QUASISYMMETRIC ROBUSTNESS OF THE COLLET-ECKMANN CONDITION IN THE QUADRATIC FAMILY

ARTUR AVILA AND CARLOS GUSTAVO MOREIRA

ABSTRACT. We consider quasisymmetric reparametrizations of the parameter space of the quadratic family. We prove that the set of quadratic maps which are either regular or Collet-Eckmann with polynomial recurrence of the critical orbit has full Lebesgue measure.

CONTENTS

1. INTRODUCTION

The intent of this work is just to be a rigorous reference for [AM2]. *It is based on the LaTeX file of* [AM1]*: the proofs are very similar and in many places differ just by change of constants. In fact the proofs in* [AM1] *give easily robustness under quasisymmetric reparametrizations with small qs-constant, there are some minor differences to remove this assumption (like introduction of fast landings and bad returns).*

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Here we consider the quadratic family, $f_a = a - x^2$, where $-1/4 \leq a \leq 2$ is the parameter, and we consider its dynamics in the invariant interval.

In [AM1], a thorough understanding of the dynamics of typical quadratic maps was obtained. More specifically, it was shown that a typical quadratic map is either regular (with a periodic attractor) or Collet-Eckmann (positive Lyapunov exponent of the critical value) with polynomial recurrence of the critical orbit. The first possibility corresponds to a hyperbolic deterministic setting, with the well known good properties of hyperbolic systems. The second is a particularly well studied case of non-uniformly hyperbolic chaotic dynamics: in the 90's such maps were shown to possess many hyperbolic-like properties like stochastic stability, exponential decay of correlations and others ([KN], [Y], [BV] and [BBM]). In particular it was possible to answer affirmatively Palis Conjecture [Pa] for the quadratic family.

It was shown in [ALM] that the parameter space of general analytic families of unimodal maps (with negative Schwarzian derivative) can be related to the parameter space of quadratic maps through a quasisymmetric 'holonomy map'. It becomes then feasible to transfer results from the quadratic family to other families, but there is one obstruction: quasisymmetric maps are not absolutely continuous.

Here we show that the set of 'good' parameters has not only full Lebesgue measure, but is resistent to a quasisymmetric reparametrization:

Theorem A. *Consider a quasisymmetric reparametrization of the parameter space of the quadratic family. The set of parameters which are either regular or Collet-Eckmann:*

$$\liminf_{n\to\infty} \frac{\ln(|Df^n(f(0))|)}{n} > 0.$$

has full Lebesgue measure.

Theorem B. *Consider a quasisymmetric reparametrization of the parameter space of the quadratic family. The set of parameters which are either regular or have polynomial recurrence of the critical orbit*

$$0 < \limsup_{n\to\infty} \frac{-\ln(f^n(0))}{\ln(n)} < \infty.$$

has full Lebesgue measure.

In [AM2] this result is used to obtain a proof of Palis Conjecture for unimodal maps with negative Schwarzian derivative. Another approach to this result (with stronger estimates) allows to obtain more general results, the proof is however more elaborate [A].

2. General definitions

2.1. **Maps of the interval.** Let $f : I \to I$ be a C^1 map defined on some interval $I \subset \mathbb{R}$. The orbit of a point $p \in I$ is the sequence $\{f^k(p)\}_{k=0}^{\infty}$. We say that p is recurrent if there exists a subsequence $n_k \to \infty$ such that $\lim f^{n_k}(p) = p$.

We say that p is a periodic point of period n of f if $f^n(p) = p$, and $n \geq 1$ is minimal with this property. In this case we say that p is hyperbolic if $|Df^n(p)|$ is not 0 or 1. Hyperbolic periodic orbits are attracting or repelling according to $|Df^n(p)| < 1$ or $|Df^n(p)| > 1$.

We will often consider the distortion of an iterate f^n restricted to some interval $J \subset I$, such that $f^n|_J$ is a diffeomorphism. In this case we will be intrested on the distortion of $f^n|_J$,

$$\operatorname{dist}(f^n|_J) = \frac{\sup_J Df^n}{\inf_J Df^n}.$$

This is always a number bigger than or equal to 1, we will say that it is small if it is close to 1.

2.2. **Trees.** We let Ω denote the set of finite sequences of non zero integers (including the empty sequence). Let Ω_0 denote Ω without the empty sequence.

We denote $\sigma^+ : \Omega_0 \to \Omega$ by $\sigma^+(j_1, ..., j_m) = (j_1, ..., j_{m-1})$ and $\sigma^- : \Omega_0 \to \Omega$ by $\sigma^-(j_1, ..., j_m) = (j_2, ..., j_m)$.

2.3. **Growth of functions.** Let X be a class of functions $g : \mathbb{N} \to \mathbb{R}$ such that $\lim_{n\to\infty} g(n) = \infty$. We say that a function $f : \mathbb{N} \to \mathbb{R}$ grows at least rate X if there exists a function $g \in X$ such that $f(n) \geq g(n)$ for n sufficiently big. We say that it grows at rate X if there are $g_1, g_2 \in X$ such that $g_1(n) \leq f(n) \leq g_2(n)$ for n sufficiently big. We say that f decreases with rate (at least) X if $1/f$ grows at rate (at least) X.

Standard classes are the following. Linear for linear functions with positive slope. Polynomial for functions $g(n) = n^k, k > 0$. Exponential for functions $g(n) = e^{kn}, k > 0$.

The standard torrential function T is defined recursively by $T(1) = 1$, $T(n+1) = 2^{T(n)}$. The torrential class is the set of functions $g(n) = T(\max\{n+k, 1\}), k \in \mathbb{Z}$.

Torrential growth can be detected from recurrent estimates easily. A sufficient condition for a function which is unbounded from above to grow at least torrentially is an estimate as

$$f(n+1) > e^{f(n)^a}$$

for some $a > 0$. Torrential growth is implied by an estimate as

$$e^{f(n)^a} < f(n+1) < e^{f(n)^b}$$

with $0 < a < b$.

2.4. **Quasisymmetric maps.** Let $\gamma \geq 1$ be given. We say that a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is γ-quasisymmetric (γ-qs) if it has a quasiconformal symmetric extension to $\mathbb{C}$ with dilatation bounded by γ. Notice that quasisymmetric maps form a group under composition and if h_1 is γ_1-qs and h_2 is γ_2-qs then $h_2 \circ h_1$ is $\gamma_2\gamma_1$-qs.

If $X \subset \mathbb{R}$ and $h : X \to \mathbb{R}$ has a γ-quasisymmetric extension to $\mathbb{R}$ we will also say that h is γ-qs.

Let $QS(\gamma)$ be the set of γ-qs maps of $\mathbb{R}$.

3. Real quadratic maps

If $\lambda \in \mathbb{C}$ we let $f_\lambda : \mathbb{C} \to \mathbb{C}$ denote the (complex) quadratic map $\lambda - z^2$. If $\lambda \in \mathbb{R}$ is such that $-1/4 \leq \lambda \leq 2$ there exists an interval $I_\lambda = [-\beta, \beta]$ with

$$\beta = \frac{-1 - \sqrt{1+4\lambda}}{2}$$

such that $f_\lambda(I_\lambda) \subset I_\lambda$ and $f_\lambda(\partial I_\lambda) \subset \partial\lambda$. For such a λ, the map $f = f_\lambda|_{I_\lambda}$ is unimodal, that is, is a self map of I_λ with a unique turning point.

We will keep the notation f_λ to refer to a quadratic map when we discuss its complex extension and f to denote a fixed quadratic map when we discuss its unimodal restriction.

3.1. **The combinatorics of unimodal maps.** In this subsection we fix a real quadratic map f and define some objects related to it.

3.1.1. *Return maps.* Given an interval I we define the first return map $R_I : X \to I$ where $X \subset I$ is the set of points x such that there exists $n > 0$ with $f^n(x) \in I$, and $R_I(x) = f^n(x)$ for the minimal n with this property.

3.1.2. *Nice intervals.* An interval I is nice if it is symmetric and the iterates of ∂I never intersect $\operatorname{int} I$. Given a nice interval I we notice that the domain of the first return map R_I decomposes in a union of intervals I^j, indexed by integer numbers. We reserve the index 0 to denote the component of 0 if it exists: $0 \in I^0$, in this case we say I is proper.

If I is nice, it follows that for all $j \in \mathbb{Z}$, $R_I(\partial I^j) \subset \partial I$. It follows that if $j \neq 0$ then $R_I|_{I^j}$ is a diffeomorphism onto I and if I is proper, $R_I|_{I^0}$ is symmetric with a unique critical point 0. As a consequence, I^0 is also a nice interval.

If $R_I(0) \in I^0$, we say that R_I is central.

3.1.3. *Landing maps.* Given a proper interval I we define the landing map $L_I : X \to I^0$ where $X \subset I$ is the set of points x such that there exists $n \geq 0$ with $f^n(x) \in I^0$, and $L_I(x) = f^n(x)$ for the minimal n with this property. We notice that $L_I|_{I^0} = \mathrm{id}$.

3.1.4. *Trees.* If I is a proper interval, the first return map to I naturally relates to the first landing to I^0 in the following way.

If $\underline{d} \in \Omega$, we define $I^{\underline{d}}$ inductively in the following way. $I^{\underline{d}} = I$ if $\underline{d}$ is empty and if $\underline{d} = (j_1, ..., j_m)$ we let $I^{\underline{d}} = (R_I|_{I_{j_1}})^{-1}(I^{\sigma^-(\underline{d})})$.

We denote $R_I^{\underline{d}} = R_I^{|\underline{d}|}|_{I^{\underline{d}}}$ which is always a diffeomorphism over I.

We denote by $C^{\underline{d}} = (R_I^{\underline{d}})^{-1}(I^0)$. We notice that the domain of the first landing map L_I coincides with the union of the $C^{\underline{d}}$, and furthermore $L_I|_{C^{\underline{d}}} = R_I^{\underline{d}}$.

Notice that this allows us to relate R_I to R_{I^0}, since $R_{I^0} = L_I \circ R_I$.

3.1.5. *Renormalization.* We say that f is renormalizable if there is an interval I with and $m > 1$ such that $f^m(T) \subset T$ and $f^j(T) \cap I = \emptyset$ for $1 \leq j < n$. The maximal such interval is called the renormalization interval of period m, it has the property that $f^m(\partial T) \subset \partial T$.

The set of renormalization periods of f gives an increasing (possibly empty) sequence of numbers m_i, each related to a unique renormalization interval $T^{(i)}$ which form a nested sequence of intervals.

We say that f is finitely renormalizable if there is a smallest renormalization interval $T^{(k)}$. We say that $f \in \mathcal{F}$ if f is finitely renormalizable and 0 is recurrent but not periodic.

