STABLE INTERSECTIONS OF REGULAR CANTOR SETS WITH LARGE HAUSDORFF DIMENSIONS

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Abstract: In this paper we prove a conjecture by J. Palis according to which the arithmetic difference of generic pairs of regular Cantor sets on the line either has zero Lebesgue measure or contains an interval. More precisely, we prove that if the sum of the Hausdorff dimensions of two regular Cantor sets is bigger than one then, in almost all cases, there are translations of them stably having intersection with positive Hausdorff dimension.

Introduction

Regular Cantor sets on the line play a fundamental role in dynamical systems and notably also in some problems in number theory. They are defined by expansive maps and have some kind of self-similarity property: small parts of them are diffeomorphic to big parts with uniformly bounded distortion (see precise definition in section I). In both settings, dynamics and number theory, a key question is whether the arithmetic difference of two such sets contains an interval when the sum of their Hausdorff dimensions is bigger than one. Some background on regular Cantor sets which are relevant to our work can be found in [PT2] and [Su].

From the dynamics side, in 1983, J. Palis and F. Takens ([PT], [PT1]) proved a the-

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orem about homoclinic bifurcations associated to a basic set that assures full density of hyperbolicity in the parameter family provided that the Hausdorff dimension of the basic set is smaller than one. A central fact used in the proof is that if K_1 and K_2 are regular Cantor sets on the real line such that the sum of their Hausdorff dimensions is smaller than one, then $K_1 - K_2 = \{x - y \mid x \in K_1, y \in K_2\}$ (the arithmetic difference between K_1 and K_2) is a set of zero Lebesgue measure (indeed of Hausdorff dimension smaller than 1). In the same year, looking for some kind of converse of this result Palis conjectured (see [P], [P1]) that for generic pairs of regular Cantor sets (K_1, K_2) of the real line either $K_1 - K_2$ has zero measure or else it contains an interval (the last statement should correspond in homoclinic bifurcations to open sets of tangencies). A slightly stronger startement is that, if K_1 and K_2 are generic regular Cantor sets and the sum of their Hausdorff dimensions is bigger than 1, then $K_1 - K_2$ contains intervals.

Another motivation for the conjecture was Newhouse's work in the seventies ([N1], [N2], [N3]), when he introduced the concept of thickness of a regular Cantor set. This allowed him to exhibit examples of pairs of Cantor sets whose arithmetic difference stably is an interval, and to show that after a homoclinic bifurcation there are always open sets of stable tangencies in the parameter line. Is to be noted that in general (as in the [PT1] case above) these sets can have zero density.

Some previous works related to Palis' conjecture are [S], [BPV], [BPMV], [L], [MO] and [PY].

An earlier and totally independent development had taken place in number theory. In 1947, M. Hall ([H]) proved that any real number can be written as the sum of two numbers whose continued fractions coefficients (of positive index) are at most 4. More precisely, $C(4)$ is the regular subset formed of such numbers in $[0,1]$, one has $C(4)+C(4) = [\sqrt{2}-1, 4(\sqrt{2}-1)]$. It follows that the Lagrange spectrum $L = \{\limsup\limits_{\substack{p,q\to\infty \\ p,q\in\mathbb{N}}} (q|q\alpha - p|)^{-1},\, \alpha \in \mathbb{R}\backslash\mathbb{Q}\}$ contains a whole half-line (for instance $[6, +\infty)$). In 1975, G. Freiman ([F], see also [CF])proved some difficult results showing that the arithmetic sum of certain (regular) Cantor sets related to continued fractions contain intervals, and used them to determine the precise beginning of Hall's ray (the biggest half-line contained in L), which is

$$\frac{2221564096 + 283748\sqrt{462}}{491993569} \cong 4,52782956616\ldots.$$

In 1993, the concept of stable intersection of two regular Cantor sets was introduced (see [M]): two Cantor sets K_1 and K_2 have stable intersection if there is a neighbourhood V of (K_1, K_2) in the set of pairs of C^{1+}-regular Cantor sets (see section 2 for a definition of the topology) such that $(\widetilde{K}_1, \widetilde{K}_2) \in V \Rightarrow \widetilde{K}_1 \cap \widetilde{K}_2 \neq \emptyset$. In the same paper conditions based on renormalizations were introduced to ensure stable intersections, and applications of stable intersections to homoclinic bifurcations were obtained, namely positive density of stable tangencies at the initial bifurcation point. It was also shown that this last phenomena can coexist with positive density of hyperbolicity in a persistent way. Besides, the following question is posed: Does there exist a dense (and automatically open) set $U \subset \Omega^\infty = \{(K_1, K_2),\ K_1$ and K_2 C^k-regular Cantor sets $\mid HD(K_1) + HD(K_2) > 1\}$ such that $(K_1, K_2) \in U \Rightarrow \exists t \in \mathbb{R}$ st. $(K_1, K_2 + t)$ has stable intersection?

A positive answer to this question implies a strong version of Palis' conjecture. In the present work we will provided an affirmative answer to it, by proving (see section §2.4) the following

Theorem: There is an open and dense set $U \subset \Omega^\infty$ such that if $(K_1, K_2) \in U$, then $I_s(K_1, K_2)$ is dense in $K_1 - K_2$ and $HD((K_1 - K_2) \backslash I_s(K_1, K_2)) < 1$, where $I_s(K_1, K_2) := \{t \in \mathbb{R} \mid K_1$ and $(K_2 + t)$ have stable intersection$\}$.

We will also prove that the same result works if we replace stable intersection by d-stable intersection, which is defined by asking that any pair $(\widetilde{K}_1, \widetilde{K}_2)$ in some neighbourhood of (K_1, K_2) satisfies $HD(\widetilde{K}_1 \cap \widetilde{K}_2) \geq d$: most pairs of Cantor sets $(K_1, K_2) \in \Omega^\infty$ have d-stable intersection for any $d < HD(K_1) + HD(K_1) - 1$ (see section §2.6).

Moreover, we will show that the open set U mentioned in the theorem above is very large in Ω^∞ in the sense that generic n-parameter families in Ω^∞ are actually contained in U (see section §3.9.1).

In a forthcoming paper, we prove the following fact concerning generic homoclinic bifurcations associated to two dimensional saddle-type hyperbolic sets (horseshoes) with Hausdorff dimension bigger than one: it yield open sets of stable tangencies in the parameter line with positive density at the initial bifurcation value. Moreover, the union of this set with the hyperbolicity set in the parameter line generically has full density at the initial bifurcation value. This extends a theorem by Palis and Yoccoz ([PY]).

§1. Regular Cantor sets: the setting.

1.1 Let $\mathbb{A}$ be a finite alphabet, $\mathcal{B}$ a subset of $\mathbb{A}^2$, and Σ the subshift of finite type of $\mathbb{A}^{\mathbb{Z}}$ with allowed transitions $\mathcal{B}$.

We will always assume that Σ is topologically mixing, and that every letter in $\mathbb{A}$ occurs in Σ.

An *expansive map of type* Σ is a map g with the following properties:

(i) the domain of g is a disjoint union $\bigcup_{\mathcal{B}} I(a,b)$, where, for each (a,b), $I(a,b)$ is a compact subinterval of $I(a) := [0,1] \times \{a\}$;

(ii) for each $(a,b) \in \mathcal{B}$, the restriction of g to $I(a,b)$ is a smooth diffeomorphism onto $I(b)$ satisfying $|Dg(t)| > 1$ for all t.

The *regular Cantor set* associated to g is the maximal invariant set

$$K = \bigcap_{n \geq 0} g^{-n}\left(\bigcup_{\mathcal{B}} I(a,b)\right).$$

Let Σ^+ be the unilateral subshift associated to Σ. There exists a unique homeomorphism $h: \Sigma^+ \to K$ such that

$$h(\underline{a}) \in I(a_0), \text{ for } \underline{a} = (a_0, a_1, \dots) \in \Sigma^+$$
$$h \circ \sigma = g \circ h.$$

Let r be a real number > 1, or $r = +\infty$. The space of C^r expansive maps of type Σ, endowed with the C^r topology, will be denoted by Ω^r_Σ. The union $\Omega_\Sigma = \bigcup_{r>1} \Omega^r_\Sigma$ is endowed with the inductive limit topology.

1.2 For $(a, b) \in \mathcal{B}$, let

$$f_{a,b} = \left[g\big|_{I(a,b)}\right]^{-1};$$

this is a contracting diffeomorphism from $I(b)$ onto $I(a, b)$. If $\underline{a} = (a_0, \dots, a_n)$ is a word of Σ, we put

$$f_{\underline{a}} = f_{a_0,a_1} \circ \cdots \circ f_{a_{n-1}a_n};$$

this is a diffeomorphism from $I(a_n)$ onto a subinterval of $I(a_0)$ that we denote by $I(\underline{a})$.

1.3 Let $\Sigma^- = \{(\theta_n)_{n \le 0}, (\theta_i, \theta_{i+1}) \in \mathcal{B} \text{ for } i < 0\}$. We equip Σ^- with the following ultrametric distance: for $\underline{\theta} \neq \underline{\widetilde{\theta}} \in \Sigma^-$, set

$$d(\underline{\theta}, \underline{\widetilde{\theta}}) = \begin{cases} 1 \quad \text{if} \quad \theta_0 \neq \widetilde{\theta}_0 \\ |I(\underline{\theta} \wedge \underline{\widetilde{\theta}})| \quad \text{otherwise} \end{cases}$$

where $\underline{\theta} \wedge \underline{\widetilde{\theta}} = (\theta_{-n}, \dots, \theta_0)$ if $\widetilde{\theta}_{-j} = \theta_{-j}$ for $0 \le j \le n$ and $\widetilde{\theta}_{-n-1} \neq \theta_{-n-1}$.

Now, let $\underline{\theta} \in \Sigma^-$; for $n > 0$, let $\underline{\theta}^n = (\theta_{-n}, \dots, \theta_0)$, and let $B(\underline{\theta}^n)$ be the affine map from $I(\underline{\theta}^n)$ onto $I(\theta_0)$ such that the diffeomorphism $k_n^{\underline{\theta}} = B(\underline{\theta}^n) \circ f_{\underline{\theta}^n}$ is orientation preserving.

We have the following well-known result (see [Su]):

Proposition. Let $r \in (1, +\infty)$, $g \in \Omega^r_\Sigma$.

1. For any $\underline{\theta} \in \Sigma^-$, there is a diffeomorphism $k^{\underline{\theta}} \in \mathrm{Diff}^r_+(I(\theta_0))$ such that $k^{\underline{\theta}}_n$ converge to $k^{\underline{\theta}}$ in $\mathrm{Diff}^{r'}_+(I(\theta_0))$, for any $r' < r$, uniformly in $\underline{\theta}$. The convergence is also uniform in a neighbourhood of g in Ω^r_Σ.

2. If r is an integer, or $r = +\infty$, $k^{\underline{\theta}}_n$ converge to $k^{\underline{\theta}}$ in $\mathrm{Diff}^r_+(I(\theta_0))$. More precisely, for every $0 \le j \le r-1$, there is a constant C_j (independent on $\underline{\theta}$ such that

$$|D^j \ \mathrm{Log}\ D\big[k^{\underline{\theta}}_n \circ (k^{\underline{\theta}})^{-1}\big](x)| \le C_j |I(\underline{\theta}^n)|.$$

It follows that $\underline{\theta} \to k^{\underline{\theta}}$ is Lipschitz in the following sense: for $\theta_0 = \widetilde{\theta}_0$, we have

$$|D^j \ \mathrm{Log}\ D\big[k^{\underline{\widetilde{\theta}}} \circ (k^{\underline{\theta}})^{-1}\big](x)| \le C_j \, d(\underline{\theta}, \underline{\widetilde{\theta}}).$$

1.4 For $\underline{\theta} \in \Sigma^-$ and $\underline{a}$ a word in Σ starting with $a_0 = \theta_0$, we denote:

$$K(\underline{a}) = K \cap I(\underline{a}),$$

$$I^{\underline{\theta}}(\underline{a}) = k^{\underline{\theta}}(I(\underline{a})),$$

$$K^{\underline{\theta}}(\underline{a}) = k^{\underline{\theta}}(K(\underline{a})).$$

The different $K^{\underline{\theta}}$ are related in the following way: let $F^{\underline{\theta}}$ be the affine map from $I(\theta_0)$ onto $I^{\sigma^{-1}(\underline{\theta})}(\theta_{-1}, \theta_0)$ with the same orientation than $f_{\theta_{-1}\theta_0}$. Then, on $I(\theta_0)$:

$$F^{\underline{\theta}} \circ k^{\underline{\theta}} = k^{\sigma^{-1}(\underline{\theta})} \circ f_{\theta_{-1}\theta_0}.$$

If we define

$$\mathcal{I} = \{(\underline{\theta}, x), \underline{\theta} \in \Sigma^-, x \in I(\theta_0)\}$$

$$\mathcal{K} = \{(\underline{\theta}, x), \underline{\theta} \in \Sigma^-, x \in K^{\underline{\theta}}(\theta_0)\}$$

$$F(\underline{\theta}, x) = (\sigma^{-1}(\underline{\theta}), F^{\underline{\theta}}(x)),$$

we have a fiber-contracting map F such that

$$\mathcal{K} = \bigcap_{n \ge 0} F^n(\mathcal{I}).$$

Moreover, if we define $H: \Sigma \to \mathcal{K}$ by

$$H(\underline{\theta}) = (\pi^-(\underline{\theta}),\ k^{\pi^-(\underline{\theta})} \circ h(\pi^+(\underline{\theta})))$$

where π^-, π^+ are the natural projections of Σ onto Σ^-, Σ^+ respectively, the map H is an homeomorphism which satisfies

$$H \circ \sigma^{-1} = F \circ H.$$

1.5 Let $r \in (1, +\infty]$. For $a \in \mathbb{A}$, denote by $\mathcal{P}^r(a)$ the space of C^r-embeddings of $I(a)$ into $\mathbb{R}$, endowed with the C^r topology. The affine group $\mathrm{Aff}(\mathbb{R})$ acts by composition on the left on $\mathcal{P}^r(a)$, the quotient space being denoted by $\overline{\mathcal{P}}^r(a)$. We also consider $\mathcal{P}(a) = \bigcup\limits_{r>1} \mathcal{P}^r(a)$ and $\overline{\mathcal{P}}(a) = \bigcup\limits_{r>1} \overline{\mathcal{P}}^r(a)$, endowed with the inductive limit topologies.

Let $(a_0, a_1) \in \mathcal{B}$, and $g \in \Omega^r_\Sigma$. We define the renormalization operator

$$T^g_{a_1 a_0}: \mathcal{P}^r(a_0) \to \mathcal{P}^r(a_1)$$
$$h \mapsto h \circ f_{a_0, a_1}.$$

If $\underline{a} = (a_0, \ldots, a_n)$ is a word in Σ, we write $T^g_{\underline{a}}: \mathcal{P}^r(a_0) \to \mathcal{P}^r(a_n)$ for $T^g_{a_n a_{n-1}} \circ \cdots \circ T^g_{a_1, a_0}$. We have

$$T^g_{\underline{a}}(h) = h \circ f_{\underline{a}}.$$

The renormalization operators commute to the action of the affine group, hence they are defined on the quotient spaces $\overline{\mathcal{P}}^r(a)$ as well.

We can reinterpret the proposition above as follows. For $\underline{\theta} \in \Sigma^-$, the sequence $T^g_{\underline{\theta}^n}(h_n)$ ($h_n \in \mathcal{P}^r(\theta_{-n})$) converge (in the $\mathcal{C}^{r'}$ topology, $r' < r$) to a limit in $\overline{\mathcal{P}}^r(\theta_0)$, provided the h_n stay bounded. The limit is independent of the h_n and the convergence is uniform in $\underline{\theta}$ and bounded subsets of the $\mathcal{P}^r(a)$.

On the limit set of the renormalization operators, we can also view the renormalization dynamics as follows. Let $\mathcal{A} = \{(\underline{\theta}, A)\}$, where $\underline{\theta} \in \Sigma^-$ and A is now an *affine* embedding of

$I(\theta_0)$ into $\mathbb{R}$. We have a canonical map

$$\mathcal{A} \to \mathcal{P}^r = \bigcup_{\mathbb{A}} \mathcal{P}^r(a)$$

$$(\underline{\theta}, A) \mapsto A \circ k^{\underline{\theta}} \quad (\in \mathcal{P}^r(\theta_0)).$$

We can lift the action of the renormalization operators to $\mathcal{A}$: if $(\theta_0, \theta_1) \in \mathcal{B}$, we set

$$T_{\theta_1,\theta_0}(\underline{\theta}, A) = (\underline{\theta}\theta_1, A \circ F^{\underline{\theta}\theta_1}).$$

At this level, the various T_{θ_1,θ_0} are just the inverse branches of the map S:

$$S(\underline{\theta}, A) = (\sigma^{-1}(\underline{\theta}), A \circ (F^{\underline{\theta}})^{-1}).$$

The affine group $\mathrm{Aff}(\mathbb{R})$ still acts by composition on the left on $\mathcal{A}$ (fiberwise onto Σ^-).

§2. Intersections of regular Cantor sets: setting and statement of the main result.

2.1 In this section, we assume we are given two sets of data $(\mathbb{A}, \mathcal{B}, \Sigma, g)$, $(\mathbb{A}', \mathcal{B}', \Sigma', g')$ defining regular Cantor sets K, K'.

We define as in §1.5 the spaces $\mathcal{P} = \bigcup_{\mathbb{A}} \mathcal{P}(a)$ and $\mathcal{P}' = \bigcup_{\mathbb{A}'} \mathcal{P}(a')$.

A pair (h, h'), $(h \in \mathcal{P}(a), h' \in \mathcal{P}'(a'))$ is called a *smooth configuration* for $K(a)$, $K'(a')$. Actually, rather than working in the product $\mathcal{P} \times \mathcal{P}'$, it is better to go to the quotient Q by the diagonal action of the affine group $\mathrm{Aff}(\mathbb{R})$. Elements of Q are called *smooth relative configurations* for $K(a)$, $K'(a')$.

We say that a smooth configuration $(h, h') \in \mathcal{P}(a) \times \mathcal{P}(a')$ is

– *linked* if $h(I(a)) \cap h'(I(a')) \neq \emptyset$;

– *intersecting* if $h(K(\underline{a})) \cap h'(K(\underline{a}')) \neq \emptyset$;

– *stably intersecting* if it is still intersecting when we perturb it in $\mathcal{P}\times\mathcal{P}'$, and we perturb (g,g') in $\Omega_\Sigma\times\Omega_{\Sigma'}$.

All these definitions are invariant under the action of the affine group, and therefore make sense for smooth relative configurations.

The following remark is obvious, but underlines the fundamental role played by the renormalization operators.

A smooth relative configuration (h_0,h'_0) is intersecting if and only if there exists a sequence (h_n,h'_n), *relatively compact in* Q, such that (h_{n+1},h'_{n+1}) is obtained from (h_n,h'_n) by applying one of the renormalization operators (associated to g or g'). In particular,this condition implies that the configuration (h_0,h'_0) is linked.

2.2 As in 1.5, we can introduce the spaces $\mathcal{A}$, $\mathcal{A}'$ associated to the limit geometries of g, g' respectively. We denote by $\mathcal{C}$ the quotient of $\mathcal{A}\times\mathcal{A}'$ by the diagonal action on the left of the affine group. An element of $\mathcal{C}$, represented by $(\underline{\theta},A)\in\mathcal{A}$, $(\underline{\theta}',A')\in\mathcal{A}'$, is called a relative configuration of the limit geometries determined by $\underline{\theta}$, $\underline{\theta}'$. We have canonical maps

$$\mathcal{A}\times\mathcal{A}'\to\mathcal{P}\times\mathcal{P}'$$

$$\mathcal{C}\to Q$$

which allow to define linked, intersecting, and stably intersecting configurations at the level of $\mathcal{A}\times\mathcal{A}'$ or $\mathcal{C}$.

Remark: For a configuration $((\underline{\theta},A),(\underline{\theta}',A'))$ of limit geometries, one could also consider the *weaker* notion of stable intersection, obtained by considering perturbations of g, g' in $\Omega_\Sigma\times\Omega_{\Sigma'}$ and perturbations of $(\underline{\theta},A)$, $(\underline{\theta}',A')$ in $\mathcal{A}\times\mathcal{A}'$. We do not know of any example of expansive maps g, g', and configurations $(\underline{\theta},A)$, $(\underline{\theta}',A')$ which are stably intersecting in the weaker sense but not in the stronger sense.

2.3 When looking for stably intersecting configurations, the following notion is crucial.

A compact set $\mathcal{L}$ in $\mathcal{C}$ is *recurrent* if for every $u \in \mathcal{L}$ we can find $v \in \operatorname{int} \mathcal{L}$ and $\ell, \ell' \geq 0$ with $\ell + \ell' > 0$ and $u = S^{\ell} S'^{\ell'}(v)$. In other terms, if $(\underline{\theta}, A)$, $(\underline{\theta}', A')$ represents u, we can find words $\underline{a} = (a_0, \dots, a_\ell)$, $\underline{a}' = (a'_0, \dots, a'_{\ell'})$ in Σ, Σ' respectively, with $a_0 = \theta_0$, $a'_0 = \theta'_0$, such that $(T_{\underline{a}}(\underline{\theta}, A), T'_{\underline{a}'}(\underline{\theta}', A')) = v$ belongs to $\operatorname{int} \mathcal{L}$.

We say that $\mathcal{L}$ is *immediately recurrent* if moreover we can take $\ell + \ell' = 1$ (for any $u \in \mathcal{L}$).

