(z,y))$, with $(z,y) \in \mathbb{R}^p \times \mathbb{R}^{n-p}$ and $G(z,y) \in \mathbb{R}^{m-p}$, in a neighbourhood V of $x_o = (z_o, y_o)$. We shall restrict our attention to this neighbourhood. We have $x = (z,y) \in C_p(F) \Leftrightarrow D_y\, G(z,y) = 0$. We can apply the results of the section 2 (Theorem 2.1, Corollary 2.3 and Remark 2.1) to the function $D_y\, G$, and obtain the decomposition $A = \bigcup_{i=1}^{\infty} A_i$, $A_i \subset \psi_i(V_i \times B_i)$, where $A = \{(z,y) \in V \mid D_y G(z,y) = 0\}$. Let us fix such an A_i.

Since $\psi_i^{-1}(A_i) = \bigcup_{m\in\mathbb{N}} \{x \in \psi_i^{-1}(A_i) \mid \varepsilon_x \geq \frac{1}{m}, K_x \leq m\}$, we may suppose $\varepsilon_x \geq \frac{1}{M}$, $K_x \leq M$, $\forall\, x \in \psi_i^{-1}(A_i)$, for some fixed M and also that V has finite Lebesgue measure $\lambda(V)$.

With these assumptions, we shall prove that there is a constant K_o such that for any $X \subset V$, $\nu > 0$, we can cover $F(A_i \cap X)$ by balls $B_{\delta_i}(p_i)$ so that $\sum_{i=1}^{\infty} \delta_i^d \leq K_o(\lambda(X) + \nu)$. For this, given a point $x \in A_i \cap X$ and an $\varepsilon < \frac{1}{2\sqrt{n}\,M}$, we can divide the cube $C_\varepsilon(x) = C_\varepsilon(z) \times C_\varepsilon(y)$ into $([\varepsilon^{1-(k+\alpha)}]+1)^p$ boxes $C_\delta(\widetilde{z}_i) \times C_\varepsilon(y)$, $\delta < \varepsilon^{k+\alpha}$. If there is some point (z_i, y_i) in $(C_\delta(\widetilde{z}_i) \times C_\varepsilon(y)) \cap (A_i \cap X)$, then for any point (z_i', y_i') in $(C_\delta(\widetilde{z}_1) \times C_\varepsilon(y)) \cap (A_i \cap X)$, we have $|F(z_i', y') - F(z_i, y_i)| \leq |F(z_i', y') - F(z_i, y')| + |F(z_i, y') - F(z_i, y_i)| \leq K'\delta + |F(z_i, y_i') - F(z_i, y_i)|$ (where K' is $\sqrt{p}$ times a Lipschitz constant of $F\big|_V$ which we may suppose to exist) $\leq K'\delta + |G(z_i, y_i') - G(z_i, y_i)|$.

Observe now that $(z_i, y_i) = (z_i, \widetilde{\psi}_i(p_1))$ and $(z_i', y_i') = (z_i, \widetilde{\psi}_i(p_2))$, for some p_1, p_2 in $\{z_i\} \times B_i$ with $|p_1 - p_2| \leq |y_i - y_i'| \leq 2\varepsilon\sqrt{n}$. Let $\gamma\colon [0,1] \to V_i \times B_i$ be a straight path joining p_1 and p_2. Then $G(z_i, y_i') - G(z_i, y_i) = \int_0^1 \frac{\partial G}{\partial y}(\widetilde{\gamma}(t)) \cdot \widetilde{\gamma}'(t)\, dt$, where $\widetilde{\gamma} := \psi_i \circ \gamma$. We have $\frac{\partial G}{\partial y}(\widetilde{\gamma}(0)) = 0$, so

$$\begin{aligned}
\left\| \frac{\partial G}{\partial y}(\widetilde{\gamma}(t)) \right\| &= \left\| \frac{\partial G}{\partial y}(\widetilde{\gamma}(t)) - \frac{\partial G}{\partial y}(\widetilde{\gamma}(0)) \right\| \\
&\leq M|p_1 - p_2|^{k+\alpha-1} \leq M(2\varepsilon\sqrt{n})^{k+\alpha-1} \Rightarrow \left\| \frac{\partial G}{\partial y}(\widetilde{\gamma}(t)) \right\| |\gamma'(t)| \\
&\leq K''\varepsilon^{k+\alpha},
\end{aligned}$$