3.1.6. *Principal nest.* Let Δ_k denote the set of all maps f which have (at least) k renormalizations and which have an orientation reversing non-attracting periodic point of period m_k which we denote p_k (that is, $f^{m_k}|_{T^{(k)}}$ has a fixed point p_k and $Df^n(p_k) \leq -1$). In this case we denote by $T_1^{(k)} = [-p_k, p_k]$. We define by induction a (possibly finite) sequence $T_i^{(k)}$, such that $T_{i+1}^{(k)}$ is the component of the domain of $R_{T_i^{(k)}}$ containing 0. If this sequence is infinite, then either it converges to a point or to an interval.

In the former case, f has a recurrent critical point which is not periodic, and it is possible to show that f is not $k+1$ times renormalizable, obviously in this case we have $f \in \mathcal{F}$, and we write $\mathcal{F}_k$ for the set of such maps. If the limit set is an interval, it is possible to show that f is $k+1$ times renormalizable.

We can of course write $\mathcal{F}$ as a disjoint union $\cup_{i=0}^{\infty} \mathcal{F}_i$. For a map $f \in \mathcal{F}_k$ we refer to the sequence $\{T_i^{(k)}\}_{i=1}^{\infty}$ as the principal nest.

It is important to notice that the domain of the first return map to $T_i^{(k)}$ is always dense in $T_i^{(k)}$. Moreover, outside a very special case, the return map has a hyperbolic structure.

Lemma 3.1. *Assume $T_i^{(k)}$ does not have a non-hyperbolic periodic orbit in its boundary. For all $T_i^{(k)}$ there exists $C > 0$, $\lambda > 1$ such that if $x, f(x), ..., f^{n-1}(x)$ do not belong to $T_i^{(k)}$ then $|Df^n(x)| > C\lambda^n$.*

This theorem is a simple consequence of a general theorem of Guckenheimer on hyperbolicity of maps of the interval without critical points and non-hyperbolic orbits (Guckenheimer assumes negative Schwarzian derivative, so this applies directly to our case, the general case is also true by Mañé's Theorem, see [MvS]). Notice that the existence or a non-hyperbolic periodic orbit in the boundary of $T_i^{(k)}$ depends on a very special combinatorial setting, in particular, all $T_j^{(k)}$ must coincide (with $[-p_k, p_k]$), and the k-th renormalization of f is in fact renormalizable of period 2.

This Lemma shows that the complement of the domains of the return map to $T_i^{(k)}$ (outside the special case) form a regular Cantor set. This has many useful consequences (for instance, the image of a regular Cantor set by a quasisymmetric map has always 0 Lebesgue measure.

3.1.7. *Lyubich's Regular or Stochastic dichotomy.* A map $f \in \mathcal{F}_k$ is called simple if the principal nest has only finitely many central returns, that is, there are only finitely many j such that $R|_{T_j^{(k)}}$ is central.

In [L2], it was proved that almost every quadratic map is either regular or simple or infinitely renormalizable. It was then shown in [L3] that infinitely renormalizable maps have 0 Lebesgue measure.

In [ALM] it is remarked that Lyubich's estimates actually prove quasisymmetric robustness of the set of regular or simple maps, that is, they are still typical after reparametrization.

3.1.8. *Strategy.* Both Theorems A and B will be proved using the same strategy.

Our strategy is to describe the dynamics of the principal nest, this is our phase analysis. From time to time, we transfer the information from the phase space to the parameter, following the description of the parapuzzle nest which we will make in the next subsection.

Due to Lyubich's results, we can completely forget about infinitely renormalizable maps, we just have to prove the claimed estimates for almost every (after reparametrization) simple map. During our discussion, for notational reasons, we will actually consider a fixed renormalization level k, that is, we will analyse maps in Δ_k.

This allow us to fix some convenient notation: given $g \in \Delta_k$ we define $I_i[g] = T_i^{(k)}[g]$, so that $\{I_i[g]\}$ is a sequence of intervals (possibly finite). We use the notation $R_i[g] = R_{I_i[g]}$, $L_i[g] = L_{I_i[g]}$ and so on. When doing phase analysis (working with fixed f) we usually drop the dependence on the map and write R_i for $R_i[f]$.

3.2. **Parameter partition.** Part of our work is to transfer information from the phase space of some map $f \in \mathcal{F}$ to a neighborhood of its parameter space. This is done in the following way. We consider the first landing map to I_i. The domains of this map partition the interval I_i, the complement of this set is a hyperbolic Cantor set $K_i = I_i \setminus \cup C_i^{\underline{d}}$. This Cantor set persists in a small parameter neighborhood J_i of f, changing in a continuous way.

Along J_i, the first landing map is topologically the same (in a way will be clear soon). However the critical value $R_i[g](0)$ moves relative to the partition (when g moves in J_i). This allows us to partition the parameter piece J_i in smaller pieces, each corresponding to a region where $R_i(0)$ belongs to some fixed domain of the first landing map.

For a discussion of the next Theorem, see [AM1].

Topological Phase-Parameter relation. *Let $f \in \mathcal{F}$ be exactly k times renormalizable. Then there is a sequence $\{J_i\}_{i \in \mathbb{N}}$ of nested parameter intervals (the principal parapuzzle nest of f) with the following properties.*

(1) *J_i is the maximal interval containing f such that for all $g \in J_i$ the interval $T_{i+1}^{(k)}[g]$ is defined and changes in a continuous way. Since the first return map to $R_i[g]$ has a central domain, so that the landing map $L_i[g] : \cup C_i^{\underline{d}}[g] \to I_i[g]$ is defined.*

(2) *$L_i[g]$ is topologically the same along J_i: there exists homeomorphisms $H_i[g] : I_i \to I_i[g]$, such that $H_i[g](C_i^{\underline{d}}) = C_i^{\underline{d}}[g]$. The maps $H_i[g]$ change continuously.*

(3) *There exists a homeomorphism $\Xi_i : I_i \to J_i$ such that $\Xi_i(C_i^{\underline{d}})$ is the set of g such that $R_i[g](0)$ belongs to $C_i^{\underline{d}}[g]$.*

The homeomorphisms H_i and Ξ_i are not uniquely defined, it is easy to see that we can modify then inside each $C_i^{\underline{d}}$ window. H_i and Ξ_i are well defined maps if restricted to K_i.

With this result we can define for any $f \in \mathcal{F}$ intervals $J_i^j = \Xi_i(I_i^j)$ and $J_i^{\underline{d}} = \Xi_i(I_i^{\underline{d}})$. From the description we gave it immediately follows that two intervals $J_i[f]$ and $J_i[g]$ associated to different maps f and g are either disjoint or nested, and the same happens for intervals J_i^j or $J_i^{\underline{d}}$.

We will concentrate on the analysis of the regularity of Ξ_i for the special class of simple maps f: one of the good properties of the class of simple maps is better control of the phase-parameter relation. Even for simple maps, however, the regularity of Ξ_i is not great: there is too much dynamical information contained in it. A solution to this problem is to forget some dynamical information. With this intent we introduce an interval which will be used to erase information.

3.2.1. *Gape interval.* If $i > 1$, we define the gape interval $\tilde{I}_{i+1}$ as follows.

We have that $R_i|_{I_{i+1}} = L_{i-1} \circ R_{i-1} = R_{i-1}^{\underline{d}} \circ R_{i-1}$ for some $\underline{d}$, so that $I_{i+1} = (R_{i-1}|_{I_i})^{-1}(C_{i-1}^{\underline{d}})$. We define the gape interval $\tilde{I}_{i+1} = (R_{i-1}|_{I_i})^{-1}(I_{i-1}^{\underline{d}})$.

We notice that for each I_i^j, the gape interval $\tilde{I}_{i+1}$ either contains or is disjoint from I_i^j.

3.2.2. *The phase-parameter relation.* As we discussed before, the dynamical information contained in Ξ_i is entirely given by $\Xi_i|_{K_i}$: a map obtained by Ξ_i by modification inside a $C_i^{\underline{d}}$ window has still the same properties. Therefore it makes sense to ask about the regularity of $\Xi_i|_{K_i}$. As we anticipated before we must erase some information to obtain good results.

Let f be a simple map and τ_i be such that $R_i(0) \in I_i^{\tau_i}$. We define two Cantor sets, $K_i^\tau = K_i \cap I_i^{\tau_i}$ which contains refined information restricted to the $I_i^{\tau_i}$ window and $\tilde{K}_i = I_i \setminus \cup I_i^j \setminus \tilde{I}_{i+1}$, which contains global information, at the cost of erasing information inside each I_i^j window and in $\tilde{I}_{i+1}$.

The following Theorem is one of the main steps in [AM1].

Phase-Parameter relation. *Let f be a simple map. For all $\gamma > 1$ there exists i_0 such that for all $i > i_0$ we have*

Quadratic PhPa1: $\Xi_i|_{K_i^\tau}$ *is* γ*-qs,*
Quadratic PhPa2: $\Xi_i|_{\tilde{K}_i}$ *is* γ*-qs,*
PaPa1: $H_i[g]|_{K_i}$ *is* γ*-qs if* $g \in J_i^{\tau_i}$*,*
PaPa2: *the map* $H_i[g]|_{\tilde{K}_i}$ *is* γ*-qs if* $g \in J_i$*.*

3.3. **Reparametrization.** To prove Theorems A and B, we have to show that for any γ_0, if h is γ_0-qs then almost every quadratic map *after reparametrization by* h have some good properties. Therefore, we fix now such γ_0 and such a reparametrization. We keep otherwise the same notation:

Reparametrized Phase-Parameter relation. *Let* f *be a simple map. For all* $\lambda > 1$ *there exists* i_0 *such that for all* $i > i_0$ *we have*
PhPa1: $h \circ \Xi_i|_{K_i^\tau}$ *is* $\lambda\gamma_0$*-qs,*
PhPa2: $h \circ \Xi_i|_{\tilde{K}_i}$ *is* $\lambda\gamma_0$*-qs,*
PaPa1: $H_i[g]|_{K_i}$ *is* λ*-qs if* $g \in J_i^{\tau_i}$*,*
PaPa2: *the map* $H_i[g]|_{\tilde{K}_i}$ *is* λ*-qs if* $g \in J_i$*.*

Notice that PaPa1 and PaPa2 did not change after reparametrization, since they are just Phase-Phase estimates.

All references to the parameter space from now on will assume this reparametrization.

4. Measure and capacities

4.1. **Quasisymmetric maps.** If $X \subset \mathbb{R}$ is measurable, let's denote $|X|$ its Lebesgue measure. Let's describe more metric properties of γ-qs maps.

To each γ, there exists a constant $k \geq 1$ such that for all $f \in QS(\gamma)$, for all $J \subset I$ intervals,

$$\frac{1}{k}\left(\frac{|J|}{|I|}\right)^k \leq \frac{|f(J)|}{|f(I)|} \leq \left(\frac{k|J|}{|I|}\right)^{1/k}.$$

Furthermore $\lim_{\gamma \to 1} k(\gamma) = 1$. So for each $\epsilon > 0$ there exists $\gamma > 1$ such that $k(2\gamma - 1) < \epsilon/3$.