Let $\mathcal{L}$ be a recurrent compact set. Then one can find finitely many compact sets $\mathcal{L}_1, \dots, \mathcal{L}_N$, and words $\underline{a}^i$, $\underline{a}'^i$, $1 \leq i \leq N$ (with, for every i, at least one having more than one letter) such that

(i) $\displaystyle\bigcup_{1 \leq i \leq N} \mathcal{L}_i$ is a neighbourhood of $\mathcal{L}$;

(ii) $T_{\underline{a}^i} \, T'_{\underline{a}'^i}$ is defined on $\mathcal{L}_i$ and sends $\mathcal{L}_i$ into $\operatorname{int} \mathcal{L}$.

From this, we deduce immediately the following properties of recurrent compact sets.

1. Any recurrent compact set is contained in an immediately recurrent compact set.
2. Any recurrent (resp. immediately recurrent) compact set for g, g' is still recurrent (resp. immediately recurrent) for $\tilde{g}, \tilde{g}'$ in a neighbourhood of (g, g') in $\Omega_\Sigma \times \Omega_{\Sigma'}$.
3. Any relative configuration contained in a recurrent compact set is intersecting.

The last property can be strengthened:

Proposition: Any relative configuration (of limit geometries) contained in a recurrent compact set is stably intersecting.

Proof: By property 1 above, we may assume that the given configuration $u = [(\underline{\theta}, A), (\underline{\theta}', A')]$ is contained in an immediately recurrent compact set $\mathcal{L}$. We first show that any small perturbation of $(A \circ k^{\underline{\theta}}, A' \circ k'^{\underline{\theta}'})$ in $\mathcal{P}(\theta_0) \times \mathcal{P}'(\theta'_0)$ is still an intersecting smooth configuration for (g, g'). Such a perturbation can be written $(h \circ A \circ k^{\underline{\theta}}, h' \circ A' \circ k'^{\underline{\theta}'})$, with h, h' close to the identity in some C^r-topology, $r > 1$. Applying one of the renormalization operators, and dividing h into a translation and a translation-free part, we obtain *in* Q $\quad (\tilde{h} \circ \widetilde{A} \circ k^{\tilde{\underline{\theta}}}, \tilde{h}' \circ \widetilde{A}' \circ$

$k'^{\underline{\widetilde{\theta}}'}$), with $\tilde{h}$, $\tilde{h}'$ closer to the identity than were h, h', and $[(\widetilde{A}, \widetilde{\theta}), (\widetilde{A}', \widetilde{\theta}')]$ close (in $\mathcal{C}$) to the image of u under the same renormalization operator.

It follows, by the last remark in 2.1, that this smooth configuration is intersecting.

Finally, the same argument works uniformly in a neighbourhood of (g, g') in $\Omega_\Sigma \times \Omega_{\Sigma'}$ (see property 2 above); this concludes the proof. □

2.4 Our main result is now the following.

THEOREM: Let g, g' be two C^∞ expansive maps as above. Assume that the associated Cantor sets satisfy

$$HD(K) + HD(K') \geq 1.$$

Then, we can find, arbitrarily close to g, g' (in the C^∞-topology), perturbations $\tilde{g}$, $\tilde{g}'$ having a *non empty recurrent compact set* of relative configurations.

Remark: In the case where both g and g' are orientation-preserving (equivalently, all periodic orbits of g and g' have positive eigenvalues), one can moreover find such a recurrent compact set which meets the two possible relative orientations (of A, A'); this will be apparent from the proof.

We consider the following subset V of $\Omega_\Sigma \times \Omega_{\Sigma'}$. A pair (g, g') belongs to V if for any $[(\underline{\theta}, A), (\underline{\theta}', A')] \in \mathcal{A} \times \mathcal{A}'$ there is a translation R_t (in $\mathbb{R}$) such that $(R_t \circ A \circ k^{\underline{\theta}}, A' \circ k'^{\underline{\theta}'})$ is a stably intersecting configuration.

THEOREM:

1. V is open in $\Omega_\Sigma \times \Omega_{\Sigma'}$, and $V \cap (\Omega_\Sigma^\infty \times \Omega_{\Sigma'}^\infty)$ is dense (for the C^∞-topology) in the set $\{(g, g'), HD(K) + HD(K') > 1\}$.
2. Let $(g, g') \in V$. There exists $d^* < 1$ such that for any $(h, h') \in \mathcal{P} \times \mathcal{P}'$, the set

$$\mathcal{I}_s = \{t \in \mathbb{R}, (R_t \circ h, h') \text{ is a stably intersecting smooth configuration for } (g, g')\}$$

is (open and) dense in

$$\mathcal{I} = \{t \in \mathbb{R}, (R_t \circ h, h') \text{ is an intersecting smooth configuration for } (g, g')\}$$

and moreover $HD(\mathcal{I} - \mathcal{I}_s) \leq d^*$. The same d^* is also valid for $(\tilde{g}, \tilde{g}')$ in a neigbourhood of (g, g') in $\Omega_\Sigma \times \Omega_{\Sigma'}$.

Proof: Let us first prove that V is open. For $\lambda \in \mathbb{R}^*$, denote by H_λ the homothety $x \mapsto \lambda x$. Let $(g, g') \in \Omega_\Sigma \times \Omega_{\Sigma'}$, and $R > 1$ be larger than the supremum of the derivatives of the expansive maps g, g'. Then (g, g') belongs to V if and only if for every $\underline{\theta} \in \Sigma^-$, $\underline{\theta}' \in \Sigma'^-$, $\lambda \in J_R = [-R, -R^{-1}] \cup [R^{-1}, R]$, there exists $t \in \mathbb{R}$ such that $(R_t \circ k^{\underline{\theta}}, H_\lambda \circ k^{\underline{\theta}'})$ is a stably intersecting configuration. Using the compactness of the set of $(\underline{\theta}, \underline{\theta}', \lambda)$ under consideration, we see not only that V is open but for $(g, g') \in V$ the following is true: there exist $\delta > 0$, a neighbourhood W of (g, g') in $\Omega_\Sigma \times \Omega_{\Sigma'}$ and a neighbourhood Z of the identity in $\bigcup\limits_{r>1} \mathrm{Diff}^r(\mathbb{R})$ such that for any $(\underline{\theta}, \underline{\theta}', \lambda) \in \Sigma^- \times \Sigma'^- \times J_R$ there is an interval J of length δ with $(R_t \circ \tilde{h} \circ k^{\underline{\theta}}, \tilde{h}' \circ k^{\underline{\theta}'})$ being a stably intersecting configuration for all $t \in J$, $\tilde{h}, \tilde{h}' \in Z$ and $(\tilde{g}, \tilde{g}') \in W$.

Assertion 2 in the theorem is now an easy consequence of the last statement. Let $(g, g') \in V$, $(h, h') \in \mathcal{P} \times \mathcal{P}'$, $t \in \mathcal{I}$. Let $\varepsilon > 0$ be small; let $\underline{a}$, $\underline{a}'$ be words in Σ, Σ' respectively such that $I(\underline{a})$, $I(\underline{a}')$ have size of order ε, and such that $h(K(\underline{a})) \cap h'(K(\underline{a}')) \neq \emptyset$. We apply (in the quotient space Q) $T_{\underline{a}} T'_{\underline{a}'}$ to $(R_t \circ h, h')$ and we get an intersecting configuration of the form $(\tilde{h} \circ k^{\underline{\theta}}, \tilde{h}' \circ H_\lambda \circ k^{\underline{\theta}'})$, where $\underline{\theta}$, $\underline{\theta}'$ end with $\underline{a}$, $\underline{a}'$ respectively and $\tilde{h}, \tilde{h}' \in Z$ if ε is small enough. The statement above now shows that there is an interval of length $c\delta\varepsilon$ (for some fixed $c > 0$) contained in $\mathcal{I}_s \cap [t - \varepsilon, t + \varepsilon]$. But this last property at the same time guarantees that $\mathcal{I}_s$ is dense in $\mathcal{I}$, and that $HD(\mathcal{I} - \mathcal{I}_s)$ is at most $d^* < 1$, where d^* depends only on $c\delta$ (cf. [PT2], p.79, prop. 6).

To show denseness of V, we will show that if (g, g') admits a non empty recurrent compact set $\mathcal{L}$ (meeting both orientations if g and g' are orientation-preserving, see remark after theorem 2.4), and if there are periodic points x, x' of period n, n' of g, g' respectively such that

$$\frac{\operatorname{Log}|Dg^n(x)|}{\operatorname{Log}|Dg'^{n'}(x')|}$$

is irrational, then $(g, g') \in V$. Indeed, let $\underline{a}$ (resp. $\underline{a}'$) the word in Σ (resp. Σ') of length $n+1$ (resp. $n'+1$) such that $f_{\underline{a}}(x) = x$ (resp. $f'_{\underline{a}'}(x') = x'$). Let N be the set of $(\underline{\theta}, \underline{\theta}', \lambda) \in \Sigma^- \times \Sigma'^- \times \mathbb{R}^*$ such that there exists $t \in \mathbb{R}$ for which $(R_t \circ k^{\underline{\theta}}, H_\lambda \circ k^{\underline{\theta}'})$ have stable intersection.

From the existence of the compact recurrent set $\mathcal{L}$, we deduce that N is non empty, and even that for any given $(\underline{\theta}, \underline{\theta'}) \in \Sigma^- \times \Sigma'^-$ there are $\lambda_1 < 0 < \lambda_2$ such that $(\underline{\theta}, \underline{\theta'}, \lambda_i) \in N$ for $i = 1, 2$. But then, setting $\bar{\underline{a}} = (\ldots \underline{a}, \underline{a}, \underline{a})$ and $\bar{\underline{a}}' = (\ldots \underline{a}', \underline{a}', \underline{a}')$, the hypothesis on the eigenvalues imply that $(\bar{\underline{a}}, \bar{\underline{a}}', \lambda) \in N$ for all $\lambda \in \mathbb{R}^*$. Using again the renormalization operators, we finally get that $N = \Sigma^- \times \Sigma'^- \times \mathbb{R}^*$. The proof is complete. $\square$

2.5 Let $\tilde{d} > 0$.

A smooth configuration $(h, h') \in \mathcal{P}(a) \times \mathcal{P}(a')$ has $\tilde{d}$-stable intersection if the Hausdorff dimension of $h(K(a)) \cap h'(K(a'))$ is $> \tilde{d}$, and this property still holds when we perturb $(h, h') \in \mathcal{P}(a) \times \mathcal{P}(a')$ and (g, g') in $\Omega_\Sigma \times \Omega_{\Sigma'}$.

In relation to this property, we say that a compact subset $\mathcal{L} \subset \mathcal{C}$ is $\tilde{d}$-recurrent if the following holds: for any $((\underline{\theta}, A), (\underline{\theta}', A'))$ representing a point $u \in \mathcal{L}$, we can find pairs $(\underline{a}^1, \underline{a}'^1), \ldots, (\underline{a}^k, \underline{a}'^k)$ for words such that

i) the first letter of $\underline{a}$ (resp. $\underline{a}'^i$) is θ_0;

ii) the intervals $I(\underline{a}^i)$ are disjoint;

iii) $(T_{\underline{a}^i}(\underline{\theta}, A), T'_{\underline{a}'^i}(\underline{\theta}', A'))$ represents a point in $\operatorname{int} \mathcal{L}$ for $1 \le i \le k$;

iv) $\sum_1^k |I^{\underline{\theta}}(\underline{a}^i)|^{\tilde{d}} > 1$.

By the compactness of $\mathcal{L}$, we can find the $\underline{a}^i$, $\underline{a}'^i$ with uniformly bounded length,

iv') $\sum_1^k |I^{\underline{\theta}}(\underline{a}^i)|^{\tilde{d}'} \ge c > 1$, with $\tilde{d}'$ slightly larger than $\tilde{d}$.

Then (iv) will still hold when we perturb (g, g'). On the other hand, from (iv'), we fairly easily deduce the

Proposition: If a relative affine configuration belongs to a $\tilde{d}$-recurrent non empty compact subset of $\mathcal{C}$, it is a $\tilde{d}$-stable intersecting configuration.

On the other hand, we will indicate in the course of the proof of the first theorem in 2.4 that slight modifications give the following stronger result.

THEOREM: Let $0 < \tilde{d} < HD(K) + HD(K') - 1$. Then we can find arbitrarily small C^∞-perturbations of (g, g') which exhibit a non empty $\tilde{d}$-recurrent compact subset.

§3. Outline of the proof of the main theorem.

3.1 Let g, g' be expansive maps as above. If $((\underline{\theta}, A), (\underline{\theta}', A'))$ is a relative configuration, the affine map $A^{-1} \circ A' \colon \mathbb{R} \times \{\theta_0'\} \to \mathbb{R} \times \{\theta_0\}$ is well defined; the coefficient of x is a number in $\mathbb{R}^*$ which is called the *relative scale* of the given relative configuration. We obtain thus the space $\mathcal{S} = \Sigma^- \times \Sigma'^- \times \mathbb{R}^*$ of relative scales, which is canonically a quotient of the space $\mathcal{C}$ of relative configurations (under the actions of translations on $\mathcal{A}$, $\mathcal{A}'$, independently).

The renormalization operators S, S' and their inverse branches T_{a_0,a_1}, $T'_{a_0',a_1'}$ still act on $\mathcal{S}$.

Actually, let $\underline{\theta} \in \Sigma^-$, $\underline{\theta}' \in \Sigma'^-$, and $\underline{a}$ be a word in Σ starting with θ_0. Let

$$T_{\underline{a}} = T_{a_n,a_{n-1}} \circ \cdots \circ T_{a_1,a_0} .$$

Then we have, for $s \in \mathbb{R}^*$

$$T_{\underline{a}}(\underline{\theta}, \underline{\theta}', s) = (\underline{\theta} a_1 \dots a_n, \underline{\theta}', \varepsilon s |I^{\underline{\theta}}(\underline{a})|^{-1})$$

where $\varepsilon = +1$ (resp. -1) if $f_{\underline{a}}$ is orientation preserving (resp. reversing).

Similarly, if $\underline{a}'$ is a word in Σ starting with θ'_0, we have

$$T'_{\underline{a}'}(\underline{\theta}, \underline{\theta}', s) = (\underline{\theta}, \underline{\theta}' a'_1 \dots a'_n, \varepsilon' s |I^{\underline{\theta}'}(\underline{a}')|).$$

3.2 The first main ingredient in the proof of the main theorem is a recurrence result for the action of renormalization on the space $\mathcal{S}$ of relative scales; we now present this result, which we call the Scale recurrence Lemma. Its proof is defferred to section 6.

We will always restrict our attention to a compact subset

$$\mathcal{S}_R = \Sigma^- \times \Sigma'^- \times ([-R, -R^{-1}] \cup [R^{-1}, R])$$

of $\mathcal{S}$, with R conveniently large.

Fix once and for all a conveniently large constant c_0. The *size* of a word $\underline{a}$ in Σ is the length $|I(\underline{a})|$; we say that $\underline{a}$ has approximate size ρ if

$$c_0^{-1}\,\rho \leq |I(\underline{a})| \leq c_0 \rho$$

and denote by $\Sigma(\rho)$ the set of those words. Its cardinality is of order ρ^{-d}, where we set

$$d = HD(K), \quad d' = HD(K').$$

We define similarly $\Sigma'(\rho)$, with cardinality of order $\rho^{-d'}$. To prove the Scale recurrence Lemma below, we need to assume that at least one of the two regular Cantor sets K, K' is non linear. More precisely, we assume that there exist $\underline{\theta}^0, \underline{\theta}^1 \in \Sigma^-$, with $\theta^0_0 = \theta^1_0$, and $x_0 \in K^{\underline{\theta}^0}(\theta^0_0)$ such that

$$|D \text{ Log } D(k^{\underline{\theta}^1} \circ (k^{\underline{\theta}^0})^{-1})(x_0)| \neq 0.$$

Then, denoting $J_R = [-R, -R^{-1}] \cup [R^{-1}, R]$, we have:

SCALE RECURRENCE LEMMA. If R, c_0 are conveniently large, we can find $c_1, c_2, c_3 > 0$ with the following properties: given $0 < \rho < 1$, and a family $E(\underline{a}, \underline{a}')$ of subsets of J_R, $(\underline{a}, \underline{a}') \in \Sigma(\rho) \times \Sigma'(\rho)$, such that

$$m(J_R - E(\underline{a}, \underline{a}')) \leq c_1\,, \qquad \forall\,(\underline{a}, \underline{a}')$$

we can find another family $E^*(\underline{a},\underline{a}')$ of subsets of J_R satisfying:

(i) for any $(\underline{a},\underline{a}')$, $E^*(\underline{a},\underline{a}')$ is contained in a $c_2\rho$-neighbourhood of $E(\underline{a},\underline{a}')$;

(ii) let $(\underline{a},\underline{a}') \in \Sigma(\rho)\times\Sigma'(\rho)$, $s \in E^*(\underline{a},\underline{a}')$; we can find at least $c_3\,\rho^{-(d+d')}$ pairs $(\underline{b},\underline{b}') \in \Sigma(\rho)\times\Sigma'(\rho)$ (with $\underline{b}$, $\underline{b}'$ starting respectively with the last letter of $\underline{a}$, $\underline{a}'$) such that, if $\underline{\theta}\in\Sigma^-$, $\underline{\theta}'\in\Sigma'^-$ end respectively with $\underline{a}$, $\underline{a}'$ and

$$T_{\underline{b}}T'_{\underline{b}'}(\underline{\theta},\underline{\theta}',s) = (\widetilde{\underline{\theta}},\widetilde{\underline{\theta}}',\tilde{s})$$

the ρ-neighbourhood of $\tilde{s}$ is contained in $E^*(\underline{b},\underline{b}')$.

Observe that $\widetilde{\underline{\theta}}$, $\widetilde{\underline{\theta}}'$ above end respectively with $\underline{b}$, $\underline{b}'$.

Let $\mathcal{S}^*(\underline{a},\underline{a}')$ be the set of $(\underline{\theta},\underline{\theta}',s)\in\mathcal{S}_R$ such that $\underline{\theta}$ ends with $\underline{a}$, $\underline{\theta}'$ with $\underline{a}'$ and $s\in E^*(\underline{a},\underline{a}')$; set $\mathcal{S}^* = \bigcup_{\underline{a},\underline{a}'}\mathcal{S}^*(\underline{a},\underline{a}')$. What property (ii) says is that there is a positive proportion of compatible $(\underline{b},\underline{b}')\in\Sigma(\rho)\times\Sigma'(\rho)$ such that the image by the inverse branch $T_{\underline{b}}T_{\underline{b}'}$ is "well inside" $\mathcal{S}^*$.

3.3 The fibers of the quotient map: $\mathcal{C}\to\mathcal{S}$ are one-dimensional and have a canonical affine structure; moreover this bundle map is trivializable; we choose now an explicit trivialization $\mathcal{C}\cong\mathcal{S}\times\mathbb{R}$ in order to have a coordinate in each fiber.

For each $a\in\mathbb{A}$ (resp. $a'\in\mathbb{A}'$), we choose $\omega(a)\in\Sigma^+$ starting with a (resp. $\omega(a')\in\Sigma'^+$ starting with a'). Given a configuration represented by $(\underline{\theta},A)$, $(\underline{\theta}',A')$, we normalize (by the action of the affine group) in order to obtain that $A'\colon\mathbb{R}\times\{\theta_0\}\to\mathbb{R}$ becomes a translation sending $k'^{\underline{\theta}'}(h'(\omega(\theta'_0)))$ to 0.

Then our coordinate $t((\underline{\theta},A),(\underline{\theta}',A'))$ on the fiber will be by definition

$$t = A(k^{\underline{\theta}}(h(\omega(\theta_0)))).$$

With the chosen normalization, $A'(I(\theta'_0))$ has size 1 while $A(I(\theta_0))$ has size $|s|$.

3.4 The first step in the proof is to get sure that the hypothesis of the Scale recurrence Lemma is satisfied (by either g or g'), and that $d+d'>1$.

So we make if necessary a small perturbation and will now assume that these hypotheses are satisfied by g, g'.

We want to perturb g, g' (actually it will be sufficient to perturb g) in the C^∞-topology in order to create a non empty recurrent compact set of relative configurations.

But a neighbourhood in the C^∞ topology is a neighbourhood in some C^k topology for finite k. We now fix such an integer $k \geq 2$.

The construction of the perturbation will depend on a scale parameter ρ; the perturbation will become close to g when ρ is small.

In the following, we denoted such a scale by ρ, and it will be assumed to be small.

3.5 The required perturbation for g will be picked by a probabilistic argument out of a family of random perturbations that we will now construct.

We first pick a subset Σ_0 of $\Sigma(\rho^{1/k})$ such that

$$K = \bigcup_{\underline{a} \in \Sigma_0} K(\underline{a})$$

is a *partition* of K into disjoint cylinders.

We then define Σ_1 as the subset of Σ_0 formed of the words $\underline{a} \in \Sigma_0$ such that no word in $\Sigma(\rho^{1/3k})$ appears twice in $\underline{a}$.

Let $c_4 > 1$ be a constant sufficiently close to 1 to have the following: let $\widehat{I}(\underline{a})$, for $\underline{a} \in \Sigma_0$, be the interval with same center as $I(\underline{a})$ and c_4^2 times the size of $I(a)$; then the $\widehat{I}(\underline{a})$, $\underline{a} \in \Sigma_0$, are pairwise disjoint.

We then choose, once and for all, a smooth even function $\chi\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$\chi(x) = 1 \quad \text{for} \quad |x| \leq c_4\,,$$

$$\chi(x) = 0 \quad \text{for} \quad |x| \geq c_4^2\,.$$

For $\underline{a} \in \Sigma_1$, we define the vectorfield $X_{\underline{a}}$, with support $\subset \widehat{I}(\underline{a})$ by

$$X_{\underline{a}}(x) = c_5\, \rho^{(1+1/2k)}\, \chi(B(x)) \frac{\partial}{\partial x}$$

where $c_5 > 0$ is a conveniently large constant, to be chosen later, and B is an affine map sending $I(\underline{a})$ onto $[-1,+1]$; there are two such maps, but as χ is even, they give the same $X_{\underline{a}}$.