for some constant K''. Indeed, $|\widetilde{\gamma}'(t)|$ is limited by a constant multiple of $|p_1 - p_2| \leq 2\sqrt{n}\varepsilon$. So

$$|G(z_1, y_i') - G(z_i, y_i)| \leq \int_0^1 |\frac{\partial G}{\partial y}(\widetilde{\gamma}(t)) \circ \widetilde{\gamma}(t)|\; dt \leq K'' \varepsilon^{k+\alpha}$$

and

$$|F(z_i', y') - F(z_i, y_i)| \leq K'\delta + K''\varepsilon^{k+\alpha} \leq K_0' \cdot \varepsilon^{k+\alpha},$$

where $K_0' = K' + K''$.

Therefore, $F(C_\delta(\widetilde{z}_i) \times C_\varepsilon(y))$ is contained in some ball $B_{\delta_i}(q_i)$, with $\delta_i \leq K_0' \cdot \varepsilon^{k+\alpha}$, so

$$\begin{aligned}\sum_i \delta_i^d &\leq ([\varepsilon^{1-(k+\alpha)}] + 1)^p (K_o' \varepsilon^{k+\alpha})^d \\ &= (K_o')^d ([\varepsilon^{1-(k+\alpha)}] + 1)^p (\varepsilon^{k+\alpha})^{p+\frac{n-p}{k+\alpha}} \\ &\leq \widetilde{K}_o \varepsilon^n\end{aligned}$$

for some constant $\widetilde{K}_o$. So, $F(C_\varepsilon(x))$ can be covered by balls $B_{\delta_i}(q_i)$ with $\sum_i \delta_i^d \leq K_o.\lambda(C_\varepsilon(x))$, and by the Lemma 3.2, remarks 3.2 and 3.4, we can conclude that $m_d(F(A_i \cap X)) \leq K_o \lambda(X)$ where $K_0 = 2^n \widetilde{K}_0$, and so we can cover $F(A_i \cap X)$ by balls $B_{\delta_i}(p_i)$ so that $\sum_{i=1}^\infty \delta_i^d \leq K_o(\lambda(X) + \nu)$, as we stated.