4.2. **Capacities and trees.** The γ-capacity of a subset X of an interval I is defined as follows.

$$p_\gamma(X|I) = \sup_{h \in QS(\gamma)} \frac{|h(X)|}{|h(I)|}.$$

This geometric quantity is well adapted to our context, since it is well behaved under tree decompositions of sets. In other words, if I^j

are disjoint subintervals of I and $X \subset \cup I_j$ then

$$p_\gamma(X|I) \leq p_\gamma(\cup I_j|I) \sup p_\gamma(X \cap I^j|I^j).$$

4.3. **A measure theoretical lemma.** Our usual procedure consists in picking a class X of maps which we show is full measure among non-regular maps and then for each map in this class we describe what happens for the principal nest, showing finally that a subset Y of X is still full measure. The first such step was to consider the class of simple maps.

We describe here our usual argument (a variation of Borel-Cantelli lemma). Assume at some point we know how to prove that almost every map belong to some set X. Let Q_n be a (bad) property that a map may have (usually related to the n-th stage of the principle nest). Suppose we prove that if $f \in X$ then the probability that a map in $J_n(f)$ has the property Q_n is bounded by $q_n(f)$ which is shown to be summable for all $f \in X$. We then conclude that almost every map does not have property Q_n for n big enough.

Sometimes we also apply the same argument, proving instead that $q_n(f)$ is summable where $q_n(f)$ is the probability that a map in $J_n^{\tau_n}(f)$ has property Q_n, where τ_n is such that $f \in J_n^{\tau_n}(f)$.

In other words, we use the following lemma.

Lemma 4.1. *Let $X \subset \mathbb{R}$ be a measurable set such that for each $x \in X$ is defined a sequence $J_i(x)$ of nested intervals converging to x such that for all $x_1, x_2 \in X$ and any i, $J_i(x_1)$ is either equal or disjoint to $J_i(x_2)$. Let Q_n be measurable subsets of $\mathbb{R}$ and $q_n(x) = |Q_n \cap J_n(x)|/|J_n(x)|$. Let Y be the set of x in X which belong to finitely many Q_n. If $\sum q_n(x)$ is finite for almost any $x \in X$ then $|Y| = |X|$.*

Proof. Let $Y_n = \{x \in X | \sum_{k=n}^{\infty} q_k(x) < 1/2\}$. It is clear that $Y_n \subset Y_{n+1}$ and $|\cup Y_n| = |X|$.

Let $Z_n = \{x \in Y_n ||Y_n \cap J_m(x)|/|J_m(x)| > 1/2, m \geq n\}$. It is clear that $Z_n \subset Z_{n+1}$ and $|\cup Z_n| = |X|$.

Let $T_n^m = \cup_{x \in Z_n} J_m(x)$. Let $K_n^m = T_n^m \cap Q_m$. Of course

$$|K_n^m| \leq \int_{T_n^m} q_m \leq 2 \int_{Y_n} q_m.$$

And of course

$$\sum_{m \geq n} \int_{Y_n} q_m \leq \frac{1}{2}|Y_n|.$$

This shows that $\sum_{m \geq n} |K_n^m| \leq |Y_n|$, so almost every point in Z_n belong to finitely many K_n^m. We conclude then that almost every point in X belong to finitely many Q_m.

□

In practice, we will estimate the capacity of sets in the phase space: that is, given a map f we will obtain subsets $\tilde{Q}_n(f)$ in the phase space, corresponding to bad branches of return or landing maps. We will the show that for some $\gamma > \gamma_0$ we have $\sum p_\gamma(\tilde{Q}_n(f)|J_n(f)) < \infty$ or $\sum p_\gamma(\tilde{Q}_n(f)|J_n^{\tau_n}(f)) < \infty$. We will then use PhPa1, PhPa2 and the measure-theoretical lemma to conclude that with total probability among non-regular maps, for all n sufficiently big, $R_n(0)$ does not belong to a bad set.

From now on when we prove that almost every non-regular parameter has some property, we will just say that with total probability (without specifying) such property holds.

5. Statistics of the principal nest

Through the end of this paper we fix some constant $\gamma > \gamma_0$. We also fix $\tilde{b} \gg 1000k(2\gamma - 1)^1000$ and $\tilde{a} = b^{-1}$, and set $b = \tilde{b}^{1000\tilde{b}}$, $a = b^{-1}$.

5.1. Decay of geometry.

Let as before $\tau_n \in \mathbb{Z}$ such that $R_n(0) \in I_n^{\tau_n}$.

An important parameter in our construction will be the scaling factor

$$c_n = \frac{|I_{n+1}|}{|I_n|}.$$

This variable of course changes inside each $J_n^{\tau_n}$ window, however, not by much. From PaPa1, for instance, we get that with total probability

$$\lim_{n \to \infty} \sup_{g_1, g_2 \in J_n^{\tau_n}} \frac{\ln(c_n[g_1])}{\ln(c_n[g_2])} = 1.$$

This variable is by far the most important on our analysis of the statistics of return maps. We will often consider other variables (say, return times): we will show that the distribution of those variables is concentrated near some average value. Our estimates will usually give a range of values near the average, and c_n will play an important role. Due (among other issues) to the variability of c_n inside the parameter windows, the ranges we select will depend on c_n up to an exponent (say, in a range from a to b). From the estimate we just obtained, for big n the variability (margin of error) of c_n will fall confortably in such range, and we won't elaborate more.

If $x \in I_n^j$ we let $j^{(n)}(x) = j$ and if $x \in C_n^{\underline{d}}$ we let $\underline{d}^{(n)}(x) = \underline{d}$.

Lemma 5.1. *With total probability, for all n sufficiently big we have*

$$p_{2\gamma-1}(|\underline{d}^{(n)}(x)| \leq k | x \in I_n) < k c_n^{\tilde{a}},$$

$$p_{2\gamma-1}(|\underline{d}^{(n)}(x)| \geq k | x \in I_n) < e^{-kc_n^{\tilde{b}}}.$$

We also have

$$p_{2\gamma-1}(|\underline{d}^{(n)}(x)| \leq k | x \in I_n^{\tau_n}) < kc_n^{\tilde{a}},$$

$$p_{2\gamma-1}(|\underline{d}^{(n)}(x)| \geq k | x \in I_n^{\tau_n}) < e^{-kc_n^{\tilde{b}}}.$$

Proof. Let's compute the first two estimates.

Since I_n^0 is in the middle of I_n, we have as a simple consequence of the real Schwarz Lemma (see [L1]) that

$$\frac{c_n}{4} < \frac{|C_n^{\underline{d}}|}{|I_n^{\underline{d}}|} < 4c_n.$$

As a consequence

$$p_{2\gamma-1}(|\underline{d}^{(n)}(x)| = m | x \in I_n) < (4c_n)^{2\tilde{a}}$$

and we get the estimate summing up on $0 \leq m \leq k$.

For the same reason, we get that

$$p_{2\gamma-1}(|\underline{d}^{(n)}(x) > m | x \in I_n) < 1 - (c_n/4)^{\tilde{b}/2} p_{2\gamma-1}(|\underline{d}^{(n)}(x)| \geq m | x \in I_n).$$

This implies

$$p_{2\gamma-1}(|\underline{d}(x)| \geq m | x \in I_n) \leq (1 - (c_n/4)^{\tilde{b}/2})^m.$$

The second estimate follows from

$$(1 - (\frac{c_n}{4})^{\tilde{b}/2})^k < (1 - c_n^{\tilde{b}})^k < ((1 - c_n^{\tilde{b}})^{c_n^{-\tilde{b}}})^{kc_n^{\tilde{b}}} < e^{-kc_n^{\tilde{b}}}.$$

The two remaining estimates are analogous. □

Transferring the result (more precisely the second pair of estimates) to the parameter in each $J_n^{\tau_n}$ window using PhPa1 we get (noticing that the measure of the complement of the set of parameters in $J_n^{\tau_n}$ such that $c_n^{-\tilde{a}/2} < s_n < c_n^{-2\tilde{b}}$ can be estimated by $2c_n^{\tilde{a}/2}$ for n big, which is summable.

Lemma 5.2. *With total probability,*

$$c_n^{-\tilde{a}/2} < \frac{\ln(s_n)}{\ln(c_n^{-1})} < c_n^{-2\tilde{b}}.$$

Remark 5.1. The parameter s_n influences the size of c_{n+1} in a determinant way. It is easy to see (using for instance the real Schwarz Lemma, see [L1]) for instance that $\ln(c_{n+1}^{-1}) > Ks_n$ for some universal constant K, which in general is bounded from below (real a priori bounds), but since we have decay of geometry, actually goes to infinity.

As an easy consequence we get

Corollary 5.3. *With total probability,*

$$\frac{\ln(\ln(c_{n+1}^{-1}))}{\ln(c_n^{-1})} > \tilde{a}/3.$$

In particular, c_n decreases at least torrentially fast.

5.2. **Fine partitions.** We use Cantor set K_n and $\tilde{K}_n$ to partition the phase space. In many circumstances we are directly concerned with intervals of this partition. However, sometimes we just want to exclude an interval of given size (usually around 0). This size does not usually correspond to a union of gaps, so we instead should substitute in applications an interval which is union of gaps, with approximately the given size. The degree of relative approximation will always be torrentially good (in n), so we usually won't elaborate on this. In this section we just give some results which will imply that the partition induced by the Cantor sets are fine enough to allow torrentially good approximations.

The following Lemma summarizes the situation. The proof is based on estimates of distortion using the real Schwarz Lemma and the Koebe Principle (see [L1]) and is very simple, so we skip it.

Lemma 5.4. *The following estimates hold.*

- $|I_n^j|/|I_n| = O(\sqrt{c_{n-2}})$,
- $|I_n^{\underline{d}}|/|I_n^{\sigma^+(\underline{d})}| = O(\sqrt{c_{n-2}})$.
- $|C_n^{\underline{d}}|/|I_n^{\underline{d}}| \approx c_n$,
- $|\tilde{I}_{n+1}|/|I_n| = O(e^{1-s_{n-1}})$.

In other words, distances in I_n can be measured with precision $\sqrt{c_{n-2}}|I_n|$ in the partition induced by $\tilde{K}_n$, due to first and last items (since $e^{1-s_{n-1}} = O(c_{n-1})$).

Distances can be measured much more precisely with respect to the partition induced by K_n, in fact we have good precision in each $I_n^{\underline{d}}$ scale. In other words, inside $I_n^{\underline{d}}$, the central gap $C_n^{\underline{d}}$ is of size $O(c_n|I_n^{\underline{d}}|)$ (third item) and the other gaps have size $O(\sqrt{c_{n-2}}|C_n^{\underline{d}}|)$ (second and third item).