The probability space underlying the familiy of random perturbations is $\Omega = [-1,1]^{\Sigma_1}$, equipped with the normalized Lebesgue measure.

For $\underline{\omega} = (\omega(\underline{a}))_{\underline{a}\in\Sigma_1} \in \Omega$, we define $\Phi_{\underline{\omega}}$ to be the time-one map of the vector field $X_{\underline{\omega}} = -\sum\limits_{\underline{a}} \omega(\underline{a}) X_{\underline{a}}$, and $g^{\underline{\omega}}$ to be $g \circ \Phi_{\underline{\omega}}$. Our task will be to find $\underline{\omega} \in \Omega$ such that $(g^{\underline{\omega}}, g')$ have a non empty compact recurrent set of relative configurations.

3.6 We make now some easy remarks on the family $g^{\underline{\omega}}$; the finer properties will be studied later.

3.6.1 By construction, the vectorfields $X_{\underline{a}}$, $\underline{a} \in \Sigma_1$, have disjoint supports.

The intervals $I(\underline{a})$, $\underline{a} \in \Sigma_1$, have approximate size $\rho^{1/k}$; therefore the affine map B entering the definition of $X_{\underline{a}}$ has derivative of the order of $\rho^{-1/k}$. It follows that, when ρ is small, $X_{\underline{a}}$ is small in the C^k-topology. Then, the $\Phi_{\underline{\omega}}$, for any $\underline{\omega} \in \Omega$, are close to the identity in the C^k-topology.

In particular, the estimates in proposition 1.3 for the $k^{\underline{\theta}}$ are valid for $g^{\underline{\omega}}$ uniformly in $\underline{\omega}$, if we consider no more than k derivatives.

3.6.2 For $a \in \mathbb{A}$, let

$$x_{\min}(a) = \inf K(a),$$

$$x_{\max}(a) = \sup K(a).$$

The finite set $\bigcup\limits_{a} \{x_{\min}(a), x_{\max}(a)\}$ is invariant under g. Let x_0 be a point in this set; if $\underline{a}_0$ is the element of Σ_0 such that $x_0 \in K(\underline{a}_0)$, $\underline{a}_0$ is the initial part of an eventually periodic word in Σ^+. Therefore $\underline{a}_0 \in \Sigma_0 - \Sigma_1$ if ρ is small enough. It follows that $g^{\underline{\omega}}$ coincides with g in $I(\underline{a}_0)$ (for any $\underline{\omega} \in \Omega$), and that the points $x_{\min}(a)$, $x_{\max}(a)$ are still the "endpoints" of the perturbed Cantor set $K_{\underline{\omega}}(a)$.

3.6.3 For $\underline{a} \in \Sigma_0$, let $\widetilde{I}(\underline{a})$ be the ρ-neighbourhood of $I(\underline{a})$.

Let $\underline{a} \in \Sigma_0$, $a_{-1} \in \mathbb{A}$ such that $(a_{-1}, a_0) \in \mathcal{B}$; let $\underline{a}'$ be the initial part of $a_{-1}\underline{a}$ that belongs to Σ_0. We have $f_{a_{-1}a_0}(I(\underline{a})) \subset I(\underline{a}')$; because the $f_{a,b}$ are contractions, there exists $c_6 > 0$ such that the $c_6\rho$-neighbourhood of $f_{a_{-1}a_0}(\widetilde{I}(\underline{a}))$ is contained in $\widetilde{I}(\underline{a}')$.

On the other hand, for the perturbed inverse branch $f^{\underline{\omega}}_{a_{-1}a_0}$, $\underline{\omega} \in \Omega$, we will have, for any $x \in \widetilde{I}(\underline{a})$

$$f_{a_{-1}a_0}(x) = \begin{cases} f_{a_{-1}a_0}(x) & \text{if} \quad \underline{a}' \in \Sigma_0 - \Sigma_1 \\ f_{a_{-1}a_0}(x) + c_5\rho^{1+1/2k}\,\omega(\underline{a}') & \text{if } \underline{a}' \in \Sigma_1 \end{cases}$$

(because $X_{\underline{a}'}$ is constant on $\widetilde{I}(\underline{a}')$). This allows to conclude that $f^{\underline{\omega}}_{a_{-1}a_0}(\widetilde{I}(\underline{a})) \subset \widetilde{I}(\underline{a}')$ (if ρ is small enough), and therefore that the perturbed Cantor set $K_{\underline{\omega}}$ will be contained in $\bigcup\limits_{\Sigma_0} \widetilde{I}(\underline{a})$.

3.7 We are now ready to give some information on the set $\mathcal{L} = \mathcal{L}_{\underline{\omega}}$ of relative configurations for which we want to prove recurrence for at least one $\underline{\omega} \in \Omega$.

As indicated, this set will depend on $\underline{\omega}$, but only as far as the fiber coordinate t is concerned. The image of $\mathcal{L}_{\underline{\omega}}$ under the projection map: $\mathcal{C} \to \mathcal{S}$ will be a subset $\widetilde{\mathcal{L}}$ of $\mathcal{S}$ independent of $\underline{\omega}$.

Let $\Sigma_{nr}(\rho^3)$ be the subset of $\Sigma(\rho^3)$ formed by words $\underline{a}$ such that:

(i) no word $\underline{b} \in \Sigma(\rho^{1/3k})$ appears twice in $\underline{a}$;

(ii) if $\underline{c} \in \Sigma(\rho^{1/6k})$ appears at the end of $\underline{a}$, then it does not appear elsewhere in $\underline{a}$.

We next define Σ^-_{nr} as the subset of Σ^- formed by $\underline{\theta}$ which end with a word in $\Sigma_{nr}(\rho^3)$. This is an open and closed subset in Σ^-.

To define $\widetilde{\mathcal{L}}$, we will apply the Scale recurrence Lemma (with $\rho^{1/2}$ instead of ρ).

A family of subsets $E(\underline{a}, \underline{a}')$ of J_R, for $(\underline{a}, \underline{a}') \in \Sigma(\rho^{1/2}) \times \Sigma'(\rho^{1/2})$ will be carefully constructed in section 4, in relation to Marstrand's theorem, and it will satisfy the hypothesis

$$m(J_R - E(\underline{a}, \underline{a}')) \le c_1\,, \quad \forall (\underline{a}, \underline{a}').$$

Then, the Scale recurrence Lemma gives us another family $E^*(\underline{a},\underline{a}') \in \Sigma(\rho^{1/2}) \times \Sigma'(\rho^{1/2})$, with the properties (i), (ii) indicated in the statement of the lemma.

The set $\widetilde{\mathcal{L}}$ is defined to be the subset of $\mathcal{S}_R$ formed by the $(\underline{\theta},\underline{\theta}',s)$ such that $\underline{\theta} \in \Sigma_{nr}^-$, and there exists $\underline{a} \in \Sigma(\rho^{1/2})$, $\underline{a}' \in \Sigma'(\rho^{1/2})$ with $s \in E^*(\underline{a},\underline{a}')$ and $\underline{\theta}$, $\underline{\theta}'$ ending with $\underline{a}$, $\underline{a}'$ respectively.

3.8 For every $(\underline{\theta},\underline{\theta}',s)$ in $\widetilde{\mathcal{L}}$, we will define in section 4, in relation to the conclusions of Marstrand's theorem, a non empty subset $L^0_{\underline{\omega}}(\underline{\theta},\underline{\theta}',s)$, depending on $\underline{\omega} \in \Omega$, of the fiber of $\mathcal{C}$ over $(\underline{\theta},\underline{\theta}',s)$.

Let

$$\mathcal{L}^0_{\underline{\omega}} = \{(\underline{\theta},\underline{\theta}',s,t),(\underline{\theta},\underline{\theta}',s) \in \widetilde{\mathcal{L}}, t \in L^0_{\underline{\omega}}(\underline{\theta},\underline{\theta}',s)\};$$

consider next the ρ-neighbourhood $\mathcal{L}^1_{\underline{\omega}}$ of $\mathcal{L}^0_{\underline{\omega}}$ in $\widetilde{\mathcal{L}} \times \mathbb{R}$:

$$\begin{gathered}\mathcal{L}^1_{\underline{\omega}} = \{(\underline{\theta},\underline{\theta}',s,t),(\underline{\theta},\underline{\theta}',s) \in \widetilde{\mathcal{L}} \text{ and } \exists\,(\underline{\theta}_0,\underline{\theta}'_0,s_0,t_0) \in \mathcal{L}^0_{\underline{\omega}} \\ \text{with } d(\underline{\theta},\underline{\theta}_0) < 2\rho^{5/2}, d(\underline{\theta}',\underline{\theta}'_0) < 2\rho^{5/2}, |s-s_0| < \rho, |t-t_0| < \rho\}.\end{gathered}$$

Fix $u = (\underline{\theta},\underline{\theta}',s,t) \in \widetilde{\mathcal{L}} \times \mathbb{R}$. We define two subsets $\Omega^0(u)$, $\Omega^1(u)$ of Ω. First,

$$\Omega^1(u) = \{\underline{\omega} \in \Omega, (\underline{\theta},\underline{\theta}',s,t) \in \mathcal{L}^1_{\underline{\omega}}\}.$$

Second, $\Omega^0(u)$ is the set of $\underline{\omega} \in \Omega$ such that there exists $\underline{b} \in \Sigma(\rho)$, $\underline{b}' \in \Sigma'(\rho)$, with $b_0 = \theta_0$, $b'_0 = \theta'_0$ and the image $T_{\underline{b}}T'_{\underline{b}'}(u) = (\widetilde{\underline{\theta}},\widetilde{\underline{\theta}}',\tilde{s},\tilde{t})$ satisfies:

(i) for any $\tilde{s}_1$ with $|\tilde{s}-\tilde{s}_1| < \frac{1}{2}\rho^{1/2}$, we have $(\widetilde{\underline{\theta}},\widetilde{\underline{\theta}}',\tilde{s}_1) \in \widetilde{\mathcal{L}}$;

(ii) $\tilde{t} \in L^0_{\underline{\omega}}(\widetilde{\underline{\theta}},\widetilde{\underline{\theta}}',\tilde{s})$.

[The renormalization operators $T_{\underline{b}}$, $T_{\underline{b}'}$ depend on $\underline{\omega}$; the dependence is not indicated in the notations to keep the text readable].

The following crucial estimate will be proven in section 5.

Proposition: Assume that c_5 is chosen conveniently large. Then there exists $c_7 > 0$, such that, for any $u \in \widetilde{\mathcal{L}} \times \mathbb{R}$, one has

$$\mathbb{P}(\Omega^1(u) - \Omega^0(u)) \leq \exp\big(-c_7 \rho^{-\frac{1}{2k}(d+d'-1)}\big).$$

Basically, the reason for this inequality is the following: the sets $L(\underline{\theta}, \underline{\theta}', s)$ are formed of values of t such that $T_{\underline{b}} T'_{\underline{b}'}(\underline{\theta}, \underline{\theta}', s, t)$ is bounded for many pairs $(\underline{b}, \underline{b}') \in \Sigma(\rho) \times \Sigma'(\rho)$. It will follow from Marstrand's theorem that the Lebesgue measure of $L(\underline{\theta}, \underline{\theta}', s)$ is uniformly bounded from below. So we can expect for fixed $(\underline{b}, \underline{b}')$ the probability (in $\underline{\omega}$) that $T_{\underline{b}} T'_{\underline{b}'}(\underline{\theta}, \underline{\theta}', s, t)$ is right to be bounded from below. When $(\underline{b}, , \underline{b}')$ vary, we will be able to find $c\, \rho^{-\frac{1}{2k}(d+d'-1)}$ such pairs for which we have rough independence, which " explains" the inequality in the proposition.

3.9 With this estimate, the end of the proof of the theorem is not difficult.

The sets $L^0_{\underline{\omega}}(\underline{\theta}, \underline{\theta}', s)$ will always satisfy

$$t \in L^0_{\underline{\omega}}(\underline{\theta}, \underline{\theta}', s) \Rightarrow |t| \leq 1 + |s| \leq 1 + R.$$

We choose finite subsets Δ_0, Δ'_0, Δ_2 of Σ^-_{nr}, Σ'^-, $[-(1+R+2\rho), (1+R+2\rho)]$ respectively such that:

- Δ_0 is $\rho^{5/2}$ dense in Σ^-_{nr}, $\#\Delta_0 \leq c_8 \rho^{-\frac{5}{2}d}$;
- Δ'_0 is $\rho^{5/2}$ dense in Σ'^-, $\#\Delta'_0 \leq c_8 \rho^{-\frac{5}{2}d'}$;
- Δ_2 is $\rho^{5/2}$ dense in $[-(1+R+2\rho), 1+R+2\rho]$, $\#\Delta_2 \leq c_8 \rho^{-\frac{5}{2}}$.

Then, for each $(\underline{\theta}, \underline{\theta}') \in \Delta_0 \times \Delta'_0$, we choose a finite subset $\Delta_1(\underline{\theta}, \underline{\theta}')$ of the fiber of $\widetilde{\mathcal{L}}$ over $(\underline{\theta}, \underline{\theta}')$, which is $\rho^{5/2}$ dense in this fiber, and has cardinality $\leq c_8 \rho^{-5/2}$. Let then

$$\Delta = \{u = (\underline{\theta}, \underline{\theta}', s, t), \underline{\theta} \in \Delta_0, \underline{\theta}' \in \Delta'_0, t \in \Delta_2, s \in \Delta_1(\underline{\theta}, \underline{\theta}')\}.$$

One has

$$\#\Delta \leq c_8^4\, \rho^{-(5+\frac{5}{2}(d+d'))} \leq c_8^4\, \rho^{-10}.$$

Now, if ρ is small enough, we have

$$c_8^4\, \rho^{-10} \exp\big(- c_7\, \rho^{-\frac{1}{2k}(d+d'-1)}\big) < 1,$$

and therefore we can find $\underline{\omega}_0 \in \Omega$ such that, for any $u \in \Delta$, we have either $\underline{\omega}_0 \notin \Omega^1(u)$ or $\underline{\omega}_0 \in \Omega^0(u)$ (or both).

3.10 Fix $\underline{\omega}_0$ as above. Define $\mathcal{L} = \mathcal{L}_{\underline{\omega}_0}$ as follows:

$$\mathcal{L} = \{(\underline{\theta}, \underline{\theta}', s, t) \in \widetilde{\mathcal{L}} \times \mathbb{R},\ \ \exists\, (\widetilde{\underline{\theta}}, \widetilde{\underline{\theta}}', \tilde{s}, \tilde{t}) \in \widetilde{\mathcal{L}} \times \mathbb{R}$$

$$\text{with } d(\underline{\theta}, \widetilde{\underline{\theta}}) \leq \rho^{5/2},\ \ d(\underline{\theta}', \widetilde{\underline{\theta}}') \leq \rho^{5/2},$$

$$|s - \tilde{s}| \leq \rho/2, \quad |t - \tilde{t}| \leq \rho/2$$

$$\text{and } \tilde{t} \in L^0_{\underline{\omega}_0}(\widetilde{\underline{\theta}}, \widetilde{\underline{\theta}}', \tilde{s})\}.$$

We claim that for $(g^{\underline{\omega}_0}, g')$, $\mathcal{L}$ is a non empty compact recurrent set of relative configurations.

First, $\mathcal{L}$ contains $\mathcal{L}^0_{\underline{\omega}_0}$, hence is non empty. It is compact by construction. Let $u = (\underline{\theta}, \underline{\theta}', s, t) \in \mathcal{L}$.

We can find $\underline{\theta}^0 \in \Delta_0$, $\underline{\theta}'^0 \in \Delta'_0$, $t_0 \in \Delta_2$ with $d(\underline{\theta}, \underline{\theta}^0) < \rho^{5/2}$, $d(\underline{\theta}', \underline{\theta}'^0) < \rho^{5/2}$, $|t - t_0| < \rho^{5/2}$. Then, by construction of $\widetilde{\mathcal{L}}$, we have still $(\underline{\theta}^0, \underline{\theta}'^0, s) \in \widetilde{\mathcal{L}}$, hence we can find $s_0 \in \Delta_1(\underline{\theta}^0, \underline{\theta}'^0)$ with $|s - s_0| < \rho^{5/2}$. Let $u_0 = (\underline{\theta}^0, \underline{\theta}'^0, s_0, t_0) \in \Delta$.

With $\widetilde{\underline{\theta}}$, $\widetilde{\underline{\theta}}'$, $\tilde{s}$, $\tilde{t}$ related to $\underline{\theta}$, $\underline{\theta}'$, s, t by the definition of $\mathcal{L}$ above, we have

$$d(\underline{\theta}^0, \widetilde{\underline{\theta}}) < 2\rho^{5/2},$$

$$d(\underline{\theta}'^0, \widetilde{\underline{\theta}}') < 2\rho^{5/2},$$

$$|s_0 - \tilde{s}| \leq \rho/2 + \rho^{5/2} < \rho,$$

$$|t_0 - \tilde{t}| \leq \rho/2 + \rho^{5/2} < \rho,$$

which means that $u_0 \in \mathcal{L}^1_{\underline{\omega}_0}$; in other words, $\underline{\omega}_0 \in \Omega^1(u_0)$.

Then by choice of $\underline{\omega}_0$, we have $\underline{\omega}_0 \in \Omega^0(u_0)$. This means that there exists $\underline{b} \in \Sigma(\rho)$, $\underline{b}' \in \Sigma'(\rho)$, with $b_0 = \theta^0_0$, $b'_0 = \theta'^0_0$ such that, if

$$T^{\omega_0}_{\underline{b}}\, T'_{\underline{b}'}(u_0) = (\widehat{\underline{\theta}}^0, \widehat{\underline{\theta}}'^0, \hat{s}_0, \hat{t}_0) = \hat{u}_0\,,$$

we have

$$(\widehat{\theta}^0, \widehat{\theta}'^0, \hat{s}'_0) \in \widetilde{\mathcal{L}} \quad \text{for} \quad |\hat{s}_0 - \hat{s}'_0| < \frac{1}{2}\rho^{1/2}$$

and $\hat{t}_0 \in L^0_{\underline{\omega}_0}(\widehat{\underline{\theta}}^0, \widehat{\underline{\theta}}'^0, \hat{s}_0)$.

Let

$$\hat{u} = T^{\omega_0}_{\underline{b}} T'_{\underline{b}'}(u) = (\widehat{\underline{\theta}}, \widehat{\underline{\theta}}', \hat{s}, \hat{t}).$$

Lemma: We have

$$d(\widehat{\underline{\theta}}, \widehat{\underline{\theta}}^0) \le c_9\, \rho^{7/2}, \qquad d(\widehat{\underline{\theta}}', \widehat{\underline{\theta}}'^0) \le c_9\, \rho^{7/2}$$

$$|\hat{s} - \hat{s}_0| \le c_9\, \rho^{5/2}, \qquad |\hat{t} - \hat{t}_0| \le c_9\, \rho^{3/2}.$$

Proof: By proposition §1.3, the C^1-distance to the identity of $k^{\underline{\theta}} \circ (k^{\underline{\theta}^0})^{-1}$, $k'^{\underline{\theta}'} \circ (k'^{\underline{\theta}'^0})^{-1}$ is at most of order $\rho^{5/2}$. The estimates from the lemma then follow immediately from the definitions of $T_{\underline{b}}$, $T_{\underline{b}'}$. $\square$

To prove that $\hat{u} \in \operatorname{int} \mathcal{L}$, we take $(\widehat{\theta}^1, \widehat{\theta}'^1, \hat{s}_1, \hat{t}_1)$ with

$$d(\widehat{\underline{\theta}}^1, \widehat{\underline{\theta}}) \le \rho^{7/2}, \qquad d(\widehat{\underline{\theta}}'^1, \widehat{\underline{\theta}}') \le \rho^{7/2}$$

$$|\hat{s}_1 - \hat{s}| \le \rho^{5/2}, \qquad |\hat{t}_1 - \hat{t}| \le \rho^{3/2}.$$

We have then

$$d(\widehat{\underline{\theta}}^1, \widehat{\underline{\theta}}^0) \le (1 + c_9)\rho^{7/2}, \qquad d(\widehat{\underline{\theta}}'^1, \widehat{\underline{\theta}}'^0) \le (1 + c_9)\rho^{7/2},$$

$$|\hat{s}_1 - \hat{s}_0| \le (1 + c_9)\rho^{5/2}, \qquad |\hat{t}_1 - \hat{t}_0| \le (1 + c_9)\rho^{3/2}$$

which imply first $(\widehat{\underline{\theta}}^1, \widehat{\underline{\theta}}'^1, \hat{s}_1) \in \widetilde{\mathcal{L}}$, and then $(\widehat{\underline{\theta}}^1, \widehat{\underline{\theta}}'^1, \hat{s}_1, \hat{t}_1) \in \mathcal{L}$.

3.11 In order to prove the theorem in §2.5, one has to modify slightly the argument as follows.

Let $0 < \tilde{d} < d + d' - 1$; choose $\tilde{d} < d^* < d + d' - 1$. We will indicate at the end of §4 how to define $\overline{L}^0_{\underline{\omega}}(\underline{\theta}, \underline{\theta}', s)$ (analogous to $L^0_{\underline{\omega}}(\underline{\theta}, \underline{\theta}', s)$), for $(\underline{\theta}, \underline{\theta}', s) \in \widetilde{\mathcal{L}}$.