We shall prove now that there is an $A_i' \subset A_i \subset V$ with $\lambda(A_i \backslash A_i') = 0$ such that $F(C_\varepsilon(x) \cap A_i)$ can be covered by balls $B_{\delta_i^{(\varepsilon)}}(W_i)$, $i \in \mathbb{N}$ with $\lim\limits_{\varepsilon \to 0} \frac{\sum_i (\delta_i^{(\varepsilon)})^d}{\lambda(C_\varepsilon(x))} = 0$, $\forall x \in A_i'$. This will imply our theorem, by the Lemma 3.2, remarks 3.2 and 3.3, since we have proved above that $m_d(F(C_\varepsilon(x) \cap A_i) \leq K_o \lambda(C_\varepsilon(x))$, $\forall \varepsilon < \frac{1}{2\sqrt{n}\, M}$. For this, since $A_i \subset \psi_i(V_i \times B_i)$, $B_i \subset \mathbb{R}^{r_i}$, $r_i \leq n-p$, we may suppose $r_i = n-p$ and $\psi_i =$ identity, because $r_i < n-p \Rightarrow \lambda(A_i) = 0$ and we can take $A_i' = A_i$. Let us take A_i' equal to the set of the density points of A_i. Given a point $x \in A_i'$, and an $\eta' > 0$, we want to find an $\varepsilon_o > 0$ such that $\varepsilon < \varepsilon_o \Rightarrow F(C_\varepsilon(x) \cap A_i)$ can be covered by balls $B_{\delta_i^{(\varepsilon)}}(W_i)$, $i \in \mathbb{N}$ such that $\sum_i (\delta_i^{(\varepsilon)})^d \leq \eta' \lambda(C_\varepsilon(x))$. Let $\eta, \tilde{\eta} > 0$, $\tilde{\varepsilon} < \frac{1}{2\sqrt{n}\, M}$ such that $\frac{\lambda(C_\varepsilon(x) \cap A_i)}{\lambda(C_\varepsilon(x))} > 1 - \tilde{\eta}^2$, $\forall \varepsilon \leq \tilde{\varepsilon}$. Divide the cube $C_\varepsilon(x) = C_\varepsilon(\widetilde{z}) \times C_\varepsilon(\widetilde{y})$, $\varepsilon < \tilde{\varepsilon}$ into $N = ([\varepsilon^{1-(K+\alpha)}\eta^{-1}] + 1)^p$ boxes $C_\delta(\widetilde{z}_i) \times C_\varepsilon(\widetilde{y})$, $\delta < \eta\, \varepsilon^{k+\alpha}$, $1 \leq i \leq N$. Then for at least $(1 - \tilde{\eta})N$ values of i, there is a $z_i \in C_\delta(\widetilde{z}_i)$ such that $\lambda(\{y \in C_\varepsilon(\widetilde{y}) \mid (z_i, y) \in A_i\} / \lambda(C_\varepsilon(\widetilde{y})) > 1 - \tilde{\eta}$ (here λ is the Lebesgue measure in $\mathbb{R}^{n-p}$), because $\lambda(C_\varepsilon(x) \cap A_i) > (1 - \tilde{\eta}^2)\lambda(C_\varepsilon(x))$. For such an i, take an y_i such that $(z_i, y_i) \in A_i$. Then, applying Theorem 2.1, Corollaries 2.2 and 2.3, given η we can choose $\tilde{\eta}$ so that $|F(z_i, y) - F(z_i, y_i)| < \eta \cdot \varepsilon^{k+\alpha} \Rightarrow |F(z, y) - F(z_i, y_i)| \leq 2K'\sqrt{n}.\eta\varepsilon^{K+\alpha} + \eta\varepsilon^{K+\alpha} \leq K''\eta\varepsilon^{k+\alpha}$, for some constant K'' and for any $z \in C_\varepsilon(\widetilde{z}_i)$, where K' is a Lipschitz constant for $F \Rightarrow F(C_\delta(\widetilde{z}_i) \times C_\varepsilon(\widetilde{y}))$ is contained in a ball $B_{\delta_i}(Q_i)$, with $\sum_i \delta_i^d$ over these values of i less than

$$([\varepsilon^{1-(K+\alpha)}\eta^{-1}]+1)^p(K''\eta\varepsilon^{k+\alpha})^d = (K'')^d([\varepsilon^{1-(k+\alpha)}.\eta^{-1}]+1)^p(\eta\varepsilon^{k+\alpha})^{p+\frac{n-p}{k+\alpha}} \leq \widetilde{K}_o\eta^{\frac{n-p}{k+\alpha}}.\varepsilon^n$$

for some constant $\widetilde{K}_o$. The union of the remaining (at most $\tilde{\eta}N$) boxes has volume at most $\tilde{\eta}\varepsilon^n \Rightarrow$ the union of the image of the intersection of A_i with the union of these boxes by F is contained in a union of balls $B_{\delta_i}(Q_i)$ with $\sum_i \delta_i^d \leq 2K_o\tilde{\eta}\varepsilon^n$, by the statement proved before, and so $F(C_\varepsilon(x))$ can be covered by balls $B_{\delta_i}(\widetilde{Q}_i)$ with $\Sigma\delta_i^d \leq (\widetilde{K}_o\eta^{\frac{n-p}{k+\alpha}} + 2K_o\tilde{\eta})\varepsilon^n$. Choosing η, $\tilde{\eta}$ so small that $\widetilde{K}_o\eta^{\frac{n-p}{k+\alpha}} + 2K_o\tilde{\eta} \leq \eta'$, we obtain the desired result with $\varepsilon_o = \tilde{\varepsilon}$. □

Remark 3.5. In the cases of functions of class C^k ($C^{k+\alpha}$ with $\alpha = 0$) we have the same result. If $k \geq 2$, it follows from the case of class $C^{k-1+(1)}$ of the theorem. If $k = 1$, $p + \frac{n-p}{k+\alpha} = n$, and the proof of the Theorem 3.3 shows that if $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is a function and $C(F) = \{x \in U \mid DF(x) \text{ exists and rank } DF(x) < n\}$ then $m_n(F(C(F))) = 0$, where m_n is the Hausdorff measure of dimension n.