Remark 5.2. We need to consider intervals which are union of gaps due to our phrasing of the phase-parameter relation, which only gives information about such gaps. However, this is not absolutely necessary, and we could have proceeded in a different way: our proof of the phase-parameter relation actually shows that there is a holonomy map with good qs estimates between phase and parameter intervals (and not only Cantor sets). While this map is not canonical, the fact that it is a holonomy map for a motion with good phase-phase estimates would allow our proofs to work.

5.3. **Initial estimates on distortion.** To deal with the distortion control we need some preliminary known results. We won't get too much in details here, those estimates are related to the estimates on gaps of Cantor sets and the Koebe Principle, and can be concluded easily.

Proposition 5.5. *For any j, if $R_n|_{I_n^j} = f^k$ then* $\mathrm{dist}(f^{k-1}|_{f(I_n^j)}) = 1 + O(c_{n-1})$.

For any $\underline{d}$, $\mathrm{dist}(R_n^{\sigma^+(\underline{d})}|_{I_n^{\underline{d}}}) = 1 + O(\sqrt{c_{n-2}})$.

We will use the following immediate consequence for the decomposition of certain branches.

Lemma 5.6. *With total probability,*

- $R_n|_{I_n^0} = \phi \circ f$ *where ϕ has torrentially small distortion,*
- $R_n^{\underline{d}} = \phi_2 \circ f \circ \phi_1$ *where ϕ_2 and ϕ_1 have torrentially small distortion and $\phi_1 = R_n^{\sigma^+(\underline{d})}$.*

5.4. **Estimating derivatives.**

Lemma 5.7. *With total probability,*

$$d(R_n(0), \partial I_n \cup \{0\}) < |I_n| n^{-\tilde{b}}.$$

In particular $R_n(0) \notin \tilde{I}_{n+1}$ for all n large enough.

Proof. This is a simple consequence of PhPa2, using that $n^{-\sqrt{\tilde{b}}}$ is summable. ☐

From now on we suppose that f satisfies the conditions of the above lemma.

Lemma 5.8. *With total probability, for n big enough and $j \neq 0$*

$$\mathrm{dist}(f|_{I_n^j}) \leq n^{\tilde{b}}.$$

Proof. Notice that the gaps of the Cantor sets K_n inside $I_n^{\underline{d}}$ which are different from $C_n^{\underline{d}}$ are torrentially (in n) smaller then $C_n^{\underline{d}}$.

Denote by $P_n^{\underline{d}}$ a $|C_n^{\underline{d}}|/n^{\tilde{b}}$ neighborhood of $C_n^{\underline{d}}$.

It is clear that the image by a γ-qs homeomorphism of $P_n^{\underline{d}} \setminus C_n^{\underline{d}}$ is of order $n^{-\sqrt{\tilde{b}}}|C_n^{\underline{d}}|$ (this is true both on I_n or restricting to $I_n^{\tau_n}$). Since $C_n^{\underline{d}}$ are disjoint,

$$p_\gamma(I_n^{\tau_n} \cap \cup (P_n^{\underline{d}} \setminus C_n^{\underline{d}}) | I_n^{\tau_n})$$

is summable.

Transferring the result to the parameter using PhPa1 we see that the critical point will never be in a $n^{-\tilde{b}}$ neighborhood of any I^j_{n+1} with $j \neq 0$. □

Applying Lemma 5.6 we get

Lemma 5.9. *With total probability, for n big enough and for all $\underline{d}$*

$$\operatorname{dist}(R^{\underline{d}}_n) \leq n^{\tilde{b}}.$$

In particular, for n big enough, $\sup_{\underline{d}\neq\emptyset} \operatorname{dist}(R^{\underline{d}}_n)$ is bounded by 2^n and $\inf |R'_n|_{\cup_{j\neq 0} I^j_n}| > 2$.

Remark 5.3. Lemma 5.8 has also an application for approximation of intervals. The result implies that if $I^j_n = (c,d)$ and $j \neq 0$, we have $1/2^n < d/c < 2^n$. As a consequence, for all symmetric interval $I_{n+1} \subset X \subset I_n$, there exists a symmetric interval $X \subset \tilde{X}$, which is union of I^j_n and such that $|\tilde{X}|/|X| < 2^n$ (approximation by union of $C^{\underline{d}}_n$, with $|tildeX|/|X|$ torrentially close to 1, follows more easily from the discussion on fine partitions).

We will also need to estimate derivatives of iterates of f, and not only of return branches.

Lemma 5.10. *With total probability, if n is sufficiently big and if $x \in \cup_{j\neq 0} I^j_n$ and $R_n|_{I^j_n} = f^r$, then for $1 \leq k \leq r$, $|(Df^k(x))| > |x| c^3_{n-1}$.*

Proof. Let n_0 be minimum such that if $n \geq n_0$ then

$$\inf |R'_n|_{\cup_{j\neq 0} I^j_n}| > 1.$$

From hyperbolicity of f, from Lemma 3.1, restricted to the complement of I_{n_0}, there exists a constant $C > 0$ such that if $f^s(x) \notin I^0_{n_0}$, $r \leq s \leq k$ then $|Df^{k-r}(f^r(x))| > C$.

If $k = r$, the result follows from Lemma 5.9. We assume $k < r$.

Let's define $d(s)$, $1 \leq s \leq k$ such that $f^s(x) \in I_{d(s)} \setminus I^0_{d(s)}$. Let $m(s) = \max_{s\leq t\leq k} d(s)$. Let $k_0 = 0$. Supposing $k_j < k$ define k_{j+1} as $k_{j+1} = \max\{k_j < s \leq k | d(s) = m_s\}$.

If $d(k_1) = n$, $k = r$ and from Lemma 5.9 $|Df^k(x)| > 1$.

We have then a sequence $0 = k_0 < k_1 < ... < k_l = k$. Let k_i be maximal with $d(k_i) \geq n_0$. We have

$$|(Df^{k-k_i}(f^{k_i}(x))| > C|f^{k_i}(x)|$$

and for $1 < j \leq i$,

$$|Df^{k_j-k_{j-1}}(f^{k_{j-1}}(x))| > |Df(f^{k_{j-1}})(x)| > c_{d(k_{j-1})}.$$

We also have $|Df^{k_1}(x)| \geq |x|$ if $i \geq 1$. If $i = 0$, $|Df^k(x)| \leq C|x|$. Combining it all we get

$$|Df^k(x)| > C|x| \prod_{1 \leq j \leq i} c_{d(k_j)} > |x| c_{n-1}^3.$$

□

6. Sequence of quasisymmetric constants and trees

6.1. **Preliminary estimates.** We will need from now on to consider not only γ-capacities with γ fixed, but a sequence γ_n converging to γ.

We define the sequences $\rho_n = (n+1)/n$ and $\tilde{\rho}_n = (2n+3)/(2n+1)$, so that $\rho_n > \tilde{\rho}_n > \rho_{n+1}$ and $\lim \rho_n = 1$. We define the sequence $\gamma_n = \gamma\rho_n$ and an intermediate sequence $\tilde{\gamma}_n = \gamma\tilde{\rho}_n$.

As we know, renormalization proccess has two phases, first R_n to L_n and then L_n to R_{n+1}. The following remarks shows why it is useful to consider the sequence of quasisymmetric constants due to losses related to distortion.

Remark 6.1. Let S be an interval contained in $I_n^{\underline{d}}$. Using Lemma 5.6 we have $R_n^{\underline{d}}|_S = \psi_2 \circ f \circ \psi_1$, where the distortion of ψ_2 and ψ_1 are torrentially small and $\psi_1(S)$ is contained in some I_n^j, $j \neq 0$. If S is contained in I_n^0 we may as well write $R_n|_S = \phi \circ f$, with $\mathrm{dist}(\phi)$ torrentially small.

In either case, if we decompose S in $2km$ intervals S_i of equal length, where k is the distortion of either $R^{\underline{d}}|_S$ or $R|_S$ and m is subtorrentially big (say $m < 2^n$), the distortion obtained restricting to any interval S_i will be bounded by $1 + 1/m$.

Remark 6.2. We have the following estimate for the effect of the pull-back of a subset of I_n by the central branch $R_n|_{I_n^0}$. With total probability, for all n sufficiently big, if $X \subset I_n$ satisfies

$$p_{\tilde{\gamma}_n}(X|I_n) < \delta < n^{-b} \leq n^{-\tilde{b}^{10\tilde{b}}}$$

then

$$p_{\gamma_{n+1}}((R_n|_{I_{n+1}})^{-1}(X)|I_n) < \delta^{\tilde{a}^3} < \delta^{-a}.$$

Indeed, let V be a $\delta^{\tilde{a}}|I_{n+1}|$ neighborhood of 0. Then $R_n|_{I_{n+1} \setminus V}$ has distortion bounded by $2\delta^{\tilde{a}}$.

Let $W \subset I_n$ an interval of size $\lambda|I_n|$. Of course

$$p_{\tilde{\gamma}_n}(X \cap W|W) < \delta\lambda^{-\tilde{b}}.$$

Let's decompose each side of $I_{n+1} \setminus V$ as a union of $n^{\tilde{b}}\delta^{-\tilde{a}}$ intervals. Let W be such an interval. From Lemma 5.7, it is clear that the image of W covers at least $\delta^{2\tilde{a}} n^{-3\tilde{b}} |I_n|$. It is clear then that

$$p_{\tilde{\gamma}_n}(X \cap R_n(W)|R_n(W)) < \delta(\delta^{2\tilde{a}} n^{-3\tilde{b}})^{-\tilde{b}} = \delta^3 n^{-3\tilde{b}^2}.$$

So we conclude (since the distortion of $R_n|_W$ is of order $1+n^{-3}$) that

$$p_{\gamma_{n+1}}((R_n|_{I_{n+1}})^{-1}(X) \cap W|W) < \delta^3 n^{-3\tilde{b}^2}$$

(we use the fact that the composition of a γ_{n+1}-qs map with a map with small distortion in $\tilde{\gamma}_n$-qs). Since

$$p_{\gamma_{n+1}}(V|I_{n+1}) < 2\delta^{\tilde{a}^2},$$

we get the required estimate.

6.2. **More on trees.** Let's see an application of the above remarks.

Lemma 6.1. *With total probability, for all n sufficiently big*

$$p_{\tilde{\gamma}_n}((R_n^{\underline{d}})^{-1}(X)|I_n^{\underline{d}}) < 2^n p_{\gamma_n}(X|I_n).$$

Proof. Decompose $I_n^{\underline{d}}$ in $n^{\ln(n)}$ intervals of equal length and apply Remark 6.1. $\square$

By induction we get

Lemma 6.2. *With total probability, for n is big enough, if $X_1, ..., X_m \subset \mathbb{Z} \setminus \{0\}$*

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) = (j_1, ..., j_m), j_i \in X_i | x \in I_n)$$
$$\leq 2^{mn} \prod_{i=1}^{m} p_{\gamma_n}(j^{(n)}(x) \in X_i | x \in I_n).$$

There is also a variation fixing the start of the sequence.