For $u \in \widetilde{\mathcal{L}} \times \mathbb{R}$, one defines $\overline{\Omega}^1(u)$ as $\Omega^1(u)$ above (using $\overline{L}^0_{\underline{\omega}}$ instead of $L^0_{\underline{\omega}}$). Then, $\overline{\Omega}^0(u)$ is the set of $\underline{\omega} \in \Omega$ such that, with $M = [\rho^{-d^*}]$, there exist pairs $(\underline{b}^1, \underline{b}'^1), \ldots, (\underline{b}^M, \underline{b}'^M) \in \Sigma(\rho) \times \Sigma'(\rho)$ (with disjoint $I(\underline{b}^i)$) for which properties (i), (ii) hold [we had $M = 1$ in §3.8].

We will see in section 5 that in this case we still have (as proposition 3.8)

$$\mathbb{P}(\overline{\Omega}^1(u) - \overline{\Omega}^0(u)) \leq \exp(-\bar{c}_7\, \rho^{-\frac{1}{2k}(d+d'-1)}).$$

The set $\overline{\mathcal{L}}_{\underline{\omega}_0}$, defined as in 3.10 with the $\overline{L}^0_{\underline{\omega}_0}$ instead of $L^0_{\underline{\omega}_0}$, is easily seen to be $\tilde{d}$-recurrent if $\underline{\omega}_0$ is selected as in 3.9.

3.12 Remark: Consider a smooth family (g_μ, g'_μ), $\mu \in [-1, +1]^r$ of maps as above such that one has $HD(K_\mu) + HD(K'_\mu) > 1$ for all μ.

We consider perturbations $(g^{\omega}_{\mu}, g'_\mu)$ constructed as above, where now μ is restricted to a ρ^3-dense finite subset Δ_3 of the parameter space $[-1, 1]^n$, with

$$\#\, \Delta_3 \leq c_n\, \rho^{-3n}.$$

As this is polynomial in ρ, we can select $\underline{\omega}_0$ such that, for any $u \in \Delta$, $\mu \in \Delta_3$ we have either $\underline{\omega}_0 \notin \Omega^1_\mu(u)$ or $\underline{\omega}_0 \in \Omega^0_\mu(u)$.

For any $\mu_0 \in \Delta_3$, this allows to construct (for $(g^{\underline{\omega}_0}_{\mu_0}, g'_{\mu_0})$) a recurrent set $\mathcal{L}_{\mu_0}$.

But, in view of the construction of this set, it is fairly clear that $\mathcal{L}_{\mu_0}$ is still recurrent for $(g^{\underline{\omega}_0}_{\mu}, g'_\mu)$, provided $||\mu - \mu_0|| < \rho^3$.

Thus the following reinforcement of the theorem in §2.4 is true: there exists open and dense sets of families as above for which one can, for *all* values of the parameter, find a recurrent compact set of configurations.

Moreover, generic families have the following property: for each value of the parameter, there are periodic points of g_μ , g'_μ respectively such that the ratio of the logarithms of the moduli of the corresponding eigenvalues is irrational.

This allows (with the same proof) the following reinforcement of the second theorem in §2.4: the open set $V \subset \Omega_\Sigma \times \Omega_{\Sigma'}$ defined in §2.4 is not only dense in $\{(g, g'), HD(K) + HD(K') > 1\}$, but its complement is infinite codimensional in the sense that generic n-parameter families (for any n) take their values in V.

§4. Application of Marstrand's theorem: construction of the sets $E(\underline{a}, \underline{a}')$ and $L^0_{\underline{\omega}}(\underline{\theta}, \underline{\theta}', s)$.

4.1 Given a point $(\underline{\theta}, \underline{\theta}', s)$ in the space $\mathcal{S}$ of relative scales, and points $x \in K(\theta_0)$, $x' \in K'(\theta'_0)$, there is a unique relative configuration above $(\underline{\theta}, \underline{\theta}', s)$ such that

$$A(k^{\underline{\theta}}(x)) = A'(k'^{\underline{\theta}'}(x')),$$

(where $(\underline{\theta}, A)$, $(\underline{\theta}', A')$ represents this configuration). Parametrizing the fiber of $\mathcal{C}$ over $(\underline{\theta}, \underline{\theta}', s)$ as in §3.3, the coordinate of this configuration is given by

$$\begin{aligned} t &:= \pi_{\underline{\theta}, \underline{\theta}', s}(x, x') \\ &= k'^{\underline{\theta}'}(x') - k'^{\underline{\theta}'}(\omega(\theta'_0)) - s(k^{\underline{\theta}}(x) - k^{\underline{\theta}}(\omega(\theta_0))). \end{aligned}$$

4.2 We equip each set $K(\theta_0)$ (resp. $K'(\theta'_0)$) with the d-dimensional (resp. d'-dimensional) Hausdorff measure μ_d (resp. $\mu_{d'}$).

Then, for $(\underline{\theta}, \underline{\theta}', s) \in \mathcal{S}$, we denote by $\mu(\underline{\theta}, \underline{\theta}', s)$ the image under $\pi_{\underline{\theta}, \underline{\theta}', s}$ of $\mu_d \times \mu_{d'}$ on $K(\theta_0) \times K'(\theta'_0)$.

It is well known that these exists constants $c_{11} > c_{10} > 0$ such that, for $\theta_0 \in \mathbb{A}$, $\theta'_0 \in \mathbb{A}'$:

$$c_{10} < \mu_d \times \mu_{d'}(K(\theta_0) \times K'(\theta'_0)) < c_{11}\,.$$

Now, Kaufman's proof of Marstrand's theorem (see [PT2]) tells us that, for fixed $(\underline{\theta}, \underline{\theta}')$, the measure $\mu(\underline{\theta}, \underline{\theta}', s)$ is absolutely continuous with respect to Lebesgue measure for Lebesgue almost every s, with L^2-density $\chi_{\underline{\theta},\underline{\theta}',s}$ satisfying

$$\int_{J_R} \left\|\chi_{\underline{\theta},\underline{\theta}',s}\right\|^2_{L^2} \, ds \leq c_{12}(R),$$

where $c_{12}(R)$ is independent on $\underline{\theta}$, $\underline{\theta}'$.

4.3 When one does control $\left\|\chi_{\underline{\theta},\underline{\theta}',s}\right\|_{L^2}$, this gives, by Cauchy-Schwarz inequality, a lower bound for the Lebesgue measure of $\pi_{\underline{\theta},\underline{\theta}',s}(X)$, X being a subset of $K \times K'$ with positive $(d + d')$-dimensional Hausdorff measure; indeed we have:

$$\begin{aligned}\mu_d \times \mu_{d'}(X) &\leq \int_{\pi_{\underline{\theta},\underline{\theta}',s}(X)} \chi_{\underline{\theta},\underline{\theta}',s}(t)\, dt \\ &\leq \left|\pi_{\underline{\theta},\underline{\theta}',s}(X)\right|^{1/2} \left\|\chi_{\underline{\theta},\underline{\theta}',s}\right\|_{L^2}\end{aligned}$$

and therefore

$$\left|\pi_{\underline{\theta},\underline{\theta}',s}(X)\right| \geq (\mu_d \times \mu_{d'}(X))^2 \left\|\chi_{\underline{\theta},\underline{\theta}',s}\right\|^{-2}_{L^2}\,.$$

4.4 Fix $(\underline{\theta}, \underline{\theta}')$ in $\Sigma^- \times \Sigma'^-$. Let $\underline{a} \in \Sigma(\rho^{1/2k})$, $a' \in \Sigma'(\rho^{1/2k})$, with $a_0 = \theta_0$, $a'_0 = \theta'_0$. as seen in 3.1, one has

$$T_{\underline{a}} T'_{\underline{a}'}(\underline{\theta}, \underline{\theta}', s) = (\underline{\theta} a_1 \dots a_n, \underline{\theta}' a_1 \dots a_{n'}\,, \varepsilon\varepsilon' s |I^{\underline{\theta}}(\underline{a})|\, |I^{\underline{\theta}'}(\underline{a}')|^{-1})$$

where $\varepsilon, \varepsilon' \in \{\pm 1\}$ and

$$c_{13}^{-1} \leq |I^{\underline{\theta}}(\underline{a})|\, |I^{\underline{\theta}'}(\underline{a}')|^{-1} \leq c_{13}\,.$$

We therefore have

$$\int_{J_R} \left\| \chi_{T_{\underline{a}} T'_{\underline{a}'}(\underline{\theta},\underline{\theta}',s)} \right\|_{L^2}^2 ds \le c'_{12}(R),$$

with $c'_{12}(R)$ independent of $\underline{\theta}$, $\underline{\theta}'$, $\underline{a}$, $\underline{a}'$. On the other hand, one has

$$\# \Sigma(\rho^{1/2k}) \le c_{14}\, \rho^{-d/2k}\,,$$

$$\# \Sigma'(\rho^{1/2k}) \le c_{14}\, \rho^{-d'/2k}\,.$$

We conclude that

$$\int_{J_R} \sum_{\underline{a},\underline{a}'} \left\| \chi_{T_{\underline{a}} T'_{\underline{a}'}(\underline{\theta},\underline{\theta}',s)} \right\|_{L^2}^2 ds \le c_{14}^2\, \rho^{-\frac{d+d'}{2k}}\, c'_{12}(R).$$

We now define, with $c_{15} > 0$ conveniently large to be determined later:

$$E(\underline{\theta},\underline{\theta}') = \Bigg\{ s \in J_R\,,\, \left\| \chi_{\underline{\theta},\underline{\theta}',s} \right\|_{L^2}^2 \le c_{15}$$

$$\text{and} \qquad \sum_{\underline{a},\underline{a}'} \left\| \chi_{T_{\underline{a}} T'_{\underline{a}'}(\underline{\theta},\underline{\theta}',s)} \right\|_{L^2}^2 \le c_{15}\, \rho^{-\frac{d+d'}{2k}} \Bigg\}.$$

For $\underline{c} \in \Sigma(\rho^{1/2})$, $\underline{c}' \in \Sigma'(\rho^{1/2})$, we define $E(\underline{c},\underline{c}')$ as the set of $s \in J_R$ such that there exists $\underline{\theta}$, $\underline{\theta}'$ ending respectively with $\underline{c}$, $\underline{c}'$ such that $s \in E(\underline{\theta},\underline{\theta}')$.

One has, for any $\underline{\theta} \in \Sigma^-$, $\underline{\theta}' \in \Sigma'^-$:

$$m(J_R - E(\underline{\theta},\underline{\theta}')) \le c_{15}^{-1}(c_{12}(R) + c_{14}^2\, c'_{12}(R));$$

therefore, provided that

$$c_{15} > c_1^{-1}(c_{12}(R) + c_{14}^2\, c'_{12}(R)),$$

we will have

$$m(J_R - E(\underline{c},\underline{c}')) \le c_1$$

for all $\underline{c} \in \Sigma(\rho^{1/2})$, $\underline{c}' \in \Sigma'(\rho^{1/2})$. This means that we can apply the Scale recurrence Lemma with the family $E(\underline{c}, \underline{c}')$ of subsets of J_R.

Let $E^*(\underline{c}, \underline{c}')$ be then the family of subsets of J_R given by the Scale recurrence Lemma (cf. 3.7). We know that each $E^*(\underline{c}, \underline{c}')$ is contained in a $c_2\, \rho^{1/2}$-neighbourhood of $E(\underline{c}, \underline{c}')$.

4.5 Until now, we have only worked with the unperturbed maps g, g' (except from a possible initial perturbation, see §3.4). We now consider the family of random perturbations $g^{\underline{\omega}}$ of §3.5, and will proceed to construct the sets $L^0_{\underline{\omega}}(\underline{\theta}, \underline{\theta}', s)$, for $(\underline{\theta}, \underline{\theta}', s) \in \widetilde{\mathcal{L}}$. For $\underline{a} \in \Sigma(\rho^{1/2k})$, let $\Sigma^-(\underline{a})$ be the open and closed subset of Σ^- formed by the $\underline{\theta}$ ending with $\underline{a}$. Choose a subset Σ_2^- of $\Sigma(\rho^{1/2k})$ such that

$$\Sigma^- = \bigcup_{\Sigma_2^-} \Sigma^-(\underline{a})$$

is a partition of Σ^-.

For $\underline{a} \in \Sigma_2^-$, define a subset $\Sigma_1(\underline{a})$ of the subset Σ_1 of §3.5 (recall $\Sigma_1 \subset \Sigma(\rho^{1/k})$), as the set of words in Σ_1 starting with $\underline{a}$. For $\underline{\theta} \in \Sigma^-(\underline{a})$, we also define $\Sigma_1(\underline{\theta}) = \Sigma_1(\underline{a})$.

Let $\underline{\theta} \in \Sigma^-$. We write

$$\Omega = [-1, +1]^{\Sigma_1(\underline{\theta})} \times [-1, +1]^{\Sigma_1 - \Sigma_1(\underline{\theta})},$$

$$\underline{\omega} = (\underline{\omega}', \underline{\omega}'')$$

and for such an $\underline{\omega}$, we set

$$\underline{\omega}^* = (\underline{0}, \underline{\omega}'').$$

This depends on $\underline{\theta}$, but nearby $\widehat{\underline{\theta}}$ (with $d(\underline{\theta}, \widehat{\underline{\theta}}) < c_0^{-1}\, \rho^{1/2k}$) will belong to the same $\Sigma^-(\underline{a})$ and give the same projection $\underline{\omega}^*$ of $\underline{\omega}$.

For $(\underline{\theta}, \underline{\theta}', s) \in \widetilde{\mathcal{L}}$, the set $L^0_{\underline{\omega}}(\underline{\theta}, \underline{\theta}', s)$ will actually only depend (as far as $\underline{\omega}$ is concerned) on the projection $\underline{\omega}^*$ of $\underline{\omega}$ associated to $\underline{\theta}$.

We will say that two words $\underline{b}^0, \underline{b}^1 \in \Sigma(\rho)$ are *independent* if there is no word $\underline{b} \in \Sigma(\rho^{1/2k})$ such that both $\underline{b}^0$ and $\underline{b}^1$ start with $\underline{b}$.

With $c_{16} > 0$ conveniently small, to be chosen in the following, let

$$N = \left[c_{16}^2 \, \rho^{-\frac{1}{2k}(d+d'-1)} \right].$$

4.6 Let $(\underline{\theta}, \underline{\theta}', s) \in \widetilde{\mathcal{L}}$, and $\underline{\omega} \in \Omega$.

We define $L^0_{\underline{\omega}}(\underline{\theta}, \underline{\theta}', s)$ to be the set of points $(\underline{\theta}, \underline{\theta}', s, t)$ in the fiber for which there exist pairs $(\underline{b}^1, \underline{b}'^1), \ldots, (\underline{b}^N, \underline{b}'^N)$ in $\Sigma(\rho) \times \Sigma'(\rho)$, with $b^i_0 = \theta_0$, $b'^i_0 = \theta'_0$ such that, if we set

$$T^{\underline{\omega}^*}_{\underline{b}^i} \, T'_{\underline{b}'^i}(\underline{\theta}, \underline{\theta}', s, t) = (\underline{\theta}^i, \underline{\theta}'^i, s_i, t_i),$$

the following hold:

(i) the words $\underline{b}^1, \ldots, \underline{b}^N$ are pairwise independent;

(ii) for $1 \le i \le N$, $\underline{\theta}^i \in \Sigma^-_{nr}$;

(iii) for $1 \le i \le N$, and $|\tilde{s} - s_i| \le \dfrac{2}{3}\rho^{1/2}$, $(\underline{\theta}^i, \underline{\theta}'^i, \tilde{s}) \in \widetilde{\mathcal{L}}$;

(iv) for $1 \le i \le N$, $|t_i| \le 2(1+R)$.

We will use also a slightly smaller set $L^{-1}_{\underline{\omega}}(\underline{\theta}, \underline{\theta}', s)$; it is defined in the same way than $L^0_{\underline{\omega}}(\underline{\theta}, \underline{\theta}', s)$, but with (iii), (iv) replaced by:

(iii)' for $1 \le i \le N$, and $|\tilde{s} - s_i| \le \dfrac{3}{4}\rho^{1/2}$, $(\underline{\theta}^i, \underline{\theta}'^i, \tilde{s}) \in \widetilde{\mathcal{L}}$

(iv)' for $1 \le i \le N$, $|t_i| \le 1 + R$.

In the end of this section, we will prove the following estimate.

Proposition: If c_{16} has been chosen sufficiently small, there exists $c_{17} > 0$ such that, for any $(\underline{\theta}, \underline{\theta}', s) \in \widetilde{\mathcal{L}}$ and any $\underline{\omega} \in \Omega$, the Lebesgue measure of $L^{-1}_{\underline{\omega}}(\underline{\theta}, \underline{\theta}', s)$ is $> c_{17}$.

4.7 In the proof of the proposition, we will use the following estimates on the family of random perturbations, to be proved in the next section.

Lemma: Let $\underline{\theta} \in \Sigma^-$, $\underline{\omega} \in \Omega$.

(i) for $\underline{a} \in \Sigma^+$, with $a_0 = \theta_0$, we have

$$|k^{\underline{\theta},\underline{\omega}}(h^{\underline{\omega}}(\underline{a})) - k^{\underline{\theta}}(h(\underline{a}))| \leq c_{18} c_5 \, \rho^{1-\frac{1}{2k}};$$

(ii) for $\underline{a}$ a word with $a_0 = \theta_0$, $|I(\underline{a})| > c_0^{-1}\rho$, we have

$$\left| \mathrm{Log} \frac{|I^{\underline{\theta},\underline{\omega}}(\underline{a})|}{|I^{\underline{\theta}}(\underline{a})|} \right| \leq c_{18} c_5 \, \rho^{1-\frac{1}{2k}}.$$

The constant c_{18} is independent of $\underline{\theta}$, $\underline{\omega}$, $\underline{a}$, ρ, and the size c_5 of the perturbation.

4.8 Let $(\underline{\theta}, \underline{\theta}', s) \in \widetilde{\mathcal{L}}$ be given.

Choose a subfamily Σ_2 of $\Sigma(\rho^{1/2k})$ of words starting with θ_0 such that

$$K(\theta_0) = \bigcup_{\Sigma_2} K(\underline{a})$$

is a partition of $K(\theta_0)$. Similarly, choose a subfamily Σ_2' of $\Sigma'(\rho^{1/2k})$ of words starting with θ_0' such that

$$K'(\theta_0') = \bigcup_{\Sigma_2'} K'(\underline{a}').$$

For each $(\underline{a}, \underline{a}') \in \Sigma_2 \times \Sigma_2'$, we have

$$c_{19}^{-1} \, \rho^{\frac{1}{2k}(d+d')} \leq \mu_d \times \mu_{d'}(K(\underline{a}) \times K'(\underline{a}')) \leq c_{19} \, \rho^{\frac{1}{2k}(d+d')}.$$

Let $J(\underline{a}, \underline{a}') := \pi_{\underline{\theta},\underline{\theta}',s}(I(\underline{a}) \times I(\underline{a}'))$ for $\underline{a} \in \Sigma_2$, $\underline{a}' \in \Sigma_2$; we have

$$c_{19}^{-1} \, \rho^{\frac{1}{2k}} \leq |J(a, a')| \leq c_{19} \, \rho^{\frac{1}{2k}}.$$

Say $(\underline{a}, \underline{a}')$ is *good* if there are no more than $c_{16}^{-1} \, \rho^{-1/2k(d+d'-1)}$ intervals $J(\underline{\tilde{a}}, \underline{\tilde{a}}')$ whose centers are distant from the center of $J(\underline{a}, \underline{a}')$ by less than $2c_{19} \, \rho^{1/2k}$.

Lemma: The number of bad pairs $(\underline{a}, \underline{a}')$ is less than $12 c_{19}^3 \, c_{16} c_{15} \, \rho^{-1/2k(d+d')}$.

Proof of lemma: By construction of $\widetilde{\mathcal{L}}$, there exists $\widetilde{\underline{\theta}}$, $\widetilde{\underline{\theta}}'$, $\tilde{s}$ with $d(\underline{\theta}, \widetilde{\underline{\theta}}) \leq (c_2 + c_0)\rho^{1/2}$,

$d(\underline{\theta}', \widetilde{\underline{\theta}}') \leq (c_2 + c_0)\rho^{1/2}$, $|s - \tilde{s}| \leq c_2\, \rho^{1/2}$ such that

$$\left\| \chi_{\widetilde{\underline{\theta}}, \widetilde{\underline{\theta}}', \tilde{s}} \right\|_{L^2}^2 \leq c_{15}\,.$$

Therefore, if we replace $\underline{\theta}$, $\underline{\theta}'$, s by $\widetilde{\underline{\theta}}$, $\widetilde{\underline{\theta}}'$, $\tilde{s}$ the endpoints of $J(\underline{a}, \underline{a}')$ only move by a distance of order at most $\rho^{1/2}$. It is thus sufficient to prove the lemma for $\widetilde{\underline{\theta}}$, $\widetilde{\underline{\theta}}'$, $\tilde{s}$ with $2c_{19}\, \rho^{1/2k}$ replaced by $5/2\, c_{19}\, \rho^{1/2k}$.