4 Examples

In this section we give some examples which show that the previous results are quite sharp. In all these examples we shall use a certain kind of functions of the real line that we shall describe below.

Definition 4.1. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of real numbers with $0 < \lambda_i < \frac{1}{2}$, $\forall i \in \mathbb{N}$. The central Cantor set $K_{\underline{\lambda}}$ is the Cantor set constructed as follows: We remove from the interval [0,1] the central open interval $U_{1,1}$ of proportion $1 - 2\lambda_1$, then we remove from the two remaining intervals the central open intervals $U_{2,1}$ and $U_{2,2}$ of proportion $1 - 2\lambda_2$, and so on. After the r-th step of the construction there will remain 2^r intervals of lenght $\lambda_1\lambda_2\ldots\lambda_r$. The intersection of all these sets is the central Cantor set $K_{\underline{\lambda}}$. The open intervals removed in the r-th step of the construction have lenght $\lambda_1\lambda_2\cdots\lambda_{r-1}(1-2\lambda_r)$.

Let $\psi\colon \mathbb{R} \xrightarrow{C^\infty} \mathbb{R}$ be a fixed function such that $\psi(\mathbb{R}) \subset [0,1]$, $\psi(x) = 0$, $\forall x \leq 0$, $\psi(x) = 1$, $\forall x \geq 1$. Given two central Cantor sets $K_{\underline{\lambda}}$ and $K_{\underline{\mu}}$, we construct the function $f_{\underline{\lambda},\underline{\mu}}\colon \mathbb{R} \to \mathbb{R}$ as follows: $f_{\underline{\lambda},\underline{\mu}}(x) = 0$, $\forall x \leq 0$, $f_{\underline{\lambda},\underline{\mu}}(x) = 1$, $\forall x \geq 1$, and if $U_{i,j} = (a,b)$ and $v_{i,j} = (c,d)$ are corresponding removed intervals in the constructions of $K_{\underline{\lambda}}$ and $K_{\underline{\mu}}$, respectively, we define $f_{\underline{\lambda},\underline{\mu}}(x) =$

$c+(d-c)\psi\left(\frac{x-a}{b-a}\right)$, $\forall\, x\in(a,b)$. We extend $f_{\underline{\lambda},\underline{\mu}}$ to $K_{\underline{\lambda}}$ by continuity, obtaining $f_{\underline{\lambda},\underline{\mu}}(K_{\underline{\lambda}})=K_{\underline{\mu}}$. It is easy to check that if $g_r:=\lambda_1\lambda_2\dots\lambda_{r-1}(1-2\lambda_r)$ and $\tilde g_r:=\mu_1\mu_2\dots\mu_{r-1}(1-2\mu_r)$ satisfy $\lim\limits_{r\to\infty}\frac{\tilde g_r}{g_r^k}=0$ then $f_{\underline{\lambda},\underline{\mu}}$ is C^k (if $k\geq 1$ is an integer). Moreover, if $q>1$, and $\underset{r}{\rightarrow}\sup\frac{\tilde g_r}{g_r^q}<\infty$ then $f_{\underline{\lambda},\underline{\mu}}$ is $C^{q-1,1}$ if q is integer and is C^q (i.e., it is $C^{[q]+\{q\}}$, where $\{q\}=q-[q]\in(0,1)$) otherwise. See [BMPV] for more details.