Lemma 6.3. *With total probability, for n is big enough, if $X_1, ..., X_m \subset \mathbb{Z} \setminus \{0\}$, and if $|\underline{d}| = (j_1, ..., j_k)$ we have*

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) = (j_1, ..., j_{m+k}), j_{i+k} \in X_i | x \in I_n^{\underline{d}})$$
$$\leq 2^{mn} \prod_{i=1}^{m} p_{\gamma_n}(j^{(n)}(x) \in X_i | x \in I_n).$$

In particular, with $\underline{d} = (\tau_n)$,

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) = (\tau_n, j_1, ..., j_m), j_i \in X_i | x \in I_n^{\tau_n})$$
$$\leq 2^{mn} \prod_{i=1}^{m} p_{\gamma_n}(j^{(n)}(x) \in X_i | x \in I_n).$$

The last part of the above Lemma will be often necessary in order to apply PhPa1.

Sometimes we are more interested in the case where the X_i are all equal.

Let $Q \subset \mathbb{Z} \setminus \{0\}$. Let $Q(m,k)$ denote the set of $\underline{d} = (j_1, ..., j_m)$ such that $\#\{i | j_i \in Q\} \cap \{1, ..., m\} \geq k$.

Define $q_n(m,k) = p_{\tilde{\gamma}_n}(\cup_{\underline{d} \in Q(m,k)} I_n^{\underline{d}} | I_n)$.

Let $q_n = p_{\gamma_n}(\cup_{j \in Q} I_n^j | I_n)$.

Lemma 6.4. *With total probability, for n large enough,*

$$q_n(m,k) \leq \binom{m}{k} (2^n q_n)^k.$$

Proof. We have the following recursive estimates for $q_n(m,k)$:

$q_n(1,0) = 1, q_n(1,1) \leq q_n \leq 2^n q_n$.

$q_n(m+1,0) = 1, q_n(m+1,k+1) \leq q_n(m,k+1) + 2^n q_n q_n(m,k)$.

Indeed, if $(j_1, ..., j_{m+1}) \in Q(m+1,k+1)$ then either $(j_1, ..., j_m) \in Q(m,k+1)$ or $(j_1, ..., j_m) \in Q(m,k)$ and $j_{m+1} \in Q$. By the estimate 2^n on the distortion of all non central branches, we get our result. $\square$

We recall that by Stirling Formula,

$$\binom{m}{qm} < \frac{m^{qm}}{(qm)!} < \left(\frac{3}{q}\right)^{qm}.$$

So we can get the following estimate. For $q \geq q_n$,

$$q_n(m, (6 \cdot 2^n) qm) < \left(\frac{1}{2}\right)^{(6 \cdot 2^n qm)}.$$

It is also used in the following form. If $q^{-1} > 6 \cdot 2^n$ (it is usually the case, since q_n will be torrentially small)

$$\sum_{k > q^{-2}} q_n(k, (6 \cdot 2^n) qk) < 2^{-n} q^{-1} \left(\frac{1}{2}\right)^{(6 \cdot 2^n) q^{-1}}.$$

7. Estimates on time

Our aim in this section is to estimate the distribution of return times to I_n: they are concentrated around c_{n-1}^{-1} up to an exponent in some range given by a and b.

The basic estimate is a large deviation estimate and is proven in the next subsection (Corollary 7.4) and states that for $k \geq 1$ the set of branches with time larger then $k c_n^{-4\tilde{b}}$ has capacity less then e^{-k}.

7.1. **A Large Deviation lemma for times.** Let $r_n(j)$ be such that $R_n|_{I_n^j} = f^{r_n(j)}$. We will also use the notation $r_n(x) = r_n(j^{(n)}(x))$, the n-th return time of x (there should be no confusion for the reader, since we consistently use j for an integer index and x for a point in the phase space.

Let

$$A_n(k) = p_{\gamma_n}(r_n(x) \geq k | x \in I_n)$$

Since f restricted to the complement of I_{n+1} is hyperbolic, from Lemma 3.1, it is clear that $A_n(k)$ decays exponentially with k.

Let ζ_n be the maximum $\zeta < c_{n-1}$ such that for all $k > \zeta^{-1}$ we have

$$A_n(k) \leq e^{-\zeta k}$$

and finally let $\alpha_n = \min_{1 \leq m \leq n} \zeta_m$.

Our main result in this section is to estimate α_n. We will show that with total probability, for n big we have $\alpha_{n+1} \geq c_n^{4\tilde{b}}$. For this we will have to do a simultaneous estimate for landing times, which we define now.

Let $l_n(\underline{d})$ be such that $L_n|_{I_n^{\underline{d}}} = f^{l_n(\underline{d})}$. We will also use the notation $l_n(x) = l_n(\underline{d}^{(n)}(x))$.

Let

$$B_n(k) = p_{\tilde{\gamma}_n}(l_n(x) > k | x \in I_n).$$

Lemma 7.1. *If* $k > c_n^{-2\tilde{b}} \alpha_n^{-2\tilde{b}}$ *then*

$$B_n(k) < e^{-c_n^{2\tilde{b}} \alpha_n^{2\tilde{b}} k}.$$

Proof. Let $k > c_n^{-2\tilde{b}} \alpha_n^{-2\tilde{b}}$ be fixed. Let $m_0 = \alpha_n^{2\tilde{b}} k$.

Notice that by Lemma 5.1

$$p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| \geq m_0 | x \in I_n) \leq e^{-c_n^{\tilde{b}} \alpha_n^{2\tilde{b}} k}.$$

Fix now $m < m_0$. Let's estimate

$$p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| = m, l_n(x) > k | x \in I_n).$$

For each $\underline{d} = (j_1, ..., j_m)$ we can associate a sequence of m positive integers r_i such that $r_i \leq r_n(j_i)$ and $\sum r_i = k$. The average value of r_i is at least k/m so we conclude that

$$\sum_{r_i \geq k/2m} r_i > k/2.$$

Recall also that

$$\frac{k}{2m} > \frac{1}{(2\alpha_n^{2\tilde{b}})} > \alpha_n^{-1}.$$

Given a sequence of m positive integers r_i as above we can do the following estimate using Lemma 6.2

$$\begin{aligned}
p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) = (j_1,...,j_m), r_n(j_i) > r_i | x \in I_n)& \\
&\leq 2^{mn} \prod_{j=1}^{m} p_{\gamma_n}(r_n(x) \geq r_j | x \in I_n) \\
&\leq 2^{mn} \prod_{r_j \geq \alpha_n^{-1}} p_{\gamma_n}(r_n(x) \geq r_j | x \in I_n) \\
&\leq 2^{mn} \prod_{r_j \geq k/2m} e^{-\alpha_n r_j} \\
&\leq 2^{mn} e^{-\alpha_n k/2}.
\end{aligned}$$

The number of sequences of m positive integers r_i with sum k is

$$\begin{aligned}
\binom{k+m-1}{m-1} &\leq \frac{1}{(m-1)!}(k+m-1)^{m-1} \\
&\leq \frac{1}{m!}(k+m)^m \leq \left(\frac{2ek}{m}\right)^m.
\end{aligned}$$

Notice that

$$\begin{aligned}
2^{mn}\left(\frac{2ek}{m}\right)^m &\leq \left(\frac{2^{n+3}k}{m}\right)^{\frac{m}{k2^{n+3}}k2^{n+3}} \\
&\leq \left(\frac{2^{n+3}k}{m_0}\right)^{\frac{m_0}{k2^{n+3}}k2^{n+3}} \quad \text{(since } x^{1/x} \text{ is decreases for } x > e) \\
&\leq \left(\frac{2^{n+3}}{\alpha_n^{2\tilde{b}}}\right)^{m_0} \leq e^{\alpha_n^{\tilde{b}} k}.
\end{aligned}$$

So we can finally estimate

$$\begin{aligned}
p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| = m, l_n(x) \geq k | x \in I_n) &\leq 2^{mn}\left(\frac{2ek}{m}\right)^m e^{-\alpha_n k/2} \\
&< e^{(\alpha_n^{\tilde{b}-1} - 1/2)\alpha_n k}.
\end{aligned}$$

Summing up on m we get

$$\begin{aligned}
p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| < m_0, l_n(x) \geq k | x \in I_n)& \\
&\leq m_0 e^{(\alpha_n^{\tilde{b}-1} - 1/2)\alpha_n k} \\
&< e^{(2\alpha_n^{\tilde{b}-1} - 1/2)\alpha_n k} (\text{since} \frac{\ln(m_0)}{k} \leq \frac{\ln(k)}{k} \leq \alpha_n^{\tilde{b}}) \\
&\leq e^{-\alpha_n k/3}.
\end{aligned}$$

As a direct consequence we get

$$B_n(k) < e^{-\alpha_n k/3} + e^{-c_n^{\tilde{b}} \alpha_n^{2\tilde{b}} k} < e^{-c_n^{2\tilde{b}} \alpha_n^{2\tilde{b}} k}.$$

□

Lemma 7.2. *With total probability, for n large enough,*

$$v_{n+1} < c_n^{-3\tilde{b}} \alpha_n^{-3\tilde{b}}.$$

Proof. Let $\underline{d}$ such that $R_n(0) \in C_n^{\underline{d}}$.

Using Lemma 7.1, with total probability, for n large enough, $l_n(\underline{d}) < n\alpha_n^{-2\tilde{b}} c_n^{-2\tilde{b}}$, and $v_{n+1} < v_n + n\alpha_n^{-2\tilde{b}} c_n^{-2\tilde{b}}$. Using torrential (and monotonic) decay of $\alpha_n c_n$ we get for n large enough $v_{n+1} < c_n^{-3\tilde{b}} \alpha_n^{-3\tilde{b}}/2$. □

Lemma 7.3. *With total probability, for n large enough,*

$$\alpha_{n+1} \geq \min\{\alpha_n^{4\tilde{b}}, c_n^{4\tilde{b}}\}.$$

Proof. Let $k \geq \max\{\alpha_n^{-4\tilde{b}}, c_n^{-4\tilde{b}}\}$. From Lemma 7.2 one immediately sees that if $r_{n+1}(j) \geq k$ then $R_n(I_{n+1}^j))$ is contained on some $C_n^{\underline{d}}$ with $l_n(\underline{d}) \geq k/2 \geq \alpha_n^{-2\tilde{b}} c_n^{-2\tilde{b}}$.

Applying Lemma 7.1 we have $B_n(k/2) < e^{-\alpha_n^{2\tilde{b}} c_n^{2\tilde{b}} k/2}$.