Let $(\underline{a}, \underline{a}')$ be bad; then there are at least $c_{16}^{-1}\, \rho^{-1/2k(d+d'-1)}$ pairs $(\tilde{a}, \tilde{a}')$ such that $J(\tilde{a}, \tilde{a}')$ is contained in the interval $J^*(\underline{a}, \underline{a}')$ of same center than $J(\underline{a}, \underline{a}')$ and length $6\, c_{19}\, \rho^{1/2k}$. This means that

$$\int_{J^*(\underline{a}, \underline{a}')} \chi_{\widetilde{\underline{\theta}}, \widetilde{\underline{\theta}}', \tilde{s}} \geq c_{19}^{-1}\, c_{16}^{-1}\, \rho^{1/2k} = \frac{1}{6} c_{19}^{-2}\, c_{16}^{-1} |J^*(\underline{a}, \underline{a}')|.$$

But then, by Cauchy-Schwarz

$$\begin{aligned} \frac{1}{6} c_{19}^{-2}\, c_{16}^{-1} |J^*(\underline{a}, \underline{a}')| \int_{J^*(\underline{a}, \underline{a}')} \chi_{\widetilde{\underline{\theta}}, \widetilde{\underline{\theta}}', \tilde{s}} &\leq \left(\int_{J^*(\underline{a}, \underline{a}')} \chi_{\widetilde{\underline{\theta}}, \widetilde{\underline{\theta}}', \tilde{s}} \right)^2 \\ &\leq |J^*(a, a')| \int_{J^*(a, a')} \chi^2_{\widetilde{\underline{\theta}}, \widetilde{\underline{\theta}}', \tilde{s}} \end{aligned}$$

and thus

$$\int_{J^*(\underline{a}, \underline{a}')} \chi^2_{\widetilde{\underline{\theta}}, \widetilde{\underline{\theta}}', \tilde{s}} \geq \frac{1}{6} c_{19}^{-2}\, c_{16}^{-1} \int_{J^*(\underline{a}, \underline{a}')} \chi_{\widetilde{\underline{\theta}}, \widetilde{\underline{\theta}}', \tilde{s}}$$

Let J^* be the union over all bad pairs $(\underline{a}, \underline{a}')$ of the intervals $J^*(\underline{a}, \underline{a}')$; one can extract a subfamily of intervals whose union is J^* and which does not cover any point more than twice.

Therefore we obtain

$$\int_{J^*} \chi^2_{\widetilde{\underline{\theta}}, \widetilde{\underline{\theta}}', \tilde{s}} \geq \frac{1}{12} c_{19}^{-2}\, c_{16}^{-1} \int_{J^*} \chi_{\widetilde{\underline{\theta}}, \widetilde{\underline{\theta}}', \tilde{s}}.$$

As $\left\| \chi_{\widetilde{\underline{\theta}}, \widetilde{\underline{\theta}}', \tilde{s}} \right\|_{L^2}^2 \leq c_{15}$, we get

$$\int_{J^*} \chi_{\widetilde{\underline{\theta}}, \widetilde{\underline{\theta}}', \tilde{s}} \leq 12\, c_{19}^2\, c_{16}\, c_{15}\,.$$

As J^* contains $J(\underline{a}, \underline{a}')$ for all bad $(\underline{a}, \underline{a}')$, the estimate of the lemma follows. $\square$

4.9 To keep the argument clearer, we now prove Proposition 4.6 when $\underline{\omega} = 0$ (for the unperturbed map g). Afterwards we will indicate how to modify the argument for general $\underline{\omega}$.

First, we construct the pairs $(\underline{b}, \underline{b}')$ amongst which the pairs $(\underline{b}^i, \underline{b}'^i)$ of 4.6 must be looked for. We make the following easy observation:

Lemma: Let $\underline{\theta} \in \Sigma_{nr}^-$. The number of words $\underline{c} \in \Sigma(\rho^{1/2})$ with $c_0 = \theta_0$, such that $\underline{\theta} c_1 \dots c_m \notin \Sigma_{nr}^-$ is $o(\rho^{-1/2d})$ as $\rho \to 0$, uniformly in $\underline{\theta}$.

Now let $(\underline{\theta}, \underline{\theta}', s) \in \widetilde{\mathcal{L}}$. It follows from conclusion (ii) of the Scale recurrence Lemma and the last observation that we can find at least $\dfrac{1}{2} c_3\, \rho^{-1/2(d+d')}$ pairs $(\underline{c}^i, \underline{c}'^i) \in \Sigma(\rho^{1/2}) \times \Sigma'(\rho^{1/2})$ such that, writing $T_{\underline{c}^i} T'_{\underline{c}'^i}(\underline{\theta}, \underline{\theta}', s) = (\underline{\theta}^i, \underline{\theta}'^i, s_i)$, we have:

- $\underline{\theta}^i \in \Sigma_{nr}^-$;
- $[s_i - \rho^{1/2}, s_i + \rho^{1/2}] \subset E^*(\underline{c}^i, \underline{c}'^i)$.

As $(\underline{\theta}^i, \underline{\theta}'^i, s_i)$ again belongs to $\widetilde{\mathcal{L}}$, we can for each i find at least $\dfrac{1}{2} c_3\, \rho^{-1/2(d+d')}$ pairs $(\underline{d}^{ij}, \underline{d}'^{ij}) \in \Sigma(\rho^{1/2}) \times \Sigma'(\rho^{1/2})$ (with the first letter of $\underline{d}^{ij}$, $\underline{d}'^{ij}$ being the last one of $\underline{c}^i$, $\underline{c}'^i$ respectively), such that writing $T_{\underline{d}^{ij}} T'_{\underline{d}'_{ij}}(\underline{\theta}^i, \underline{\theta}'^i, s_i) = (\underline{\theta}^{ij}, \underline{\theta}'^{ij}, s_{ij})$, we have

- $\underline{\theta}^{ij} \in \Sigma_{nr}^-$;
- $[s_{ij} - \rho^{1/2}, s_{ij} + \rho^{1/2}] \in E^*(\underline{d}^{ij}, \underline{d}'^{ij})$.

Concatenation of the $\underline{c}^i$, $\underline{c}'^i$ and $\underline{d}^{ij}$, $\underline{d}'^{ij}$ gives a family of words $(\underline{b}^{ij}, \underline{b}'^{ij})$ in $\Sigma(\rho) \times \Sigma'(\rho)$ with at least $\dfrac{1}{4} c_3^2\, \rho^{-(d+d')}$ elements.

Let $(\widetilde{\underline{\theta}}, \widetilde{\underline{\theta}}', \tilde{s})$ be such that $d(\underline{\theta}, \widetilde{\underline{\theta}}) \le (c_0+c_2)\rho^{1/2}$, $d(\underline{\theta}', \widetilde{\underline{\theta}}') \le (c_0+c_2)\rho^{1/2}$, $|s-\tilde{s}| \le c_2\, \rho^{1/2}$ and $\tilde{s} \in E(\widetilde{\theta}, \widetilde{\theta}')$.

Lemma: If c_{16} has been chosen sufficiently small, there are at least $\dfrac{1}{6} c_3\, c_{19}^{-2}\, \rho^{-\frac{d+d'}{2k}}$ pairs

$(\underline{a},\underline{a}') \in \Sigma_2 \times \Sigma_2'$ which are good and satisfy

$$\left\| \chi_{T_{\underline{a}} T'_{\underline{a}'}(\widetilde{\theta},\widetilde{\theta}',\widetilde{s})} \right\|_{L^2}^2 \le c_{16}^{-1}$$

and such that at least $\dfrac{1}{6} c_3 c_{19}^{-1}\, \rho^{-(d+d')(\frac{1}{2}-\frac{1}{2k})}$ pairs $(\underline{c}^i,\underline{c}'^i)$ start with $(\underline{a},\underline{a}')$.

Proof: By definition of $E(\widetilde{\underline{\theta}},\widetilde{\underline{\theta}}')$, the number of pairs $(\underline{a},\underline{a}') \in \Sigma_2 \times \Sigma_2'$ such that

$$\left\| \chi_{T_{\underline{a}} T'_{\underline{a}'}(\widetilde{\theta},\widetilde{\theta}',\widetilde{s})} \right\|_{L^2}^2 > c_{16}^{-1}$$

is at most $c_{16} c_{15}\, \rho^{-\frac{1}{2k}(d+d')}$. Call $(\underline{a},\underline{a}')$ very good if it is good and satisfy

$$\left\| \chi_{T_{\underline{a}} T'_{\underline{a}'}(\widetilde{\theta},\widetilde{\theta}',\widetilde{s})} \right\|_{L^2}^2 \le c_{16}^{-1}$$

The number of pairs $(\underline{a},\underline{a}')$ which are not very good is thus $\le c_{15}c_{16}(1+12c_{19}^3)\rho^{-\frac{1}{2k}(d+d')}$.

But for each $(\underline{a},\underline{a}') \in \Sigma_2 \times \Sigma_2'$, the number of pairs $(\underline{c},\underline{c}') \in \Sigma(\rho^{1/2}) \times \Sigma'(\rho^{1/2})$ starting with $(\underline{a},\underline{a}')$ is at most $c_{19}^2\, \rho^{(d+d')(\frac{1}{2k}-\frac{1}{2})}$.

It follows, if c_{16} is small enough, that at least $\dfrac{1}{3} c_3\, \rho^{-1/2(d+d')}$ pairs $(\underline{c}^i,\underline{c}'^i)$ start with a very good $(\underline{a},\underline{a}') \in \Sigma_2 \times \Sigma_2'$.

The total number of pairs $(\underline{a},\underline{a}')$ is at most $c_{19}\, \rho^{-\frac{d+d'}{2k}}$; if we discard those pairs for which less than $\dfrac{1}{6} c_3 c_{19}^{-1}\, \rho^{-(d+d')(\frac{1}{2}-\frac{1}{2k})}$ pairs $(\underline{c}^i,\underline{c}'^i)$ begin with $(\underline{a},\underline{a}')$, we are still left with $\dfrac{1}{6} c_3\, \rho^{-1/2(d+d')}$ pairs $(\underline{c}^i,\underline{c}'^i)$, of which no more than $c_{19}^2\, \rho^{-(d+d')(\frac{1}{2}-\frac{1}{2k})}$ may begin with the same $(\underline{a},\underline{a}')$. $\square$

4.10 Let us call excellent those pairs $(\underline{a},\underline{a}')$ given by the last lemma.

Recall that we have defined

$$J(\underline{a},\underline{a}') = \pi_{\underline{\theta},\underline{\theta}',s}(I(\underline{a}) \times I(\underline{a}')),$$

and define similarly $J(\underline{c}^i, \underline{c}'^i)$, $J(\underline{b}^{ij}, \underline{b}'^{ij})$. We only consider those i for which $(\underline{c}^i, \underline{c}'^i)$ starts with an excellent $(\underline{a}, \underline{a}')$.

The following general lemmas will be used repeteadly:

Lemma 1: Let $(J_\alpha)_{\alpha\in A}$, $(J'_\alpha)_{\alpha\in A}$ be two finite families of intervals, and let $\lambda > 1$, $\varepsilon > 0$, $\nu > 0$.

Assume that for every α, we have:

(i) $\varepsilon < |J_\alpha| < \lambda\varepsilon$, $\varepsilon < |J'_\alpha|$;

(ii) the centers of J_α, J'_α are distant by no more than $\nu\varepsilon$.

Then

$$m\Big(\bigcup_A J'_\alpha\Big) \geq \frac{1}{\lambda(4\nu+4)} m\Big(\bigcup_A J_\alpha\Big).$$

Proof: Let $A' \subset A$ be a minimal family with

$$\bigcup_{A'} J_\alpha = \bigcup_A J_\alpha .$$

Then each point in $\bigcup\limits_A J_\alpha$ belongs at most to two intervals J_α, $\alpha \in A'$. Replace each J'_α, $\alpha \in A'$, by the interval $\widetilde{J}'_\alpha$ with same center and length ε. One has

$$\sum_{A'} |\widetilde{J}'_\alpha| \geq \lambda^{-1} \sum_{A'} |J_\alpha| \geq \lambda^{-1} |\bigcup_A J_\alpha|.$$

On the other hand, let $x \in \mathbb{R}$; if $x \in \widetilde{J}'_\alpha$, for $\alpha \in A'$, then the center of $\widetilde{J}'_\alpha$ belongs to $\big[x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}\big]$ and the center of J_α is contained in $\big[x - \big(\nu + \frac{1}{2}\big)\varepsilon, x + \big(\nu + \frac{1}{2}\big)\varepsilon\big]$. Replacing the J_α by intervals $\widetilde{J}_\alpha$ with same center and length ε, we see that there are at most $(4\nu+4)$ $\alpha \in A'$ such that $x \in \widetilde{J}'_\alpha$. Then we have:

$$\Big|\bigcup_A J'_\alpha\Big| \geq \Big|\bigcup_{A'} \widetilde{J}'_\alpha\Big| \geq \frac{1}{4\nu+4} \Sigma |\widetilde{J}'_\alpha|. \qquad \square$$

Lemma 2: Let $(J_\alpha)_{\alpha\in A}$ be a finite family of intervals, and $\lambda > 1$. For each α, let K_α be a subset of J_α with $m(K_\alpha) \geq \lambda^{-1}|J_\alpha|$. Then

$$m\Big(\bigcup_A K_\alpha\Big) \geq \frac{\lambda^{-1}}{2} m\Big(\bigcup_A J_\alpha\Big).$$

Proof: We extract a subfamily A' such that $\bigcup_{A'} J_\alpha = \bigcup_A J_\alpha$ and each x belongs to at most two J_α , $\alpha \in A'$. Then we have

$$\begin{aligned} m\Big(\bigcup_A K_\alpha\Big) \geq m\Big(\bigcup_{A'} K_\alpha\Big) &\geq \frac{1}{2}\sum_{A'} m(K_\alpha) \\ &\geq \frac{\lambda^{-1}}{2}\sum_{A'} |J_\alpha|. \qquad \square \end{aligned}$$

4.11 Now let $(\underline{a}, \underline{a}')$ be an excellent pair. For i such that $(\underline{c}^i, \underline{c}'^i)$ starts with $(\underline{a}, \underline{a}')$, let $\widetilde{J}(\underline{c}^i, \underline{c}'^i) = \pi_{\underline{\widetilde{\theta}}, \underline{\widetilde{\theta}}', \tilde{s}}(I(\underline{c}^i) \times I(\underline{c}'^i))$, and let $\widetilde{J}_1(\underline{a}, \underline{a}')$ be the union of those $\widetilde{J}(\underline{c}^i, \underline{c}'^i)$. The $\mu_d \times \mu_{d'}$-measure of the union of the $K(\underline{c}^i) \times K'(\underline{c}'^i)$ is $\geq c_{20}\, \rho^{-\frac{1}{2k}(d+d')}$ (because $(\underline{a}, \underline{a}')$ is excellent), and follows then from 4.3 that

$$|\widetilde{J}_1(\underline{a}, \underline{a}')| \geq c_{21}\, c_{16}\, \rho^{-1/2k}.$$

When we come back to $(\underline{\theta}, \underline{\theta}', s)$ and define $J_1(\underline{a}, \underline{a}')$ to be the union of the $J(\underline{c}^i, \underline{c}'^i)$ starting with $(\underline{a}, \underline{a}')$, the sizes of the $J(\underline{c}^i, \underline{c}'^i)$ are still of order $\rho^{1/2}$ and the centers are distant from those of $\widetilde{J}(\underline{c}^i, \underline{c}'^i)$ by distances of order at most $\rho^{1/2}$. Then, by Lemma 1 above, we get

$$|J_1(\underline{a}, \underline{a}')| \geq c_{22}^{-1}\, c_{16}\, \rho^{1/2k}.$$

Similarly, for each $(\underline{c}^i, \underline{c}'^i)$ (starting with $(\underline{a}, \underline{a}')$), let $J_1(\underline{c}^i, \underline{c}'^i)$ be the union of the $J(\underline{b}^{ij}, \underline{b}'^{ij})$ starting with $(\underline{c}^i, \underline{c}'^i)$. Then are at least $\frac{1}{2} c_3\, \rho^{-1/2(d+d')}$ such $(\underline{b}^{ij}, \underline{b}'^{ij})$, and $(\underline{\theta}^i, \underline{\theta}'^i, s_i) = T_{\underline{c}^i} T'_{\underline{c}'^i}(\underline{\theta}, \underline{\theta}', s)$ belongs to $\widetilde{\mathcal{L}}$.

Therefore, using again 4.3, we conclude that

$$|J_1(\underline{c}^i,\underline{c}'^i)| \geq c_{23}^{-1}|J(\underline{c}^i,\underline{c}'^i)|.$$

[The argument is the same than above: we first consider $(\widetilde{\underline{\theta}}^i,\widetilde{\underline{\theta}}'^i,\tilde{s}_i)$, with $d(\underline{\theta}^i,\widetilde{\underline{\theta}}^i)$, $d(\underline{\theta}'^i,\widetilde{\underline{\theta}}'^i)$, $|s_i-\tilde{s}_i|$ of order $\rho^{1/2}$, such that $\left\|\chi_{\widetilde{\underline{\theta}}^i,\widetilde{\underline{\theta}}'^i,\tilde{s}_i}\right\|_{L^2}^2 \leq c_{15}$].

Finally, let $J_2(\underline{a},\underline{a}')$ be the union over those $(\underline{c}^i,\underline{c}'^i)$ starting with $(\underline{a},\underline{a}')$ of the sets $J_1(\underline{c}^i,\underline{c}'^i)$. By Lemma 2 above, we obtain that

$$|J_2(\underline{a},\underline{a}')| \geq \frac{1}{2}c_{23}^{-1}\,c_{22}^{-1}\,c_{16}\,\rho^{1/2k}.$$

Let φ (resp. φ_2) be the sum, over excellent pairs $(\underline{a},\underline{a}')$, of the characteristic functions of $J(\underline{a},\underline{a}')$ (resp. $J_2(\underline{a},\underline{a}')$).

The number of excellent pairs is $\geq \frac{1}{6}c_3c_{19}^{-2}\,\rho^{-\frac{d+d'}{2k}}$ by 4.8, and therefore

$$\int \varphi_2 \geq \frac{1}{12}c_{23}^{-1}\,c_{22}^{-1}\,c_3\,c_{19}^2\,c_{16}\,\rho^{-\frac{1}{2k}(d+d'-1)}.$$

On the other hands, because excellent pairs are good, one has

$$\varphi_2 \leq \varphi \leq c_{16}^{-1}\,\rho^{-1/2k(d+d'-1)}.$$

We conclude that

$$m\big(\{\varphi_2 \geq c_{16}^2\,\rho^{-\frac{1}{2k}(d+d'-1)}\}\big) \geq c_{24}\,c_{16}^2 := c_{17}$$

if $c_{16}>0$ is small enough (recall that the support of φ_2 is contained in an interval of length $1+R$).

But, if $\varphi_2(t) \geq c_{16}^2\,\rho^{-\frac{1}{2k}(d+d'-1)}$, this means that there are at least N excellent pairs $(\underline{a}^1,\underline{a}'^1),\ldots,(\underline{a}^N,\underline{a}'^N)$, and for each of these pairs $(\underline{a}^\ell,\underline{a}'^\ell)$, an element $(\underline{b}^{ij},\underline{b}'^{ij})$ starting with $(\underline{a}^\ell,\underline{a}'^\ell)$, such that $t\in J(\underline{b}^{ij},\underline{b}'^{ij})$; renaming $(\underline{b}^{ij},\underline{b}'^{ij})=(\underline{b}^\ell,\underline{b}'^\ell)$, and setting

$$T_{\underline{b}^\ell}T'_{\underline{b}'^\ell}(\underline{\theta},\underline{\theta}',s,t) = (\underline{\theta}^\ell,\underline{\theta}'^\ell,s_\ell,t_\ell),$$

conditions (i), (ii) of 4.6 are satisfied; one has by the choice of the $\underline{b}^{ij}$, $\underline{b}'^{ij}$:

(iii)" $(\underline{\theta}^\ell, \underline{\theta}'^\ell, \tilde{s}) \in \widetilde{\mathcal{L}}$ if $|\tilde{s} - s_\ell| \leq \rho^{1/2}$;

finally, $t \in J(\underline{b}^\ell, \underline{b}'^\ell) \Rightarrow |t_\ell| \leq 1 + R$.

Hence (i), (ii), (iii)", (iv)' are satisfied and $t \in L_{\underline{0}}^{-1}(\underline{\theta}, \underline{\theta}', s)$. We have proved that

$$|L_{\underline{0}}^{-1}(\underline{\theta}, \underline{\theta}', s)| \geq c_{17}\,.$$

4.12 We now consider the random perturbation $g^{\underline{\omega}}$ instead of g. By definition, we have to work with $g^{\underline{\omega}^*}$.

By 4.7, doing this, the endpoints of each $J(\underline{a}, \underline{a}')$ are moved by no more than $c_{18} c_5\, \rho^{1-\frac{1}{2k}}$. So if ρ is small enough, and $(\underline{a}, \underline{a}')$ is good, there will be (for $g^{\underline{\omega}^*}$) no more than $c_{16}^{-1}\, \rho^{-\frac{1}{2k}(d+d'-1)}$ intervals $J(\tilde{a}, \tilde{a}')$ whose centers are distant from the center of $J(a, a')$ by less than $\dfrac{7}{4} c_{19}\, \rho^{\frac{1}{2k}}$ [instead of $2c_{19}\, \rho^{1/2k}$ in the unperturbed situation].

When we consider the pairs $(\underline{c}^i, \underline{c}'^i)$, $(\underline{d}^{ij}, \underline{d}'^{ij})$, $(\underline{b}^{ij}, \underline{b}'^{ij})$ of the beginning of 4.8, the conditions $\underline{\theta}^i \in \Sigma_{nr}^-$, $\underline{\theta}^{ij} \in \Sigma_{nr}^-$ are independent of $\underline{\omega}$. If we write

$$T_{\underline{c}_i}^{\underline{\omega}^*} T'_{\underline{c}'_i}(\underline{\theta}, \underline{\theta}', s) = (\underline{\theta}^i, \underline{\theta}'^i, s_i(\underline{\omega}^*)),$$

we have $|s_i(\underline{\omega}^*) - s_i| \leq c_5 c'_{18}\, \rho^{1-\frac{1}{2k}}$ by Lemma 4.7 (ii); writing

$$T_{\underline{d}_{ij}}^{\underline{\omega}^*} T'_{\underline{d}'_{ij}}(\underline{\theta}^i, \underline{\theta}'^i, s_i) = (\underline{\theta}^{ij}, \underline{\theta}'^{ij}, s_{ij}^0(\underline{\omega}^*)),$$

$$T_{\underline{d}_{ij}}^{\underline{\omega}^*} T'_{\underline{d}'_{ij}}(\underline{\theta}^i, \underline{\theta}'^i, s_i(\underline{\omega}^*) = (\underline{\theta}^{ij}, \underline{\theta}'^{ij}, s_{ij}(\underline{\omega}^*)),$$

we have

$$|s_{ij}(\underline{\omega}^*) - s_{ij}^0(\underline{\omega}^*)| \leq c_5 c''_{18}\, \rho^{1-\frac{1}{2k}},$$

$$|s_{ij}^0(\underline{\omega}^*) - s_{ij}| \leq c_5 c'_{18}\, \rho^{1-\frac{1}{2k}},$$

and thus

$$|s_{ij} - s_{ij}(\underline{\omega}^*)| \leq c_5 (c'_{18} + c''_{18}) \rho^{1-\frac{1}{2k}}.$$

This will guarantee condition (iii)' of 4.6 later.