Example 4.1. Let $\lambda_n=\frac{1}{2}-\frac{1}{2n}$, $\mu_n\equiv a$. Then $\lim\limits_{n\to\infty}\frac{\tilde g_n}{g_n^q}=0$, $\forall\, q<\frac{-\log a}{\log 2}$, and so $f_{\underline{\lambda},\underline{\mu}}$ is C^q, $\forall\, q<\frac{-\log a}{\log 2}$. On the other hand, $m_d(K_{\underline{\mu}})=1$ where $d=\frac{-\log 2}{\log a}$ (see [PT]). Moreover, since $a\in\left(0,\frac{1}{2}\right)$, $\lim\limits_{n\to\infty}\frac{\tilde g_n}{g_n}=0$, and so $f'_{\underline{\lambda},\underline{\mu}}(x)=0$, $\forall\, x\in K_{\underline{\lambda}}$. If $F\colon\mathbb{R}^{n+p}\to\mathbb{R}^{n+p}$ is given by $F(x_1,x_2,\dots,x_n,x_{n+1},\dots,x_{n+p})=(f_{\underline{\lambda},\underline{\mu}}(x_1),f_{\underline{\lambda},\underline{\mu}}(x_2)\dots,f_{\underline{\lambda},\underline{\mu}}(x_n),x_{n+1}\dots,x_{n+p})$, then $F(C_p(F))=F(K_{\underline{\lambda}}\times K_{\underline{\lambda}}\times\cdots\times K_{\underline{\lambda}}\times\mathbb{R}^p)=K_{\underline{\mu}}\times K_{\underline{\mu}}\times\cdots\times K_{\underline{\mu}}\times\mathbb{R}^p$ that is a set with positive $(nd+p)$-measure. This shows that given $q>1$, $p>0$ and $n>p$ there is a map $F\colon\mathbb{R}^n\to\mathbb{R}^n$ such that $m_d(F(C_p(F)))>0$, where $d=p+\frac{n-p}{q}$, and F is of class $C^{q'}$ for each $q'<q$.

Remark 4.1. If $a=\frac{1}{2^n}$, $F(x_1,x_2,\dots,x_n)=f_{\underline{\lambda},\underline{\mu}}(x_1)+2f_{\underline{\lambda},\underline{\mu}}(x_2)+\cdots+2^{n-1}f_{\underline{\lambda},\underline{\mu}}(x_n)$ gives an example of a function $F\colon\mathbb{R}^n\to\mathbb{R}$ which is of class C^q, $\forall\, q<n$ $(q\in\mathbb{R})$ such that $F(\mathbb{R}^n)$ contains an open set, since $K_{\underline{\mu}}+2K_{\underline{\mu}}+\cdots+2^{n-1}K_{\underline{\mu}}=[0,2^n-1]$, which can be proved easily using representation in basis 2^n.

Example 4.2. Let $\lambda_n=\frac{1}{2}-\frac{1}{3n^2}$, $\mu_n=a-\frac{a}{3n}$, $a\in(0,1/2]$. Then $\lim\limits_{n\to\infty}\frac{\tilde g_n}{g_n^q}=0$, where $q=\frac{-\log a}{\log 2}$, and so $f_{\underline{\lambda},\underline{\mu}}$ is C^q. On the other hand we have $HD(K_{\underline{\mu}})\geq\frac{-\log 2}{\log a}$. Indeed, if $b<a$ and $\theta_n\equiv b$, $f_{\underline{\mu},\underline{\theta}}$ is clearly C^1, and $f_{\underline{\mu},\underline{\theta}}(K_{\underline{\mu}})=K_{\underline{\theta}}\Rightarrow HD(K_{\underline{\mu}})\geq HD(K_{\underline{\theta}})=\frac{-\log 2}{\log b}$, $\forall\, b<a$. If $F\colon\mathbb{R}^{n+p}\to\mathbb{R}^{n+p}$ is given by $F(x_1,x_2,\dots,x_n,x_{n+1},\dots,x_{n+p})=(f_{\underline{\lambda},\underline{\mu}}(x_1),\dots,f_{\underline{\lambda},\underline{\mu}}(x_n),x_{n+1},\dots,x_{n+p})$ then $F(C_p(F))=K_{\underline{\mu}}\times\cdots\times K_{\underline{\mu}}\times\mathbb{R}^p$, that is a set with Hausdorff dimension $nd+p$, where $d=\frac{-\log 2}{\log a}$. This shows that given $q\geq 1$, $p>0$ and $n>p$ there is a map $F\colon\mathbb{R}^n\to\mathbb{R}^n$ such that $HD(F(C_p(F)))=p+\frac{n-p}{q}$, and F is of class C^q.

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