Applying Remark 6.2 we get

$$A_{n+1}(k) < e^{-k\alpha_n^{2\tilde{b}} c_n^{2\tilde{b}}/\tilde{a}^3} < e^{-k \min\{\alpha_n^{4\tilde{b}}, c_n^{4\tilde{b}}\}}.$$

□

Since c_n decreases torrentially, we get

Corollary 7.4. *With total probability, for n large enough $\alpha_{n+1} \geq c_n^{4\tilde{b}}$.*

Remark 7.1. In particular, using Lemma 7.2, for n big, $v_n < c_{n-1}^{-6\tilde{b}}$.

7.2. **Consequences.** The Lemma bellow is just a convenient way to summarize our results on the distribution of times. We also take the opportunity to state them in terms of constants a and b.

Lemma 7.5. *With total probability, for all n sufficiently large we have*

(1) $p_{\tilde{\gamma}_n}(l_n(x) < c_n^{-s} | x \in I_n) < c_n^{\sqrt{\tilde{a}}-s} < c_n^{a-s}$, *with* $s > 0$,
(2) $p_{\tilde{\gamma}_n}(l_n(x) < c_n^{-s} | x \in I_n^{\tau_n}) < c_n^{a-s}$, *with* $s > 0$,
(3) $p_{\tilde{\gamma}_n}(l_n(x) > c_n^{-s} | x \in I_n) < e^{-c_n^{b-s}}$, *with* $s > b$,
(4) $p_{\tilde{\gamma}_n}(l_n(x) > c_n^{-s} | x \in I_n^{\tau_n}) < e^{-c_n^{b-s}}$, *with* $s > b$,
(5) $p_{\gamma_n}(r_n(x) < c_{n-1}^{-s} | x \in I_n) < c_{n-1}^{\sqrt{\tilde{a}}-s} < c_{n-1}^{a-s}$, *with* $s > 0$,
(6) $p_{\gamma_n}(r_n(x) > c_{n-1}^{-s} | x \in I_n) < e^{-c_{n-1}^{\sqrt{\tilde{b}}-s}} < e^{-c_{n-1}^{b-s}}$ *with* $s > b$.
(7) $c_{n-1}^{-a} < r_n(\tau_n) < c_{n-1}^{-b}$.

(8) $c_{n-1}^{-a} < v_n < c_{n-1}^{-b}$.

Proof. The first and third estimates are contained in Lemma 5.1 (after noticing $l_n(x) \geq |\underline{d}^{(n)}(x)|$) and the second follows from Lemma 7.1. 6 is contained in Corollary 7.4, while 5 can be obtained by pulling back estimate 1 (using Remark 6.2). 5 and 6 imply 7 by PhPa2. In view of 7, 4 follows from the proof of Lemma 7.1 (7 is needed to avoid $r_n(\tau_n)$ to be too big). 8 is easily obtained from Lemma 5.2 and Remark 7.1. $\square$

Remark 7.2. It is clear (using Lemma 5.7) that 8 in Lemma 7.5 implies that with total probability, for all n big enough, $c_{n+1} \geq 4^{-c_n^{-b}}$. This implies together with Corollary 5.3 that

$$a < \frac{\ln(\ln(c_n^{-1}))}{\ln(c_{n-1}^{-1})} < b,$$

so c_n^{-1} grows torrentially.

8. Dealing with hyperbolicity

In this section we show by an inductive proccess that the great majority of branches are reasonably hyperbolic. In order to do that, in the following subsection, we define some classes of branches with 'good' distribution of times and which are not too close to the critical point. The definition of 'good' distribution of times has an inductive component: they are composition of many 'good' branches of the previous level. The fact that most branches are good is related to the validity of some kind of Law of Large Numbers estimate.

8.1. **Some kinds of branches and landings.**

8.1.1. *Standard landings.* We define the set of standard landings at time n, $LS(n) \subset \Omega$ as the set of all $\underline{d} = (j_1, ..., j_m)$ satisfying the following.

LS1: (m not too small or large) $c_n^{-a/2} < m < c_n^{-2b}$,
LS2: (No very large times) $r_n(j_i) < c_{n-1}^{-3b}$ for all i.
LS3: (Short times are sparse in not too small initial segments) For $c_{n-1}^{-2b} \leq k \leq m$

$$\#\{r_n(j_i) < c_{n-1}^{-a/2}\} \cap \{1, ..., k\} < (6 \cdot 2^n) c_{n-1}^{a/2} k.$$

We also define the set of fast landings at time n, $LF(n) \subset \Omega$ by the following conditions

LF1: (m small) $m < c_n^{-a/2}$.
LS2: (no very large times) $r_n(j_i) < c_{n-1}^{-3b}$ for all i.

Lemma 8.1. *With total probability, for all n sufficiently big,*

(1) $p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LS(n)|x \in I_n) < c_n^{a/3}/2$,
(2) $p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LS(n) \cup LF(n)|x \in I_n) < c_n^{n^2}/2$,
(3) $p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LS(n)|x \in I_n^{\tau_n}) < c_n^{a/3}/2$,
(4) $p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LS(n) \cup LF(n)|x \in I_n^{\tau_n}) < c_n^{n^2}/2$.

Proof. The proof is immediate from our time estimates, which can be applyed directly to estimate the losses of LS1, LS2, LF1, or under large deviation form for LS3, following §6.2. Let's estimate the complement of the sets which do not satisfy some properties:

LS1: $c_{n-1}^{2a/5}$,
LS1+LF1: $e^{c_n^b} \ll c_n^{n^2}$,
LS2: $e^{c_{n-1}^{-2b}} \ll c_n^{n^2}$,
LS3: let $q = (6 \cdot 2^n) c_{n-1}^{a/2}$, we get

$$\sum_{k > c_{n-1}^{-2b}} \left(\frac{1}{2}\right)^{qk} \ll c_n^{n^2}.$$

This gives immediately 1 and 2. For 3 and 4 the same estimates hold, using (for LS2) the estimate on $r_n(\tau_n)$. □

8.1.2. *Very good returns, bad returns and excellent landings.* Define the set of very good returns, $VG(n_0, n) \subset \mathbb{Z} \setminus \{0\}$, $n_0, n \in \mathbb{N}$ and the set of bad returns, $B(n_0, n) \subset \mathbb{Z} \setminus \{0\}$, $n_0, n \in \mathbb{N}$, $n \geq n_0$ by induction as follows. We let $VG(n_0, n_0) = \mathbb{Z} \setminus \{0\}$,$B(n_0, n_0) = \emptyset$ and supposing $VG(n_0, n)$ and $B(n_0, n)$ defined, define the set of excellent landings $LE(n_0, n) \subset LS(n)$ satisfying the following extra assumptions.

LE1: (Not very good moments are sparse in not too small initial segments) For all $c_{n-1}^{-2b} < k \leq m$

$$\#\{i | j_i \notin VG(n_0, n)\} \cap \{1, ..., k\} < 6 \cdot 2^n c_{n-1}^{a^2} k,$$

LE2: (Bad moments are sparse in not too small initial segments) For all $c_n^{-1/n} < k \leq m$

$$\#\{i | j_i \notin B(n_0, n)\} \cap \{1, ..., k\} < 6 \cdot 2^n c_{n-1}^{n} k,$$

We define $VG(n_0, n+1)$ as the set of j such that $R_n(I_{n+1}^j) = C_n^{\underline{d}}$ with $\underline{d} \in LE(n_0, n)$ and the extra condition.

VG: (distant from 0) The distance of I_{n+1}^j to 0 is bigger than $c_n^{n^2} |I_{n+1}|$.

And we define $B(n_0, n+1)$ as the set of $j \notin VG(n_0, n+1)$ such that $R_n(I_{n+1}^j) = C_n^{\underline{d}}$ with $\underline{d} \notin LF(n)$.

Lemma 8.2. *With total probability, for all n_0 sufficiently big,*

1. $p_{\gamma_n}(j^{(n)}(x) \notin VG(n_0,n) | x \in I_n) < c_{n-1}^{a^2}$,
2. $p_{\gamma_n}(j^{(n)}(x) \in B(n_0,n) | x \in I_n) < 2c_{n-1}^{2n}$,
3. $p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LE(n_0,n) | x \in I_n) < c_n^{2a/5}$,
4. $p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LE(n_0,n) \cup LF(n) | x \in I_n) < c_n^{bn}$,
5. $p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LE(n_0,n) | x \in I_n^{\tau_n}) < c_n^{2a/5}$.

Proof. First notice that the validity for a given value of n of 1 and 2 implies 3, 4 and 5, using the large deviation technique of §6.2. More precisely, we estimate the complement

LE1: let $q = 6 \cdot 2^n c_{n-1}^{a^2}$,

$$\sum_{k > c_{n-1}^{-2b}} \left(\frac{1}{2}\right)^{qk} \ll c_n^{n^2},$$

LE2: let $q = 6 \cdot 2^n c_{n-1}^{n}$,

$$\sum_{k > c_n^{-1/n}} \left(\frac{1}{2}\right)^{qk} \ll c_n^{n^2}.$$

The validity of 3 and 4 imply 1 and 2 for $n+1$ by pulling back, using Remark 6.2. The proof then follows by induction (the first step is trivial). $\square$

Using PhPa2 we get

Lemma 8.3. *With total probability, for all n_0 big enough, for all n big enough, $\tau_n \in VG(n_0,n)$.*

We also have the following trivial estimate for a very good return time

Lemma 8.4. *With total probability, for all n_0 big enough and for all $n \geq n_0$, if $j \in VG(n_0,n+1)$ then*

$$m < r_{n+1}(j) < m c_{n-1}^{-4b},$$

where as usual, m is such that $R_n(I_{n+1}^j) = C_n^{\underline{d}}$ and $\underline{d} = (j_1, ..., j_m)$.

Proof. The estimate from below is obvious, the estimate from above follows from LS2 and the estimate on v_n. $\square$

And the following estimate for returns which are not very good or bad

Lemma 8.5. *With total probability for all n_0 sufficiently big, if $n > n_0$, if $j \notin VG(n_0,n) \cup B(n_0,n)$ then $r_n(j) < c_{n-1}^{-a/2} c_{n-2}^{-4b}$.*

Proof. If $j \notin VG(n_0, n) \cup B(n_0, n)$ then $R_{n-1}(I_n^j) \in LF(n_0, n-1)$. The estimate follows since a branch in $LF(n_0, n-1)$ has time bounded by $c_{n-1}^{a/2} c_{n-2}^{-3b}$ and $v_{n-1} < c_{n-2}^{-b}$. □

Let $j \in VG(n_0, n+1)$. We can write $R_{n+1}|_{I_{n+1}^j} = f^{r_{n+1}(j)}$, that is, a big iterate of f. One may consider which proportion of those iterates belong to very good branches of the previous level. More generally, we can truncate the return R_{n+1}, that is, we may consider $k < r_{n+1}(j)$ and ask which proportion of iterates up to k belong to very good branches.