Keeping the notion of excellent pairs to be the same, we consider for the perturbed map $g^{\underline{\omega}^*}$ and excellent pair $(\underline{a}, \underline{a}')$ the subsets $J_1^{\underline{\omega}^*}(\underline{a}, \underline{a}')$, $J_2^{\underline{\omega}^*}(\underline{a}, \underline{a}')$ of $J^{\underline{\omega}^*}(\underline{a}, \underline{a}')$.

By the estimates of Lemma 4.7 and the Lemma 1 of 4.10, we will still have

$$|J_1^{\underline{\omega}^*}(\underline{a}, \underline{a}')| \geq c_{22}'^{-1}\, c_{16}\, \rho^{\frac{1}{2k}}.$$

For the same reasons, we will get

$$|J_1^{\underline{\omega}^*}(\underline{c}^i, \underline{c}'^i)| \geq c_{23}'^{-1} |J^{\underline{\omega}^*}(\underline{c}^i, \underline{c}'^i)|$$

and this allows to prove in exactly the same way that

$$m\big(\{\varphi_2^{\underline{\omega}^*} \geq c_{16}^2\, \rho^{-\frac{1}{2k}(d+d'-1)}\}\big) \geq c_{17}'\,.$$

The proof of

$$\varphi_2^{\underline{\omega}^*}(t) \geq c_{16}^2\, \rho^{-\frac{1}{2k}(d+d'-1)} \Rightarrow t \in L_{\underline{\omega}}^{-1}(\underline{\theta}, \underline{\theta}', s)$$

is the same, except that we have

$$|s_\ell(\underline{\omega}^*) - s_\ell| \leq c_5(c_{18}' + c_{18}'')\rho^{1-\frac{1}{2k}},$$

which gives (iii)' instead of (iii)".

4.13 If we want to prove the reinforcement of theorem 2.4 given in 2.5, one has to modify the definition of $L_{\underline{\omega}}^0(\underline{\theta}, \underline{\theta}', s)$ (for $(\underline{\theta}, \underline{\theta}', s) \in \widetilde{\mathcal{L}}$, $\underline{\omega} \in \Omega$) as follows.

Let $\tilde{d} < d + d' - 1$; choose $\tilde{d} < d^* < d + d' - 1$, and k large enough to have

$$\left(1 - \frac{1}{k}\right)(d + d' - 1) > d^*.$$

Let $M = [\rho^{-d^*}]$. We define $\overline{L}_{\underline{\omega}}^0(\underline{\theta}, \underline{\theta}', s)$ to be set of points $(\underline{\theta}, \underline{\theta}', s, t)$ in the fiber for which there exist pairs $(\underline{b}_m^\ell, \underline{b}_m'^\ell)$, $1 \leq \ell \leq N$, $1 \leq m \leq M$ such that conditions (ii), (iii), (iv) of §4.6 are satisfied for all ℓ, m and moreover (i') for any $1 \leq m \leq M$, the words $\underline{b}_m^1, \ldots, \underline{b}_m^N$ are independent.

We also ask that

(v) $\qquad I(\underline{b}_m^\ell) \cap I(b_{\underline{m}'}^{\ell'}) = \phi$ if $(\ell, m) \neq (\ell', m')$.

Replacing (iii), (iv) by (ii)', (iv)' we also define $\overline{L}_{\underline{\omega}}^{-1}(\underline{\theta}, \underline{\theta}', s)$.

We now claim that proposition 4.6 still holds:

$$|\overline{L}_{\underline{\omega}}^{-1}(\underline{\theta}, \underline{\theta}', s)| > \bar{c}_{17}\,.$$

To prove this, we proceed as above: for an excellent pair $(\underline{a}, \underline{a}')$ and $(\underline{c}^i, \underline{c}^{'i})$ starting with $(\underline{a}, \underline{a}')$, we now define $\overline{J}_1(\underline{c}^i, \underline{c}^{'i})$ as the set of t which belong to at least $\rho^{-\frac{1}{2}d^*}$ intervals $J(\underline{b}^{ij}, \underline{b}^{'ij})$ (with $(\underline{b}^{ij}, \underline{b}^{'ij})$ starting with $(\underline{c}^i, \underline{c}^{'i})$). As $\frac{1}{2}\, d^* < \frac{1}{2}\,(d + d' - 1)$, one easily sees that we still have

$$|\overline{J}_1(\underline{c}^i, \underline{c}^{'i})| \geq \bar{c}_{23}^{-1} |J(\underline{c}^i, \underline{c}^{'i})|.$$

Next we define $\overline{J}_2(\underline{a}, \underline{a}')$ as the set of t which belong to at least $\rho^{-1/2\, d^*}$ sets $\overline{J}_1(\underline{c}^i, \underline{c}^{'i})$ (with $(\underline{c}^i, \underline{c}^{'i})$ starting with $(\underline{a}, \underline{a}')$). Because $\frac{1}{2}\, d^* < \left(\frac{1}{2} - \frac{1}{2k}\right)(d + d' - 1)$, we will still have

$$|\overline{J}_2(\underline{a}, \underline{a}')| \geq \bar{c}_{23}^{-1}\, \rho^{1/2k}.$$

If $\bar{\varphi}_2$ is the sum, over excellent pairs, of the characteristic functions of the $\overline{J}_2(\underline{a}, \underline{a}')$, the same argument than for φ_2 shows that

$$m(\{\bar{\varphi}_2 \geq N\}) \geq \bar{c}_{17}\,.$$

If $\bar{\varphi}_2(t) \geq N$, it is now easy to construct the $(\underline{b}_m^\ell, \underline{b}_m^{'\ell})$ as required: the index ℓ refers to the excellent pairs such that $t \in \overline{J}_2(\underline{a}, \underline{a}')$, and the index $M \leq \rho^{-\frac{1}{2}\, d^*} \cdot \rho^{-\frac{1}{2}\, d^*}$ for the possibilities for fixed $(\underline{a}, \underline{a}')$.

§5. The family of random perturbations.

5.1 In this section, we will prove the estimates of lemma §4.7 and the proposition §3.8. We begin with lemma §4.7, part (i).

We are given $\underline{\theta} \in \Sigma^-$ and $\underline{a} \in \Sigma^+$, with $a_0 = \theta_0$. Let $\underline{\omega} \in \Omega$; we want to show that

$$|k^{\underline{\theta},\underline{\omega}}(h^{\underline{\omega}}(\underline{a})) - k^{\underline{\theta}}(h(\underline{a}))| \leq c_{18}\, c_5\, \rho^{1-\frac{1}{2k}}.$$

Let $\hat{a} = (a_0, \ldots, a_m)$ be an initial part of $\underline{a}$ which belongs to $\Sigma(\rho^{1-1/2k})$. The sizes of $I^{\underline{\theta}}(\hat{a})$ and $I^{\underline{\theta},\underline{\omega}}(\hat{a})$ have order $\rho^{1-1/2k}$, and we have to show that these two intervals are distant by no more than $c_5 c_{18}\, \rho^{1-1/2k}$.

To simplify notation, we just write $\tilde{f}_{aa'}$ for the inverse branches associated to $g^{\underline{\omega}}$.

Because the C^0-distance between $f_{aa'}$ and $\tilde{f}_{aa'}$ is $c_5\, \rho^{1+1/2k}$, and these maps are contractions, the endpoints of $I^{\underline{\omega}}(\hat{a})$ are distant from those of $I(\hat{a})$ by no more than a multiple of $c_5\, \rho^{1+1/2k}$. So it is sufficient to see that the C^0-distance between $k^{\underline{\theta}}$ and $k^{\underline{\theta},\underline{\omega}} := \tilde{k}^{\underline{\theta}}$ is no more than $c_{18} c_5\, \rho^{1-1/2k}$.

5.2 Let $z \in I(\theta_0)$; set

$$x_0 = \tilde{x}_0 = 0, \quad y_0 = \tilde{y}_0 = 1, \quad z_0 = \tilde{z}_0 = z,$$

and define by induction

$$x_n = f_{\theta_{-n},\theta_{-n+1}}(x_{n-1}), \quad y_n = f_{\theta_{-n},\theta_{-n+1}}(y_{n-1})$$
$$z_n = f_{\theta_{-n},\theta_{-n+1}}(z_{n-1}), \quad \tilde{x}_n = \tilde{f}_{\theta_{-n},\theta_{-n+1}}(\tilde{x}_{n-1}),$$
$$\tilde{y}_n = \tilde{f}_{\theta_{-n},\theta_{-n+1}}(\tilde{y}_{n-1}), \quad \tilde{z}_n = \tilde{f}_{\theta_{-n},\theta_{-n+1}}(\tilde{z}_{n-1}).$$

We will just write f_n, $\tilde{f}_n$ for $f_{\theta_{-n},\theta_{-n+1}}$, $\tilde{f}_{\theta_{-n},\theta_{-n+1}}$. We have

$$k^{\underline{\theta}}(z) = \operatorname*{Lim}_{n\to+\infty} \frac{z_n - x_n}{y_n - x_n},$$

$$\tilde{k}^{\underline{\theta}}(z) = \operatorname*{Lim}_{n\to+\infty} \frac{\tilde{z}_n - \tilde{x}_n}{\tilde{y}_n - \tilde{x}_n}.$$

Because f_n , $\tilde{f}_n$ are C^0-distant by $c_5\,\rho^{1+1/2k}$, we will always have

$$|z_n - \tilde{z}_n| \leq c c_5\,\rho^{1+1/2k},$$

$$|x_n - \tilde{x}_n| \leq c c_5\,\rho^{1+1/2k},$$

$$|y_n - \tilde{y}_n| \leq c c_5\,\rho^{1+1/2k}.$$

Thus, as long as $|y_n - x_n| \geq c_0^{-3}\,\rho^{-1/k}$, one has

$$\left|\frac{z_n - x_n}{y_n - x_n} - \frac{\tilde{z}_n - \tilde{x}_n}{\tilde{y}_n - \tilde{x}_n}\right| \leq c c_5\,\rho^{1-\frac{1}{2k}}.$$

Let n_0 be the largest such integer; we will have $|y_n - x_n| \leq c_0^{-2}\,\rho^{-1/k}$ for $n \geq n_0$. Therefore on an interval containing the six points x_n , y_n , z_n , $\tilde{x}_n$, $\tilde{y}_n$, $\tilde{z}_n$, we will have (cf. 3.6.3) either

$$\tilde{f}_{n+1}(u) = f_{n+1}(u)$$

or

$$\tilde{f}_{n+1}(u) = f_{n+1}(u) + c_5\,\omega_{n+1}\,\rho^{1+1/2k}$$

(where ω_{n+1} is a component of $\underline{\omega}$).
Thus we will have, for $n \geq n_0$:

$$z_{n+1} - x_{n+1} = (z_n - x_n)\int_0^1 Df_{n+1}(x_n + t(z_n - x_n))\,dt,$$

$$\tilde{z}_{n+1} - \tilde{x}_{n+1} = (\tilde{z}_n - \tilde{x}_n)\int_0^1 Df_{n+1}(\tilde{x}_n + t(\tilde{z}_n - \tilde{x}_n))\,dt,$$

and similarly for $y_{n+1} - x_{n+1}$, $\tilde{y}_{n+1} - \tilde{x}_{n+1}$.

Write

$$\frac{z_{n+1} - x_{n+1}}{y_{n+1} - x_{n+1}} = \frac{z_n - x_n}{y_n - x_n}\ \alpha_n\ ,$$

$$\frac{\tilde{z}_{n+1} - \tilde{x}_{n+1}}{\tilde{y}_{n+1} - \tilde{x}_{n+1}} = \frac{\tilde{z}_n - \tilde{x}_n}{\tilde{y}_n - \tilde{x}_n}\ \tilde{\alpha}_n\ ,$$

with

$$\alpha_n = \frac{\int_0^1 Df_{n+1}(x_n + t(z_n - x_n))dt}{\int_0^1 Df_{n+1}(x_n + t(y_n - x_n))dt},$$

$$\tilde{\alpha}_n = \frac{\int_0^1 Df_{n+1}(\tilde{x}_n + t(\tilde{z}_n - \tilde{x}_n))dt}{\int_0^1 Df_{n+1}(\tilde{x}_n + t(\tilde{y}_n - \tilde{x}_n))dt}.$$

We have

$$|\alpha_n - 1| \le c|y_n - x_n|, \quad |\tilde{\alpha}_n - 1| \le c|\tilde{y}_n - \tilde{x}_n|$$

and also

$$|\alpha_n - \tilde{\alpha}_n| \le cc_5\, \rho^{1+1/2k}.$$

We obtain, setting

$$\frac{\tilde{z}_{n+1} - \tilde{x}_{n+1}}{\tilde{y}_{n+1} - \tilde{x}_{n+1}} - \frac{z_{n+1} - x_{n+1}}{y_{n+1} - x_{n+1}} = w_{n+1}$$

that

$$\begin{aligned} |w_{n+1}| &\le |w_n|(1 + c|y_n - x_n|) \\ &\quad + c \text{ Min } (c_5\, \rho^{1+1/2k}, |x_n - y_n| + |\tilde{x}_n - \tilde{y}_n|), \end{aligned}$$

starting with

$$|w_{n_0}| \le cc_5\, \rho^{1-1/2k}.$$

It is thus clear that $|w_n| \le cc_5\, \rho^{1-1/2k}$ for all $n \ge n_0$, which gives part (i) of Lemma §4.7.

5.3 We now turn to the second part of Lemma §4.7.

Let $\underline{\theta} \in \Sigma^-$, and $\underline{a} = a_0 \dots a_m$ a word in Σ with $a_0 = \theta_0$ and $|I(\underline{a})| > c_0^{-1}\, \rho$. We define

$$\begin{aligned} u_{-m} &= \tilde{u}_{-m} = 0, & v_{-m} &= \tilde{v}_{-m} = 1 \\ u_{-j} &= f_{a_j a_{j+1}}(u_{-j-1}), & v_{-j} &= f_{a_j a_{j+1}}(v_{-j-1}) \\ \tilde{u}_{-j} &= \tilde{f}_{a_j a_{j+1}}(\tilde{u}_{-j-1}), & \tilde{v}_{-j} &= \tilde{f}_{a_j a_{j+1}}(\tilde{v}_{-j-1}), \end{aligned}$$

for $0 \leq j < m$. Then u_0, v_0 (resp. $\tilde{u}_0$, $\tilde{v}_0$) are the endpoints of $I(\underline{a})$ (resp. $I^{\underline{\omega}}(\underline{a})$). For $n \geq 0$, we define $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$, $(u_n)_{n\geq 0}$, $(v_n)_{n\geq 0}$, $(\tilde{x}_n)_{n\geq 0}$, $(\tilde{y}_n)_{n\geq 0}$, $(\tilde{u}_n)_{n\geq 0}$, $(\tilde{v}_n)_{n\geq 0}$ as in 5.2. One has

$$\frac{|I^{\underline{\theta},\underline{\omega}}(\underline{a})|}{|I^{\underline{\theta}}(\underline{a})|} = \operatorname*{Lim}_{n\to+\infty} \frac{|\tilde{v}_n - \tilde{u}_n|}{|v_n - u_n|} \frac{|y_n - x_n|}{|\tilde{y}_n - \tilde{x}_n|}.$$

As above, we have

$$\mathrm{Max}(|v_n - \tilde{v}_n|, |u_n - \tilde{u}_n|) \leq cc_5\, \rho^{1+1/2k}, \;\; \forall\, n \geq -m$$

$$\mathrm{Max}(|x_n - \tilde{x}_n|, |y_n - \tilde{y}_n|) \leq cc_5\, \rho^{1+1/2k}, \;\; \forall\, n \geq 0.$$

Let n_1 be the largest integer for which $|v_n - u_n| \geq c_0^{-3}\, \rho^{-1/k}$; we have

$$\left| \mathrm{Log}\, \frac{|\tilde{v}_{n_1} - \tilde{u}_{n_1}|}{|v_{n_1} - u_{n_1}|} \right| \leq cc_5\, \rho^{1-1/2k}$$

and, for $n \geq n_1$

$$\frac{\tilde{v}_{n+1} - \tilde{u}_{n+1}}{v_{n+1} - u_{n+1}} = \frac{\tilde{v}_n - \tilde{u}_n}{v_n - u_n}\; \lambda_n\,,$$

with

$$\lambda_n = \frac{\int_0^1 Df_{n+1}(\tilde{u}_n + t(\tilde{v}_n - \tilde{u}_n))dt}{\int_0^1 Df_{n+1}(u_n + t(v_n - u_n))dt}$$

so that

$$|\lambda_n - 1| \leq cc_5\, \rho^{1+1/2k}.$$

This means that for $n_2 = [\rho^{-1/k}] \gg n_1$, we will still have

$$\left| \mathrm{Log}\, \frac{|\tilde{v}_{n_2} - \tilde{u}_{n_2}|}{|v_{n_2} - u_{n_2}|} \right| \leq cc_5\, \rho^{1-1/2k}.$$

We can show in the same way that

$$\left| \mathrm{Log}\, \frac{|\tilde{y}_{n_2} - \tilde{x}_{n_2}|}{|y_{n_2} - x_{n_2}|} \right| \leq cc_5\, \rho^{1-1/2k}$$

For $n \geq n_2$, we write as in 5.2

$$\frac{|v_{n+1} - u_{n+1}|}{|y_{n+1} - x_{n+1}|} = \frac{|v_n - u_n|}{|y_n - x_n|} \alpha_n$$

with $|\alpha_n - 1| \leq c|y_n - x_n|$; this allows to get

$$\operatorname*{Lim}_{n \to +\infty} \frac{|v_n - u_n|}{|y_n - x_n|} = \frac{|v_{n_2} - u_{n_2}|}{|y_{n_2} - x_{n_2}|} \alpha,$$

with $|\alpha - 1| \leq c|y_{n_2} - x_{n_2}|$ (because $|y_n - x_n|$ is decreasing geometrically). Similarly

$$\operatorname*{Lim}_{n \to +\infty} \frac{|\tilde{v}_n - \tilde{u}_n|}{|\tilde{y}_n - \tilde{x}_n|} = \frac{|\tilde{v}_{n_2} - \tilde{u}_{n_2}|}{|\tilde{y}_{n_2} - \tilde{x}_{n_2}|} \tilde{\alpha},$$

with $|\tilde{\alpha} - 1| \leq c|\tilde{y}_{n_2} - \tilde{x}_{n_2}|$. As we have

$$\operatorname{Max}(|y_{n_2} - x_{n_2}|, |\tilde{y}_{n_2} - \tilde{x}_{n_2}|) \ll \rho^{1-1/2k}.$$

we obtain the required estimate.

5.4 In the rest of this section, we will prove the Proposition §3.8.

We first recall the setting.

We are given $u = (\underline{\theta}, \underline{\theta}', s, t) \in \widetilde{\mathcal{L}} \times \mathbb{R}$.

The set $\Omega_1(u)$ is the set of parameters $\underline{\omega} \in \Omega$, such that there exists $\tilde{u} = (\widetilde{\underline{\theta}}, \widetilde{\underline{\theta}}', \tilde{s}, \tilde{t}) \in \widetilde{\mathcal{L}} \times \mathbb{R}$ with

$$d(\underline{\theta}, \widetilde{\underline{\theta}}) \leq 2\rho^{5/2}, \quad d(\underline{\theta}', \widetilde{\underline{\theta}}') \leq 2\rho^{5/2}, \quad |s - \tilde{s}| < \rho, \quad |t - \tilde{t}| < \rho$$

and $\tilde{t} \in L^0_{\underline{\omega}}(\widetilde{\theta}, \widetilde{\theta}', \tilde{s})$.

The set $\Omega_0(u)$ is the set of parameters $\underline{\omega} \in \Omega$ for which there exists $\underline{b} \in \Sigma(\rho)$, $\underline{b}' \in \Sigma'(\rho)$ (with $b_0 = \theta_0$, $b'_0 = \theta'_0$) such that the image

$$T^{\omega}_{\underline{b}} T'_{\underline{b}'}(u) = \hat{u} = (\widehat{\underline{\theta}}, \widehat{\underline{\theta}}', \hat{s}, \hat{t})$$

satisfies

$$(\widehat{\theta}, \widehat{\theta}', \hat{s}') \in \mathcal{L} \text{ for } |\hat{s} - \hat{s}'| < \frac{1}{2}\rho^{1/2}$$

$$\hat{t} \in L^0_{\underline{\omega}}(\widehat{\theta}, \widehat{\theta}', \hat{s}).$$

We have to prove that, provided c_5 is large enough

$$\mathbb{P}(\Omega^1(u) - \Omega^0(u)) \leq \exp(-c_7\, \rho^{-\frac{1}{2k}(d+d'-1)})$$

where $\mathbb{P}$ is normalized Lebesgue measure on Ω.

5.5 Recall the decomposition of 4.5:

$$\Omega = [-1, +1]^{\Sigma_1(\underline{\theta})} \times [-1, +1]^{\Sigma_1 - \Sigma_1(\underline{\theta})}$$

$$\underline{\omega} = (\underline{\omega}', \underline{\omega}'')$$

which only depends on $\underline{\theta}$ through an endword in $\Sigma(\rho^{1/2k})$. We have set $\underline{\omega}^* = (\underline{0}, \underline{\omega}'')$ in §4.5, and $L^0_{\underline{\omega}}(\underline{\theta}, \underline{\theta}', s)$ actually only depend on $\underline{\omega}^*$ (or $\underline{\omega}''$), *not* on $\underline{\omega}'$.

Therefore, the property $\underline{\omega} \in \Omega^1(u)$ depends only on $\underline{\omega}''$ (one has $\Sigma_1(\widetilde{\underline{\theta}}) = \Sigma_1(\underline{\theta})$ as $d(\underline{\theta}, \widetilde{\underline{\theta}}) \leq 2\rho^{5/2}$).

We will fix $\underline{\omega}'' \in \Omega'' = [-1, +1]^{\Sigma_1 - \Sigma_1(\underline{\theta})}$. Then either $\underline{\omega}^* = (\underline{0}, \underline{\omega}'') \notin \Omega^1(u)$, and then $([-1, +1]^{\Sigma_1(\underline{\theta})} \times \{\underline{\omega}''\}) \cap \Omega^1(u)$ is empty, or $\underline{\omega}^* \in \Omega^1(u)$. In this last case, we will prove that

$$\mathbb{P}_{\Omega'}(\Omega' - \Omega'^0(u)) \leq \exp(-c_7\, \rho^{-\frac{1}{2k}(d+d'-1)}),$$

where

$$\Omega' = [-1, +1]^{\Sigma_1(\underline{\theta})},$$

$$\Omega'^0(u) = \{\omega' \in \Omega', (\omega', \omega'') \in \Omega^0(u)\},$$

and $\mathbb{P}_{\Omega'}$ is Lebesgue measure normalized on Ω'. Proposition §3.8 will then follow by Fubini's theorem.

5.6 Now on, $\underline{\omega}''$ is fixed, with $\underline{\omega}^* \in \Omega^1(\underline{u})$. This means that there exists $\tilde{u} = (\widetilde{\theta}, \widetilde{\theta}', \tilde{s}, \tilde{t})$ in $\widetilde{\mathcal{L}} \times \mathbb{R}$ such that

$$d(\underline{\theta}, \underline{\widetilde{\theta}}) \leq 2\rho^{5/2}, \quad d(\underline{\theta}', \underline{\widetilde{\theta}}') \leq 2\rho^{5/2}, \quad |s - \tilde{s}| < \rho, \quad |t - \tilde{t}| < \rho$$

$$\text{and } \tilde{t} \in L^0_{\underline{\omega}^*}(\widetilde{\theta}, \widetilde{\theta}', \tilde{s}).$$

By definition of $L^0_{\underline{\omega}^*}$, there exist pairs $(\underline{b}^1, \underline{b}'^1), \dots, (\underline{b}^N, \underline{b}'^N)$ (with $b_0^i = \widetilde{\theta}_0 = \theta_0$, $b_0'^i = \widetilde{\theta}'_0 = \theta'_0$), with $N = [c_{16}^2\, \rho^{-\frac{1}{2k}(d+d'-1)}]$, such that, writing

$$\begin{aligned} T^{\underline{\omega}^*}_{\underline{b}^i} T'_{\underline{b}'^i}(\underline{\widetilde{\theta}}, \underline{\widetilde{\theta}}', \tilde{s}, \tilde{t}) &=: \tilde{u}_i \\ &=: (\underline{\widetilde{\theta}}^i, \underline{\widetilde{\theta}}'^i, \tilde{s}_i, \tilde{t}_i) \end{aligned}$$

we have

(i) $\underline{b}^1, \dots, \underline{b}^N$ are pairwise independent;

(ii) $\underline{\widetilde{\theta}}^i \in \Sigma^-_{nr}$ for $1 \leq i \leq N$;

(iii) $(\underline{\widetilde{\theta}}^i, \underline{\widetilde{\theta}}'^i, \tilde{s}'_i) \in \mathcal{L}^-$ for $1 \leq i \leq N$, $|\tilde{s}_i - \tilde{s}'_i| \leq \dfrac{2}{3}\rho^{1/2}$;

(iv) $|\tilde{t}_i| \leq 2(1+R)$ for $1 \leq i \leq N$.

For $1 \leq i \leq N$, consider

$$u_i = (\underline{\theta}^i, \underline{\theta}'^i, s_i, t_i) := T^{\underline{\omega}^*}_{\underline{b}^i} T'_{\underline{b}'^i}(\underline{\theta}, \underline{\theta}', s, t),$$

and more generally, for $\underline{\omega}' \in \Omega'$, $\underline{\omega} = (\underline{\omega}', \omega'')$:

$$u_i(\underline{\omega}') = (\underline{\theta}^i, \underline{\theta}'^i, s_i(\underline{\omega}'), t_i(\underline{\omega}')) := T^{\underline{\omega}}_{\underline{b}^i} T'_{\underline{b}'^i}(\underline{\theta}, \underline{\theta}', s, t).$$

Lemma: We have, for $1 \le i \le N$, $\underline{\theta}^i \in \Sigma_{nr}^-$ and, $\forall \underline{\omega}' \in \Omega'$, $(\underline{\theta}^i, \underline{\theta}'^i, s'_i) \in \widetilde{\mathcal{L}}$ if $|s'_i - s_i(\omega')| < \frac{1}{2}\rho^{1/2}$.

Proof: From $d(\underline{\theta}, \widetilde{\underline{\theta}}) \le 2\rho^{5/2}$, it follows that $d(\underline{\theta}^i, \widetilde{\underline{\theta}}^i) \le c\,\rho^{7/2}$; but $\widetilde{\underline{\theta}}^i \in \Sigma_{nr}^-$ and this set is determined by a condition on an endword in $\Sigma(\rho^3)$. Therefore $\underline{\theta}^i \in \Sigma_{nr}^-$. Moreover, $\widetilde{\mathcal{L}}$ has the same fiber over $(\widetilde{\underline{\theta}}^i, \widetilde{\underline{\theta}}'^i)$. By proposition §1.3, $k^{\underline{\theta}^i}$ and $k^{\widetilde{\underline{\theta}}^i}$ are at a C^1 distance of order at most $\rho^{5/2}$, and similarly for $k'^{\underline{\theta}'^i}$ and $k'^{\widetilde{\underline{\theta}}'^i}$. Therefore

$$|\tilde{s}_i - s_i| \le c|\tilde{s} - s| + c\,\rho^{5/2} \le c\,\rho.$$

Also, by Lemma 4.6, part (ii):

$$|\tilde{s}_i(\underline{\omega}') - \tilde{s}_i| \le cc_5\,c_{18}\,\rho^{1-1/2k} \le \frac{1}{10}\rho^{1/2}$$

if ρ is small enough.
By property (iii) above, we therefore will have

$$(\underline{\theta}^i, \underline{\theta}'^i, s'_i) \in \widetilde{\mathcal{L}} \text{ if } |s'_i - \tilde{s}_i(\underline{\omega}')| < \frac{1}{2}\rho^{1/2}. \qquad \square$$

5.7 Let $\underline{\omega}' \in \Omega'$; in view of the lemma above, if there exists $1 \le i \le N$ with

$$t_i(\underline{\omega}') \in L^0_{\underline{\omega}}(\underline{\theta}^i, \underline{\theta}'^i, s_i(\underline{\omega}'))$$

then $\underline{\omega}' \in \Omega'^0(u)$.
Let $\underline{a} \in \Sigma_2^-$, $\underline{a} = (\theta_{-r}, \dots, \theta_0)$ be the endword of $\underline{\theta}$ in $\Sigma_2^- (\subset \Sigma(\rho^{1/2k}))$. For $1 \le i \le N$, let

$$\underline{a}^i = (\theta_{-r}, \dots, \theta_0 = b_0^i, b_1^i, \dots, b_{r_i}^i)$$

be the element in $\Sigma_1(\underline{\theta})$ obtained by concatenation of $\underline{a}$ and the beginning of $\underline{b}^i$. Because $\underline{b}^1, \dots, \underline{b}^N$ are pairwise independent, the elements $\underline{a}^1, \dots, \underline{a}^N$ of $\Sigma_1(\underline{\theta})$ are distinct.

Let $\widetilde{\Sigma}_1(\underline{\theta}) = \Sigma_1(\underline{\theta}) - \{\underline{a}^1, \dots, \underline{a}^N\}$; we will just denote by ω_i $(1 \le i \le N)$ the coordinate of $\underline{\omega}'$ corresponding to $\underline{a}^i$. We thus have

$$\Omega' = [-1,+1]^N \times \widetilde{\Omega}', \quad \widetilde{\Omega}' = [-1,+1]^{\widetilde{\Sigma}_1(\theta)}$$
$$\underline{\omega}' = ((\omega_1, \dots, \omega_N), \underline{\tilde{\omega}}').$$

We will prove that for any $\underline{\tilde{\omega}}' \in \widetilde{\Omega}'$, the set of $(\omega_1, \dots, \omega_N)$ for which there exists $1 \le i \le N$ with $t_i(\underline{\omega}') \in L^0_{\underline{\omega}}(\underline{\theta}^i, \underline{\theta}'^i, s_i(\underline{\theta}'))$ fills $[-1,+1]^N$ except for a set of normalized Lebesgue measure $< \exp(-c_7\, \rho^{-\frac{1}{2k}(d+d'-1)})$ (provided c_5 is large enough). By Fubini's theorem, this will imply what we want.

5.8 For $\underline{\tilde{\omega}}' \in \widetilde{\Omega}'$, set $\underline{\hat{\omega}} = ((\underline{0}, \underline{\tilde{\omega}}'), \underline{\omega}'')$.

For $1 \le i \le N$, define

$$L^i := L^{-1}_{\underline{\hat{\omega}}}(\underline{\theta}^i, \underline{\theta}'^i, s_i(\hat{\omega})) \quad (\text{cf. } \S 4.5).$$

Choose $\underline{\overline{\theta}} \in \Sigma^-$ with $d(\underline{\overline{\theta}}, \underline{\theta}) < \rho^{5/2}$, such that the endword $\underline{a} \in \Sigma_2^-$ of $\underline{\theta}$ (and $\underline{\overline{\theta}}$) does not appear elsewhere in $\underline{\overline{\theta}}$. For $\underline{\omega}' = (\omega_1, \dots, \omega_N, \tilde{\omega}')$, $\underline{\omega} = (\underline{\omega}', \underline{\omega}'')$, consider

$$T^{\underline{\omega}}_{\underline{b}^i} T'_{\underline{b}'^i}(\underline{\overline{\theta}}, \underline{\theta}', s, t) = (\underline{\overline{\theta}}^i, \underline{\theta}'^i, \bar{s}_i(\underline{\omega}'), \bar{t}_i(\underline{\omega}')).$$

Lemma 1: $\bar{t}_i(\underline{\omega}')$ depends only (for fixed $\underline{\omega}''$) on ω_i, not on $\tilde{\omega}'$ and ω_j, $j \ne i$. Moreover, there exists $c'_7 > 0$ such that

$$m\{\omega_i, \bar{t}_i(\omega_i) \in L^i\} \ge c'_7\,.$$

Lemma 2: Let $\underline{\omega}' = (\omega_1, \dots, \omega_N, \underline{\tilde{\omega}}') \in \Omega'$ and $1 \le i \le N$. If $\bar{t}_i(\omega_i) \in L^i$, then $t_i(\underline{\omega}') \in L^0_{\underline{\omega}}(\underline{\theta}^i, \underline{\theta}'^i, s_i(\underline{\omega}'))$, and thus $\underline{\omega}' \in \Omega'_0(u)$.

Clearly the two lemmas imply Proposition §3.8, if we recall that $N = [c_{16}^2\, \rho^{-1/2k(d+d'-1)}]$.

5.9 Proof of Lemma 1.

Let $1 \le i \le N$. We want to understand how the endpoints of $K^{\underline{\theta},\underline{\omega}}(\underline{b}^i)$ depend on $\underline{\omega}'$. Consider first the sequence defined by

$$x_0 = x_0(\underline{\omega}') = 0, \quad x_n(\underline{\omega}') = f^{\underline{\omega}}_{\overline{\theta}_{-n}\overline{\theta}_{-n+1}}(x_{n-1}(\underline{\omega}')) \text{ for } n > 0.$$

Claim: for $n \ge 0$, x_n does not depend on $\underline{\omega}'$.

By §3.6.3, it is sufficient to prove that for $n > 0$, $x_n(\underline{\omega}')$ does not belong to any $\widehat{I}(\underline{\tilde{a}})$, $\underline{\tilde{a}} \in \Sigma_1(\underline{\theta})$.

If $|I(\overline{\theta}_{-n}, \dots, \overline{\theta}_0)| \le \rho^{\frac{3}{5k}}$ this is true because elements of $\Sigma_1(\underline{\theta})$ start with $\underline{a}$, and there is only one occurence of $\underline{a}$ in $\overline{\underline{\theta}}$: $\underline{a} = (\overline{\theta}_{-r}, \dots, \overline{\theta}_0) \in \Sigma(\rho^{1/2k})$.

If $|I(\overline{\theta}_{-n}, \dots, \overline{\theta}_0)| > \rho^{3/5k}$, $x_n(\underline{\omega}')$ cannot in fact belong to any $\widehat{I}(\underline{\tilde{a}})$, $\tilde{a} \in \Sigma_1$, at least if ρ is small enough; indeed, the endpoints of $K(\theta_0)$ are preperiodic under g, so $\tilde{a}$ would end with a preperiodic word (independent of ρ) of approximate size $\rho^{2/5k}$, which precludes $\tilde{a} \in \Sigma_1$. $\square$

In the same way, one proves that the sequence

$$y_0 = y_0(\underline{\omega}') = 1, \qquad y_n(\underline{\omega}') = f^{\underline{\omega}}_{\overline{\theta}_{-n},\overline{\theta}_{-n+1}}(y_{n-1}, (\underline{\omega}')),$$

is independent of $\underline{\omega}'$.

Consider now, with $\underline{b}^i = (b^i_0, \dots, b^i_m)$, the sequence

$$\begin{aligned} u_{-m} &= u_{-m}(\underline{\omega}') = 0, \\ u_{-j}(\underline{\omega}') &= f^{\underline{\omega}}_{b^i_j, b^i_{j+1}}(u_{-j-1}(\underline{\omega}')),\ 0 \le j < m \\ u_n(\underline{\omega}') &= f^{\underline{\omega}}_{\overline{\theta}_{1-n},\overline{\theta}_{-n+1}}(u_{n-1}(\underline{\omega}')),\ \ n > 0. \end{aligned}$$

For the same reasons than above (using that $\underline{a}$ does not appear twice in $\overline{\underline{\theta}}^i$), we have that $u_n(\underline{\omega}')$ is independent of $\underline{\omega}'$ for $-m \le n < r$. Then we have, by §3.6.3

$$u_r(\underline{\omega}') = u_r(\underline{0}) + c_5\, \omega_i\, \rho^{1+1/2k}.$$

For $n > r$, $u_n(\underline{\omega}')$ again cannot belong to any $\widehat{I}(\tilde{a})$, $\tilde{a} \in \Sigma_1(\underline{\theta})$. Setting

$$\underline{\overline{\overline{\theta}}} = \sigma^{-r}(\overline{\underline{\theta}}) = (\dots \overline{\theta}_{-r-1}, \overline{\theta}_{-r})$$

we see that the endpoint of $I^{\overline{\underline{\theta}},\underline{\omega}}(\underline{b}^i)$ determined by (u_n) satisfies

$$u(\underline{\omega}') = B \circ k^{\underline{\overline{\overline{\theta}}},\hat{\omega}}(u_r(0) + c_5 \omega_i\, \rho^{1+1/2k}),$$

where B is the affine map (with the appropriate orientation) sending $I(\underline{a})$ onto $I(\theta_0)$. Neither B nor $k^{\underline{\overline{\overline{\theta}}},\hat{\omega}}$ depend on $\underline{\omega}'$.

The other endpoint of $I^{\overline{\underline{\theta}},\underline{\omega}}(\underline{b}^i)$ satisfies a similar formula.

This means that $\bar{t}_i$, as a function of $\underline{\omega}'$, only depends on ω_i. Moreover, as $k^{\underline{\overline{\overline{\theta}}},\hat{\omega}}$ is bounded in the C^1-topology and the derivative of B is of order $\rho^{-1/2k}$, we have

$$c^{-1} c_5 < \left| \frac{\partial \bar{t}_i}{\partial \omega_i} \right| < c c_5\, .$$

On the other hand, we have $|L_i| > c_{17}$ by proposition §4.5, and $L_i \subset [-(1+R), 1+R]$. So to end the proof of Lemma 1, we have just to control $\bar{t}_i(\underline{0})$. Let

$$T_{\underline{b}^i}^{\underline{\omega}^*} T'_{\underline{b}'^i}(\overline{\underline{\theta}}, \underline{\theta}', s, t) = (\overline{\underline{\theta}}^i, \underline{\theta}'^i, s_i^*, t_i^*).$$

We have just seen that in fact $t_i^* = \bar{t}_i(0)$. On the other hand, we have

$$T_{\underline{b}^i}^{\underline{\omega}^*} T'_{\underline{b}'^i}(\widetilde{\underline{\theta}}, \widetilde{\underline{\theta}}', \tilde{s}, \tilde{t}) = (\widetilde{\underline{\theta}}^i, \widetilde{\underline{\theta}}'^i, \tilde{s}_i, \tilde{t}_i),$$

with $d(\overline{\underline{\theta}}, \widetilde{\underline{\theta}}) \leq 3\rho^{5/2}$, $d(\underline{\theta}', \widetilde{\underline{\theta}}') \leq 2\rho^{5/2}$, $|s - \tilde{s}| \leq \rho$, $|t - \tilde{t}| < \rho$ and

$$|\tilde{t}_i| \leq 2(1+R).$$

Then $|\tilde{t}_i - t_i^*| \leq cR$, which implies

$$|\bar{t}_i(0)| \leq 2(1+R) + cR;$$

this concludes the proof of Lemma 1.

5.10 Proof of Lemma 2.

Let $\underline{\omega}' = (\omega_1, \ldots, \omega_N, \underline{\tilde{\omega}}')$, $\underline{\omega} = (\underline{\omega}', \underline{\omega}'')$, $\underline{\hat{\omega}} = ((0, \tilde{\omega}'), \underline{\omega}'')$. Let also $1 \leq i \leq N$. We assume that $\bar{t}_i(\omega_i) \in L^{-1}_{\underline{\hat{\omega}}}(\underline{\theta}^i, \underline{\theta}'^i, s_i(\hat{\omega}))$ and we want to show that

$$t_i(\underline{\omega}') \in L^0_{\underline{\omega}}(\underline{\theta}^i, \underline{\theta}'^i, s_i(\underline{\omega}')).$$

The hypothesis means that there exist pairs $(\underline{c}^1, \underline{c}'^1), \ldots, (\underline{c}^N, \underline{c}'^N)$ (with $c_0^j = \theta_0^i$, $c_0'^j = \theta_0'^i$ in $\Sigma(\rho) \times \Sigma'(\rho)$ such that, if

$$T^{\underline{\hat{\omega}}(i)}_{\underline{c}^j} T_{\underline{c}'^j}(\underline{\theta}^i, \underline{\theta}'^i, s_i(\hat{\omega}), \bar{t}_i(\omega_i)) = (\underline{\theta}^{(j)}, \underline{\theta}'^{(j)}, \hat{s}_{(j)}, \hat{t}_{(j)})$$

(where $\underline{\hat{\omega}}(i)$ is obtained from $\hat{\omega}$ by setting the coordinates in $\Sigma_1(\underline{\theta}^i)$ equal to 0), one has

(i) $\underline{c}^1, \ldots, \underline{c}^N$ are pairwise independent;

(ii) $\underline{\theta}^{(j)} \in \Sigma^-_{nr}$ for $1 \leq j \leq N$;

(iii)' $(\underline{\theta}^{(j)}, \underline{\theta}'^{(j)}, \hat{s}'_{(j)}) \in \widetilde{\mathcal{L}}$ for $|\hat{s}_{(j)} - \hat{s}'_{(j)}| < \dfrac{3}{4}\rho^{1/2}$;

(iv)' $|\hat{t}_{(j)}| \leq 1 + R$ for $1 \leq j \leq N$.

Let

$$T^{\underline{\omega}(i)}_{\underline{c}^j} T'_{\underline{c}'^j}(\underline{\theta}^i, \underline{\theta}'^i, s_i(\underline{\omega}'), t_i(\underline{\omega}')) = (\underline{\theta}^{(j)}, \underline{\theta}'^{(j)}, s_{(j)}, t_{(j)}),$$

where $\underline{\omega}(i)$ is obtained from $\underline{\omega}$ by setting the coordinates in $\Sigma_1(\underline{\theta}^i)$ equal to 0.

We will show tht for $1 \leq j \leq N$:

(iii) $(\underline{\theta}^{(j)}, \underline{\theta}'^{(j)}, s'_{(j)}) \in \widetilde{\mathcal{L}}$ for $|s'_{(j)} - s_{(j)}| < \dfrac{2}{3}\rho^{1/2}$,

(iv) $|t_{(j)}| \leq 2(1 + R)$,

and this implies Lemma 2.

It is thus sufficient to show that

$$|s_{(j)} - \hat{s}_{(j)}| < \frac{1}{12}\rho^{1/2},$$

$$|t_{(j)} - \hat{t}_{(j)}| < 1 + R.$$

The first inequality is easy: by Lemma §4.7, part (ii), we have

$$|s_i(\underline{\omega}') - s_i(\hat{\omega})| \leq c_5 c_{18}\, \rho^{1-1/2k}$$

and applying a second time the same estimate from this lemma (to compare $T_{\underline{c}^j}^{\underline{\hat{\omega}}(i)}$ and $T_{\underline{c}^j}^{\underline{\omega}(i)}$) will give

$$|s_{(j)} - \hat{s}_{(j)}| < cc_5 c_{18}\, \rho^{1-1/2k}$$

which is $< \dfrac{1}{12}\rho^{1/2}$ for small enough ρ.