Lemma 8.6. *With total probability, for all n_0 big enough and for all $n \geq n_0$, the following holds.*

Let $j \in VG(n_0, n+1)$, as usual let $R_n(I_{n+1}^j) = I_n^{\underline{d}}$ and $\underline{d} = (j_1, ..., j_m)$. Let m_k be biggest possible with

$$v_n + \sum_{j=1}^{m_k} r_n(j_i) \leq k$$

(the amount of full returns to level n before time k) and let

$$\beta_k = \sum_{\substack{1 \leq i \leq m_k, \\ j_i \in VG(n_0, n)}} r_n(j_i).$$

(the total time spent in full returns to level n which are very good before time k) Then $1 - \beta_k/k < c_{n-1}^{a^2/3}$ if $k > c_n^{-2/n}$.

Proof. Let's estimate first the time i_k which is not spent on full returns:

$$i_k = k - \sum_{j=1}^{m_k} r_n(j_i).$$

This corresponds exactly to v_n plus some incomplete part of the return $j_{m_{k+1}}$. This part can be bounded by $c_{n-1}^{-b} + c_{n-1}^{-3b}$ (use the estimate of v_n and LS2 to estimate the incomplete part).

Using LS2 we conclude now that

$$m_k > (k - c_{n-1}^{-b}) c_{n-1}^{3b} > c_n^{-1/n}$$

so m_k is not too small.

Let's now estimate the contribution h_k from bad full returns j_i. The number of such returns must be less than $c_{n-1}^{n/2} m_k$, using LS2 their total time is at most $c_{n-1}^{(n/2)-3b} m_k < m_k$.

The non very good full returns on the other hand can be estimated by LE1 (given the estimate on m_k), they are at most $c_{n-1}^{a^2} m_k$. So we

can estimate the total time l_k of non very good full returns with time less then $c_{n-1}^{-a/2} c_{n-2}^{-4b}$ by
$$c_{n-1}^{a^2} c_{n-1}^{-a/2} c_{n-2}^{-4b} m_k,$$
while β_k can be estimated from below by
$$(1 - c_{n-1}^{a/4}) c_{n-1}^{-a/2} m_k.$$

It is easy to see then that $i_k/\beta_k \ll c_{n-1}^{a/5}$, $h_k/\beta_k \ll c_{n-1}^{a/5}$. We also have
$$l_k/\beta_k < 2c_{n-1}^{a^2/2}.$$
So $(i_k + h_k + l_k)/\beta_k$ is less then $c_{n-1}^{a^2/3}$. Since $i_k + h_k + l_k + \beta_k = k$ we have $1 - \beta_k/k < (i_k + h_k + l_k)/\beta_k$. ☐

8.1.3. *Cool landings.* Let's define the set of cool landings $LC(n_0, n) \subset \Omega$, $n_0, n \in \mathbb{N}$, $n \geq n_0$ as the set of all $\underline{d} = (j_1, ..., j_m)$ in $LE(n_0, n)$ satisfying.

LC1: (Starts very good) $j_i \in VG(n_0, n)$, $1 \leq i \leq c_{n-1}^{-a^2/2}$.

LC2: (Short times are sparse in not too small initial segments) For $c_{n-1}^{-a/2} \leq k \leq m$
$$\#\{r_n(j_i) < c_{n-1}^{-a/2}\} \cap \{1, ..., k\} < (6 \cdot 2^n) c_{n-1}^{a/3} k,$$

LC3: (Not very good moments are sparse in not too small initial segments) For all $c_{n-1}^{-a^2/4} < k \leq m$
$$\#\{i | j_i \notin VG(n_0, n)\} \cap \{1, ..., k\} < (6 \cdot 2^n) c_{n-1}^{a^2} k,$$

LC4: (Bad times are sparse in not too small initial segments) For $c_{n-1}^{-n/3} \leq k \leq m$
$$\#\{i | j_i \in B(n_0, n)\} \cap \{1, ..., k\} < (6 \cdot 2^n) c_{n-1}^{n/6} k,$$

LC5: (Starts with no bad times) $j_i \notin B(n_0, n)$, $1 \leq i \leq c_{n-1}^{-n/2}$.

As usual we obtain

Lemma 8.7. *With total probability, for all n_0 sufficiently big and all $n \geq n_0$,*
$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LC(n_0, n) | x \in I_n) < c_{n-1}^{a^2/3}$$
and for all n big enough
$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LC(n_0, n) | x \in I_n^{\tau_n}) < c_{n-1}^{a^2/3}.$$

Proof. **LC1:** $2^n c_{n-1}^{a^2/2} c_{n-1}^{a^2} < c_{n-1}^{a^2/3}$,

LC2: let $q = 6 \cdot 2^n c_{n-1}^{a/3}$,

$$\sum_{k > c_n^{-a/2}} \left(\frac{1}{2}\right)^{qk} \ll c_{n-1}^{n^2}.$$

LC3: let $q = 6 \cdot 2^n c_{n-1}^{a^2}$,

$$\sum_{k > c_{n-1}^{-a^2/4}} \left(\frac{1}{2}\right)^{qk} \ll c_{n-1}^{n^2},$$

LC4: let $q = 6 \cdot 2^n c_{n-1}^{n/6}$,

$$\sum_{k > c_n^{-n/3}} \left(\frac{1}{2}\right)^{qk} \ll c_n^{n^2}.$$

LC5: $2^n c_{n-1}^{-n/2} c_{n-1}^{2n} \ll c_{n-1}^{n}$.

This gives the first estimate, for the second we must use Lemma 8.3 and the estimates on $r_n(\tau_n)$.

□

Using PhPa1 we get

Lemma 8.8. *With total probability, for all n_0 big enough, for all n big enough we have $R_n(0) \in LC(n_0, n)$.*

8.2. Hyperbolicity.

8.2.1. *Preliminaries.* For $j \neq 0$, we define

$$\lambda_n^{(j)} = \inf_{x \in I_n^j} \frac{\ln(|R_n'(x)|)}{r_n(j)}.$$

And $\lambda_n = \inf_{j \neq 0} \lambda_n^{(j)}$. As a consequence of the exponential estimate on distortion, together with hyperbolicity of f in the complement of I_n^0 we immediately have the following.

Lemma 8.9. *With total probability, for all n sufficiently big, $\lambda_n > 0$.*

8.2.2. *Good branches.* We define the set of good returns $G(n_0, n) \subset \mathbb{Z} \setminus \{0\}$, $n_0, n \in \mathbb{N}$, $n \geq n_0$ as the set of all j such that

G1: (hyperbolic return)

$$\lambda_n^{(j)} \geq \lambda_{n_0} \frac{1 + 2^{n_0 - n}}{2},$$

G2: (hyperbolicity in partial return) for $c_{n-1}^{-3/(n-1)} \leq k \leq r_n(j)$ we have

$$\inf_{I_n^j} \frac{\ln(|Df^k|)}{k} \geq \lambda_{n_0} \frac{1+2^{n_0-n+1/2}}{2} - c_n^{2/(n-1)}.$$

Notice that since c_n decreases torrentially, for n sufficiently big G2 implies for $c_{n-1}^{-3/(n-1)} \leq k \leq r_n(j)$ we have

$$\inf_{I_n^j} \frac{\ln(|Df^k|)}{k} \geq \lambda_{n_0} \frac{1+2^{n_0-n}}{2}.$$

Lemma 8.10. *With total probability, for n_0 big enough, for all $n > n_0$, $VG(n_0, n) \subset G(n_0, n)$.*

Proof. Let's prove that if G1 is satisfied for all $j \in VG(n_0, n)$, then $VG(n_0, n+1) \subset G(n_0, n+1)$. Let's fix such a j. Notice that by definition of λ_{n_0} the hypothesis is satisfied for n_0.

Let $a_k = \inf_{I_n^j} \ln(|Df^k|)/k$.

Recall the estimate from Lemma 8.4 for $t = r_{n+1}(j)$, we as usually let $R_n(I_{n+1}^j) = C_n^{\underline{d}}$, $\underline{d} = (j_1, ..., j_m)$. Let's say that j_i was completed before k if $v_n + r_n(j_1) + ... + r_n(j_i) \leq k$. We let the queue be defined as

$$q_k = \inf_{C_n^{\underline{d}}} \ln(|Df^{k-r} \circ f^r|)$$

where $r = v_n + r_n(j_1) + ... + r_n(j_{m_k})$ with j_{m_k} the last complete return.

Recall that $v_n < c_{n-1}^{-b}$ and from VG it is clear $|R_n'|_{I_{n+1}^j}| > c_n^{n^2}$. Notice also that using Lemma 5.9, for any $k_0 \leq m$, the derivative of $R_n^{k_0}$ in $C_n^{\underline{d}}$ is at least 2^{k_0}. So for $m_0 = c_{n-1}^{-2b}$ we have that the derivative of $R_n^{k_0+1}$ in I_{n+1}^j is at least 1. Notice that from LS2

$$k_0 = \sum_{i=1}^{m_0} r_n(j_i) < c_{n-1}^{-2b} c_{n-1}^{-3b} \ll k.$$

It is clear that any complete return before k gives derivative at least 1 from Lemma 5.9. The queue can be bound by $\ln(c_n c_{n-1}^3)$ using Lemma 5.10. We have $-q_k/k \ll c_n^{2/n}$.

Now we use Lemma 8.6 and get

$$\begin{aligned} a_k &> \frac{\beta_k - k_0}{k} \frac{\lambda_{n_0}(1+2^{n_0-n})}{2} - \frac{-q_k}{k} \\ &\geq \frac{\lambda_{n_0}(1+2^{n_0-n-1/2})}{2} - \frac{-q_k}{k}. \end{aligned}$$

Which gives G2. If $k = r_{n+1}(j)$, $q_k = 0$ which gives G1. ∎

8.2.3. *Hyperbolicity in cool landings.*

Lemma 8.11. *With total probability, if n_0 is sufficiently big, for all n sufficiently big, if $\underline{d} \in LC(n_0, n+1)$ then for all $c_{n-1}^{-4/(n-1)} < k \leq l_n(\underline{d})$,*

$$\inf_{C_n^{\underline{d}}} \frac{\ln(|Df^k|)}{k} \geq \frac{\lambda_{n_0}}{2}.$$

Proof. Fix such $\underline{d} \in LC(n_0, n+1)$.

Let

$$a_k = \inf_{C_n^{\underline{d}}} \frac{\ln(|Df^k|)}{k}.$$

Consider the sequence $r_i = r_n(j_i)$ where as usual $\underline{d} = (j_1, ..., j_m)$.

As in Lemma 8.6, we define m_k as the biggest such that

$$\sum_{i=1}^{m_k} r_i \leq k.$$

We define

$$\beta_k = \sum_{\substack{1 \leq i \leq m_k, \\ j_i \in VG(n_0, n+1)}} r_i,$$

(counting the time up to k spent in complete very good returns) and

$$i_k = k - \sum_{i=1}^{m_k} r_i.$$

(counting the time in the incomplete return at k).

Let's then consider two cases: small m_k ($m_k < c_{n-1}^{-a^2/2}$) and otherwise.

The idea of the first case is that all full returns are very good by LC1, and the incomplete time is also part of a very good return.

Since full very good returns are very hyperbolic by G1 and very good returns are good, we just have to worry with possibly losing hyperbolicity in the incomplete time. To control this, we introduce the queue $q_k = \inf_{C_{n-1}^{\underline{d}}} \ln(|Df^{i_k} \circ f^{k-i_k}|)$. We have $-q_k < -\ln(c_{n-1}^{1/3} c_{n-1}^3)$ by Lemma 5.10

If the incomplete time is big (more than $c_{n-1}^{-4/(n-1)}$), we can use G2 to estimate the hyperbolicity of the incomplete time (which is part of a very good return). The reader can easily check the estimate in this case.

If the incomplete time is not big, we can not use G2 to estimate q_k, but in this case i_k is much less than k: since $k > c_{n-1}^{-4/(n-1)}$, at least one return was completed ($m_k \geq 1$), and since it must be very good we conclude that $k > c_{n-1}^{-a/2}$ by LS1, so

$$a_k > \lambda_{n_0} \frac{(1 + 2^{n-n_0})}{2} \cdot \frac{k - i_k}{k} - \frac{-q_k}{k} > \frac{\lambda_{n_0}}{2}.$$

Let's consider now the case $m_k > c_{n-1}^{-a^2/2}$. For an incomplete time we still have $-q_k < -\ln(c_n c_{n-1}^3)$, so $-q_k/k < c_{n-1}^{-a^2/3}$.

Arguing as in Lemma 8.6, we split $k - \beta_k - i_k$ (time of full returns which are not very good) in part relative to bad returns h_k and in part relative to fast returns (not very good or bad) l_k. Using LC4 and LC5 we get

$$h_k < c_{n-1}^{-3b} c_{n-1}^{n/7} m_k,$$

and using LC1 and LC3 we have

$$l_k < c_{n-1}^{-a/2} c_{n-2}^{-4b} (6 \cdot 2^n) c_{n-1}^{a^2} m_k,$$

By LC1 and LC3 we have $\beta_k > c_{n-1}^{-a/2} m_k/2$ so we have

$$\frac{h_k + l_k}{\beta_k} < c_{n-1}^{a^2/2}.$$

By LC1, $\beta_k > c_{n-1}^{a/2} m_k$. Now $\beta_k > c_{n-1}^{-a/2} c_{n-1}^{-a^2}/2$ by hypothesis on m_k. If $i_k < c_{n-1}^{-a/2} c_{n-2}^{-4b}$, i_k/β_k is small and we are done. Otherwise by Lemma 8.5, i_k must be either very good or bad. If i_k is very good we can reason as before that $G2$ applies for the estimate of the queue and we are done.

If i_k is bad, by LC5 we have that $m_k > c_{n-1}^{n/2}$, but $i_k < c_{n-1}^{3b}$ by LS2, so i_k/β_k is very small again and we are done. $\square$

9. Main Theorems

9.1. **Proof of Theorem A.** We must show that with total probability, f is Collet-Eckmann. We will use the estimates on hyperbolicity of cool landings to show that if the critical point always fall in a cool landing then there is uniform control of the hyperbolicity along the critical orbit.

Let

$$a_k = \frac{\ln(|Df^k(f(0))|)}{k}$$

and $e_n = a_{v_n - 1}$.

It is easy to see that if n_0 is big enough such that both Lemmas 8.8 and 8.10 are valid and using Lemma 5.7 so that $|R_n(0)| > |I_n|/2^n$ for

n big enough, we obtain using Lemma 8.11 that

$$e_{n+1} \geq e_n \frac{v_n - 1}{v_{n+1} - 1} + \frac{\lambda_{n_0}}{2} \frac{v_{n+1} - v_n}{v_{n+1} - 1}$$

and so

$$\liminf e_n \geq \frac{\lambda_{n_0}}{2}.$$

Let now $v_n - 1 < k < v_{n+1} - 1$. Define

$$q_k = \ln(|Df^{k-v_n}(f^{v_n}(0))|).$$

If $k < v_n + c_{n-1}^{-4/(n+1)}$ from LC1 and LS1 we know that the time of the R_n branch of $R_n(0)$ is at least $c_{n-1}^{-a/2}$, so k is in the middle of this branch. Using $|R_n(0)| > |I_n|/2^n$, we get that $-q_k < -\ln(2^{-n} c_{n-1} c_{n-1}^{-3})$. We then get from $v_n > c_{n-1}^{-a}$ that

$$a_k \geq e_n \frac{v_n - 1}{k} - \frac{-q_k}{k} > (1 - 1/2^n) e_n - 1/2^n.$$

If $k > v_n + c_{n-1}^{-4/(n+1)}$ using Lemma 8.11 we get

$$a_k \geq e_n \frac{v_n - 1}{k} + \frac{\lambda_{n_0}}{2} \cdot \frac{k - v_n + 1}{k}.$$

Those two estimates imply that $\liminf a_k \geq \lambda_{n_0}/2$ and so f is Collet-Eckmann.

9.2. **Proof of Theorem B.** We must obtain, with total probability, upper and lower (polynomial) bounds for the recurrence of the critical orbit. It will be easier to first study the recurrence with respect to iterates of return branches, and then estimate the total time of those iterates.

9.2.1. *Recurrence in terms of return branches.*

Lemma 9.1. *With total probability, for n big enough and for $1 \leq i \leq c_{n-1}^{-2}$,*

$$\frac{\ln(|R_n^i(0)|)}{\ln(c_{n-1})} < b^2.$$

Proof. Notice that due to torrential (and monotonic) decay of c_n, we can estimate $|I_n| = c_{n-1}^{1+\delta_n}$, with δ_n decaying torrentially fast.

For $i = 1$ it follows from Lemma 5.7.

Let $X \subset I_n$ be a $c_{n-1}^{b^2}$ neighborhood of 0. For n big, we can estimate (due to the relation between $|I_n|$ and c_{n-1})

$$\frac{|X|}{|I_n|} < c_{n-1}^{b^2-2}$$

(we of course consider X_n a union of $C^{\underline{d}}_n$, so that its size is near the required size, the precision is high enough for our purposes due to Lemma 5.3).

We have to make sure that the critical point does not land in X for $1 < i \leq c_{n-1}^{-2}$. This requirement can be translated on $R_n(0)$ not belonging to a certain set $Y \subset I_n$ such that

$$Y = \bigcup_{1 \leq |\underline{d}| < c_{n-1}^{-2}} (R^{\underline{d}}_n)^{-1}(X).$$

It is clear that

$$p_\gamma(I_n^{\tau_n} \cap Y | I_n^{\tau_n}) \leq c_{n-1}^{-2} c_{n-1}^{(b^2-2)/b} < c_{n-1}^{b-3}.$$

Applying PhPa1, the probability that for $1 \leq i \leq c_{n-1}^{-2}$ we have $|R_n^i(0)| < c_{n-1}^{b^2}$ is at most c_{n-1}^{b-3}, which is summable. $\square$

Lemma 9.2. *With total probability, for n big enough and for $c_{n-1}^{-2} < i \leq s_n$,*

$$\frac{\ln(|R_n^i(0)|}{\ln(c_{n-1}^{-1})} < b^2 \left(1 + \frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right).$$

Proof. The argument is the same as for the previous Lemma, but the decomposition has a slight different geometry.

For $j \geq 0$, let $X_j \subset I_n$ be a

$$c_{n-1}^{b^2(b^j+1)}$$

neighborhood of 0 (approximated as union of $C^{\underline{d}}_n$, if $X_j \subset I_{n+1}$, it won't be relevant for the proof, the reader can take then $X_j = \emptyset$). Let $Y_j \subset I_n$ be such that

$$Y_j = \bigcup_{c_{n-1}^{-b^j} \leq |\underline{d}| < c_{n-1}^{-b^{j+1}}} (R^{\underline{d}}_n)^{-1}(X_j).$$

It is clear that

$$p_\gamma(I_n^{\tau_n} \cap Y_j | I_n^{\tau_n}) \leq c_{n-1}^{-b^{j+1}} c_{n-1}^{b^{j+2}} < c_{n-1}^{b^{j+1}}$$

and

$$p_\gamma(I_n^{\tau_n} \cap \cup Y_j | I_n^{\tau_n}) < \sum_{j=0}^{\infty} c_{n-1}^{-b^j} < 2c_{n-1}.$$

Applying PhPa1, with total probability, the critical point does not belong to any Y_j. Let's see what this means. If $c_{n-1}^{-b^j} < i \leq c_{n-1}^{-b^{j+1}}$, $R_n^i(0) \notin X_j$, that is $|R_n^i(0)| > c_{n-1}^{b^{j+3}}$. This clearly implies the statement. $\square$

Corollary 9.3. *With total probability, for n big enough and for $1 \leq i \leq s_n$,*

$$\frac{\ln(|R_n^i(0)|)}{\ln(c_{n-1})} < b^2 \left(1 + \frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right).$$

9.2.2. *Total time of full returns.* For $1 \leq i \leq s_n$, let k_i such that $R_n^i|_{I_{n+2}} = f^{k_i}$.

Lemma 9.4. *With total probability, for n big enough and for $c_{n-1}^{-1} < i < s_n$, $k_i/i > c_{n-1}^{-a/2}/2$.*

Proof. This follows from condition LC2 and Lemma 8.8. □

Using that $v_n > c_{n-1}^{-a}$ and $k_i > v_n$ we get

Corollary 9.5. *With total probability, for n big enough and for $1 \leq i \leq s_n$,*

$$\frac{\ln(k_i)}{\ln(c_{n-1}^{-1})} > a/3 \left(1 + \frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right).$$

9.2.3. *Upper and lower bounds.* Considering $|R_n(0)| = |f^{v_n}(0)| < c_{n-1}$ and using $v_n < c_{n-1}^{-b}$ we get

$$\limsup_{n \to \infty} \frac{-\ln |f^n(0)|}{\ln(n)} > a.$$

Let now $v_n \leq k < v_{n+1}$. If $|f^k(0)| < k^{-3b^3}$ we have $f^k(0) \in I_n$ and so $k = k_i$ for some i. It follows from Corollaries 9.3 and 9.5 that

$$|f^k(0)| > k^{-3b^3}.$$

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IMPA – Estr. D. Castorina 110, 22460-320 Rio de Janeiro – Brazil.
E-mail address: `avila@impa.br, gugu@impa.br`