To prove the second inequality, we first compare $\bar{t}_i(\omega_i)$ and $t_i(\underline{\omega}')$. By definition, $\bar{t}_i(\omega_i)$ is the t-coordinate of $T_{\underline{b}^i}^{\underline{\omega}} T_{\underline{b}'^i}(\underline{\overline{\theta}}, \underline{\theta}', s, t)$, while $t_i(\underline{\omega}')$ is the t-coordinate of $T_{\underline{b}^i}^{\underline{\omega}} T_{\underline{b}'^i}(\underline{\theta}, \underline{\theta}', s, t)$. As $d(\underline{\theta}, \underline{\overline{\theta}}) \leq \rho^{5/2}$, the C^0-distance between $k^{\underline{\theta}}$, $k^{\underline{\overline{\theta}}}$ is at most of order $\rho^{5/2}$, and this imply

$$|\bar{t}_i(\omega_i) - t_i(\underline{\omega}')| \leq \rho^{3/2}.$$

We next prove the

Claim: $|s_i(\underline{\hat{\omega}}) - s_i(\underline{\omega}')| \leq cc_5\, \rho^{1+1/2k}$
(a reforcement of Lemma §4.7 part (ii) in this case).

Proof: Let

$$T_{\underline{b}^i}^{\underline{\omega}} T'_{\underline{b}'^i}(\underline{\overline{\theta}}, \underline{\theta}', s, t) = (\underline{\overline{\theta}}^i, \underline{\theta}'^i, \bar{s}_i(\underline{\omega}'), \bar{t}_i(\underline{\omega}')).$$

As $d(\underline{\theta}, \underline{\overline{\theta}}) \leq \rho^{5/2}$, $k^{\underline{\theta}, \underline{\omega}}$ and $k^{\underline{\overline{\theta}}, \underline{\omega}}$ are at a C^1-distance of order no more than $\rho^{5/2}$. This implies that

$$|s_i(\underline{\hat{\omega}}) - \bar{s}_i(\underline{\hat{\omega}})| \leq c\, \rho^{5/2}$$

$$|s_i(\underline{\omega}') - \bar{s}_i(\underline{\omega}')| \leq c\, \rho^{5/2}.$$

To compare $\bar{s}_i(\underline{\hat{\omega}})$, $\bar{s}_i(\underline{\omega}')$, we go back to the proof of Lemma 1. The endpoints of $I^{\underline{\overline{\theta}}, \underline{\omega}}(\underline{b}^i)$ depend, for fixed $\underline{\omega}''$, only on ω_i:

$$u(\underline{\omega}') = B \circ k^{\underline{\overline{\overline{\theta}}}, \hat{\omega}}(u_r(0) + c_5 \omega_i\, \rho^{1+1/2k}),$$

$$v(\underline{\omega}') = B \circ k^{\underline{\overline{\overline{\theta}}}, \hat{\omega}}(v_r(0) + c_5 \omega_i\, \rho^{1+1/2k}).$$

As $k^{\underline{\bar{\bar{\theta}}},\hat{\omega}}$ is C^2-bounded, we obtain

$$|\bar{s}_i(\underline{\hat{\omega}}) - \bar{s}_i(\underline{\omega}')| \leq cc_5\,\rho^{1+1/2k},$$

proving the claim. $\square$

Finally, let

$$T^{\underline{\hat{\omega}}(i)}_{\underline{c}^j}\, T'_{\underline{c}'^j}(\underline{\theta}^i, \underline{\theta}'^i, s_i(\underline{\omega}'), t_i(\underline{\omega}')) = (\underline{\theta}^{(j)}, \underline{\theta}'^{(j)}, \tilde{s}_{(j)}, \tilde{t}_{(j)}).$$

Lemma 3: We have $|t_{(j)} - \tilde{t}_{(j)}| < cc_5\,\rho^{1/2k}$.

The lemma is proved in the next section. Using it, we finish the proof of Lemma 2.

Let $\tilde{\tilde{t}}_{(j)}$ be the t-coordinate of

$$T^{\underline{\hat{\omega}}(i)}_{\underline{c}^j}\, T'_{\underline{c}'^j}(\underline{\theta}^i, \underline{\theta}'^i, s_i(\underline{\hat{\omega}}), t_i(\underline{\omega}')).$$

Because $|\bar{t}_i(\omega_i) - t_i(\underline{\omega}')| \leq c\,\rho^{3/2}$, we have

$$|\hat{t}_{(j)} - \tilde{\tilde{t}}_{(j)}| \leq c\,\rho^{1/2}.$$

Because $|s_i(\underline{\omega}') - s_i(\underline{\hat{\omega}})| \leq cc_5\,\rho^{1+1/2k}$, we have

$$|\tilde{\tilde{t}}_{(j)} - \tilde{t}_{(j)}| \leq cc_5\,\rho^{\frac{1}{2k}}.$$

By Lemma 3, the same estimate holds for $|\tilde{t}_{(j)} - t_{(j)}|$. We conclude that for ρ small enough, we have indeed

$$|t_{(j)} - \hat{t}_{(j)}| < 1 + R,$$

as was required. $\square$

5.11 Proof of Lemma 3.

We will compare, for $\bar{s} \in J_R$ and $\bar{t} \in \mathbb{R}$, the t-coordinates of

$$T_{\underline{c}^j}^{\underline{\omega}(i)}\, T'_{\underline{c}'^j}(\underline{\theta}^i, \underline{\theta}'^i, \bar{s}, \bar{t})$$

and

$$T_{\underline{c}^j}^{\underline{\hat{\omega}}(i)}\, T'_{\underline{c}'^j}(\underline{\theta}^i, \underline{\theta}'^i, \bar{s}, \bar{t}).$$

The two elements $\underline{\omega}(i)$, $\underline{\hat{\omega}}(i)$ have different coordinates only at $\underline{a}^1, \ldots, \underline{a}^N$ where those of $\underline{\omega}(i)$ are arbitrary and those of $\underline{\hat{\omega}}(i)$ are 0. We observe also that, because $\underline{\theta}^i \in \Sigma_{nr}^-$, we have

$$\Sigma_1(\underline{\theta}) \cap \Sigma_1(\underline{\theta}^i) = \emptyset,$$

and that the coordinates of $\underline{\omega}(i)$ (or $\underline{\hat{\omega}}(i)$) in $\Sigma_1(\underline{\theta}^i)$ are 0.

Thus, we have to understand how the t-coordinate of $T_{\underline{c}^j}^{\underline{\omega}(i)}\, T'_{\underline{c}'^j}(\underline{\theta}^i, \underline{\theta}'^i, \bar{s}, \bar{t})$ depends on $(\omega_1, \ldots, \omega_N)$, which amounts to understand how the endpoints of $I^{\underline{\theta}^i, \underline{\omega}(i)}(\underline{c}^j)$ depend on $(\omega_1, \ldots, \omega_N)$.

The scheme is the same as in the proof of Lemma 1. We replace $\underline{\theta}^i$ by $\overline{\underline{\theta}}^i$; as now $d(\underline{\theta}^i, \overline{\underline{\theta}}^i) < c\,\rho^{7/2}$, this will only move the endpoints of $I^{\underline{\theta}^i, \underline{\omega}(i)}(\underline{c}^j)$ by at most $c\,\rho^{7/2}$, and the corresponding t-coordinates by $c\,\rho^{5/2}$.

In the iteration process of §5.9, again the only time where things depend on $(\omega_1, \ldots, \omega_N)$ is when we apply $f^{\underline{\omega}(i)}_{\overline{\theta}_{-r}\overline{\theta}_{-r+1}}$ (we use here that $\underline{\theta}^{(j)} \in \Sigma_{nr}^-$). We will see as in §5.9 that the endpoints $w^{\pm}$ of $I^{\overline{\underline{\theta}}^i, \underline{\omega}(i)}(\underline{c}^j)$ do not depend on ω_k, $k \neq i$, and the dependence on ω_i is as follows

$$w^{\pm}(\omega_i) = \frac{\bar{k}(w_r^{\pm} + c_5\omega_i\rho^{1+1/2k}) - \bar{k}(u_r + c_5\omega_i\rho^{1+1/2k})}{\bar{k}(v_r + c_5\omega_i\rho^{1+1/2k}) - \bar{k}(u_r + c_5\omega_i\rho^{1+1/2k})}\,,$$

where $\bar{k} = k^{\overline{\overline{\underline{\theta}}}, \underline{\hat{\omega}}(i)}$ does not depend on $\omega_1, \ldots, \omega_N$.

Writing

$$w^{+}(\omega_i) = \frac{w_r^{+} - u_r}{v_r - u_r}\, \frac{\int_0^1 D\bar{k}(u_r + t(w_r^{+} - u_r) + c_5\omega_i\rho^{1+1/2k})dt}{\int_0^1 D\bar{k}(u_r + t(v_r - u_r) + c_5\omega_i\rho^{1+1/2k})dt}\,,$$

we see that

$$\left|\frac{\partial w^+(\omega_i)}{\partial \omega_i}\right| \leq c c_5\, \rho^{1+1/2k},$$

as $\bar{k}$ is C^2-bounded. We thus have the control on the endpoints of $I^{\underline{\overline{\theta}}^i,\underline{\omega}(i)}(\underline{c}^j)$ which guarantees the estimate of the lemma. $\square$

5.12 For the theorem in 2.5, we need to prove the probability estimate in 3.11. But, with the notations of 4.13 and fixed $1 \leq m \leq M$, the same proof than above (with the $\overline{L}_\omega$ instead of L_ω) shows that

$$\mathbb{P}(\overline{\Omega}^1(u) - \Omega_m^0(u)) \leq \exp\big(- \overline{\overline{c}}_7\, \rho^{-\frac{1}{2k}(d+d'-1)}\big)$$

where in the definition of $\Omega_m^0(u)$ we require $(\underline{b}, \underline{b}')$ to be one the $\underline{b}_m^{\ell}, \underline{b}_{m'}^{\ell'}$. Thus we obtain

$$\mathbb{P}(\overline{\Omega}^1(u) - \overline{\Omega}^0(u)) \leq M\, \exp\big(- \overline{\overline{c}}_7\, \rho^{-\frac{1}{2k}(d+d'-1)}\big) \leq \exp\big(- \bar{c}_7\, \rho^{-\frac{1}{2}(d+d'-1)}\big).$$

§6. Proof of the Scale recurrence Lemma.

6.1 We will first formulate and prove a result, in a slightly more general setting, which contains the key argument in the proof of Scale recurrence Lemma.

We consider a finite alphabet A and a finite set Λ with a map

$$\Lambda \mapsto A \times A$$

$$\lambda \mapsto (\alpha(\lambda), \omega(\lambda)).$$

Define

$$\Lambda_i^j = \{\lambda, \alpha(\lambda) = i, \omega(\lambda) = j\},$$

$$\Lambda_i = \{\lambda, \alpha(\lambda) = i\},$$

$$\Lambda^j = \{\lambda, \omega(\lambda) = j\},$$

$$\Lambda^{(2)} = \{(\lambda, \lambda') \in \Lambda^2,\ \alpha(\lambda') = \omega(\lambda)\}.$$

Set also

$$N_i^j = \#\,\Lambda_i^j,\ N_i = \#\,\Lambda_i\,,\ p_i^j = N_i^{-1}\,N_i^j\,.$$

We will assume that there exists $p^* > 0$ such that $p_i^j \geq p^*$ for all i, j.

The stochastic matrix (p_i^j) has a left eigenvector $(p^i)_{i\in A}$ satisfying

$$\sum_i p^i p_i^j = p^j \geq p^*.$$

$$\sum_i p^i = 1.$$

If we set

$$p_\lambda^{\lambda'} = \begin{cases} 0 & \text{if} \quad \omega(\lambda) \neq \alpha(\lambda') \\ N_i^{-1} & \text{if}\ \omega(\lambda) = \alpha(\lambda') = i \in A, \end{cases}$$

$$p^\lambda = N_i^{-1} p^i,\ \ \lambda \in \Lambda_i$$

then $(p_\lambda^{\lambda'})_{(\lambda,\lambda')\in\Lambda^2}$ is again a stochastic matrix with left eigenvector (p^λ). Indeed we have (with $\lambda' \in A$, $\alpha(\lambda') = j$)

$$\begin{aligned} \sum_\lambda p^\lambda p_\lambda^{\lambda'} &= \sum_i \sum_{\lambda\in\Lambda_i^j} N_i^{-1}\,p^i\,N_j^{-1} \\ &= \sum_i p_i^j\,p^i\,N_j^{-1} = p^j\,N_j^{-1} = p^{\lambda'}. \end{aligned}$$

For $z = (z_\lambda)_{\lambda\in\Lambda} \in \mathbb{C}^\Lambda$, define

$$\begin{aligned} ||z||^2 &= \sum_\Lambda p^\lambda |z_\lambda|^2 \\ &= \sum_i p^i\,\frac{1}{N_i}\Big(\sum_{\Lambda_i} |z_\lambda|^2\Big). \end{aligned}$$

Observe that, if $(\theta_\lambda^{\lambda'})$ is a family of complex numbers of modulus ≤ 1 and we define

$$w_\lambda = \sum_{\lambda'} p_\lambda^{\lambda'}\,\theta_\lambda^{\lambda'}\,z_{\lambda'}$$

we have

$$|w_\lambda|^2 \le \sum_{\lambda'} p_\lambda^{\lambda'} \sum_{\lambda'} p_\lambda^{\lambda'} |z_{\lambda'}|^2 = \sum_{\lambda'} p_\lambda^{\lambda'} |z_{\lambda'}|^2$$

hence

$$||w||^2 \le ||z||^2.$$

6.2 In the setting above, we consider a family $(E_\lambda)_{\lambda\in\Lambda}$ of bounded measurable subsets of $\mathbb{R}$, and a family $a_\lambda^{\lambda'}$ of real numbers. Let also $\rho > 0$ be a small positive number. For $\lambda \in \Lambda$, $x \in \mathbb{R}$, $\omega(\lambda) = i$, let

$$n_\lambda(x) = N_i^{-1} \# \{\lambda' \in \Lambda_i\,, [x-\rho, x+\rho] \subset E_{\lambda'} - a_\lambda^{\lambda'}\}.$$

We will assume that the following holds: there are $\Delta_0 > 0$, $0 < \kappa_0 < 1$ such that for any $\xi \in [1, \Delta_0 \rho^{-1}]$, the operator

$$w_\lambda = \sum_{\lambda'} p_\lambda^{\lambda'} \, e^{i\xi a_\lambda^{\lambda'}} \, z_{\lambda'}$$

acting on $(\mathbb{C}^\Lambda, ||\ \ ||)$ has norm $\le \kappa_0$ (we know already that the norm is ≤ 1).

Proposition: Under this hypothesis, there exists $0 < \kappa_1 < 1$, $\varepsilon > 0$, $0 < \tau < 1$, *depending only on* Δ_0, κ_0 such that, if $|E_\lambda| \le \varepsilon$ for all $\lambda \in \Lambda$, then

$$\sum p^\lambda |E_\lambda^*| \le \kappa_1 \sum p^\lambda |E_\lambda|$$

where

$$E_\lambda^* = \{n_\lambda(x) > \tau\}.$$

6.3 Proof of proposition.

For $\lambda \in \Lambda$, $\omega(\lambda) = i$, let:

$$X_\lambda = \mathbf{1}_{E_\lambda},$$

$$Y_\lambda = \frac{1}{N_i} \sum_{\lambda' \in \Lambda_i} X_{\lambda'}(x + a_\lambda^{\lambda'}),$$

$$Z_\lambda(x) = \frac{1}{2\rho} \int_{-\rho}^{\rho} Y_\lambda(x+t)\, dt.$$

We have $Z_\lambda \geq n_\lambda$ hence

$$\sum_\lambda p^\lambda |Z_\lambda|_{L^2}^2 \geq \tau^2 \sum_\lambda p^\lambda |E_\lambda^*|.$$

We will show, under the hypotheses of the proposition that

$$\sum_\lambda p^\lambda |Z_\lambda|_{L^2}^2 \leq \kappa_2 \sum p^\lambda |X_\lambda|_{L^2}^2 = \kappa_2 \sum_\lambda p^\lambda |E_\lambda|,$$

for some $\kappa_2 < 1$ depending only on κ_0, Δ_0 and this will imply the proposition (taking $\kappa_1 = \tau = \kappa_2^{1/3}$). To prove this inequality, we will consider the Fourier transforms of X_λ, Z_λ, Y_λ. We normalize the Fourier transform as

$$\widehat{X}_\lambda(\xi) = \int_{-\infty}^{+\infty} X_\lambda(x) e^{-i\xi x}\, dx.$$

We then have

$$\begin{aligned} \widehat{Y}_\lambda(\xi) &= \frac{1}{N_i} \sum_{\lambda' \in \Lambda_i} \widehat{X}_{\lambda'}(\xi)\, e^{i a_\lambda^{\lambda'} \xi} \\ &= \sum_{\lambda'} p_\lambda^{\lambda'}\, e^{i a_\lambda^{\lambda'} \xi}\, \widehat{X}_{\lambda'}(\xi), \end{aligned}$$

and

$$\widehat{Z}_\lambda(\xi) = \frac{\sin \rho\xi}{\rho\xi} \widehat{Y}_\lambda(\xi).$$

We estimate $\sum p^\lambda |\widehat{Z}_\lambda(\xi)|^2$ in various ways, depending on ξ:

a) if $|\xi| \le 1$, we use

$$\begin{aligned} |\widehat{Z}_\lambda(\xi)| \le |\widehat{Y}_\lambda(\xi)| &\le \int_{-\infty}^{+\infty} Y_\lambda(x)\,dx \\ &= \sum_{\lambda'} p_\lambda^{\lambda'} |E_{\lambda'}| \end{aligned}$$

to get

$$\begin{aligned} \sum_\lambda p^\lambda |\widehat{Z}_\lambda(\xi)|^2 &\le \sum_\lambda p^\lambda \Big(\sum_{\lambda'} p_\lambda^{\lambda'} |E_{\lambda'}|^2 \Big) \\ &\le \sum_{\lambda'} p^{\lambda'} |E_{\lambda'}|^2\,; \end{aligned}$$

b) if $1 \le |\xi| \le \Delta_0 \rho^{-1}$, we use again

$$|\widehat{Z}_\lambda(\xi)| \le |\widehat{Y}_\lambda(\xi)|$$

and by the hypothesis of the proposition

$$\sum p^\lambda |\widehat{Y}_\lambda(\xi)|^2 \le \kappa_0^2 \sum p^\lambda |\widehat{X}_\lambda(\xi)|^2;$$

c) if $|\xi| \ge \Delta_0 \rho^{-1}$, from 6.1, we have

$$\sum p^\lambda |\widehat{Y}_\lambda(\xi)|^2 \le \sum p^\lambda |\widehat{X}_\lambda(\xi)|^2,$$

but now

$$\left| \frac{\sin \rho\xi}{\rho\xi} \right| \le \kappa_3 = \operatorname*{Max}_{u \ge \Delta_0} \left| \frac{\sin u}{u} \right| < 1,$$

hence $\sum p^\lambda |\widehat{Z}_\lambda(\xi)|^2 \le \kappa_3^2 \sum p^\lambda |\widehat{X}_\lambda(\xi)|^2$. Putting these estimates together gives

$$\begin{aligned} \int_{-\infty}^{+\infty} \sum_\lambda p^\lambda |\widehat{Z}_\lambda(\xi)|^2\,d\xi \le [\mathrm{Max}(\kappa_0, \kappa_3)]^2 &\int_{-\infty}^{+\infty} \sum_\lambda p^\lambda |\widehat{X}_\lambda(\xi)|^2\,d\xi \\ &+ 2 \sum p^\lambda |E_\lambda|^2. \end{aligned}$$

Now we have

$$||Z_\lambda||^2_{L^2} = \frac{1}{2\pi}||\widehat{Z}_\lambda||^2_{L^2}$$

$$||X_\lambda||^2_{L^2} = |E_\lambda| = \frac{1}{2\pi}||\widehat{X}_\lambda||^2_{L^2}\,,$$

and therefore

$$\sum_\lambda p^\lambda |Z_\lambda|^2_{L^2} \leq \kappa_2 \sum p^\lambda |E_\lambda|,$$

with $\kappa_2 = [\mathrm{Max}(\kappa_0, \kappa_3)]^2 + \pi^{-1}\varepsilon$. If ε is small enough, we have $\kappa_2 < 1$ which concludes the proof of the proposition. $\square$

Remarks: The main point of the proposition is that τ, ε, κ_1 depend neither on the E_λ nor on the $a_\lambda^{\lambda'}$ or the combinatorics.

6.4 In the same setting that 6.2, let $\Delta_1 > 0$ and

$$\hat{n}_\lambda(x) = N_i^{-1}\,\#\{\lambda' \in \Lambda_i, [x - \Delta_1\rho, x + \Delta_1\rho] \cap (E_{\lambda'} - a_\lambda^{\lambda'}) \neq \emptyset\}.$$

We denote by $V_\delta(E)$ the δ-neighbourhood of a subset $E \subset \mathbb{R}$ (i.e. the set of points at distance $< \delta$ from E).

Corollary: There exists $0 < \kappa_4 < 1$, $\varepsilon_1 > 0$, $0 < \tau < 1$, depending only on Δ_0, κ_0, and there exists $\Delta > 0$, depending only on Δ_0, Δ_1, κ_0 such that, if $|V_{\Delta\rho}(E_\lambda)| \leq \varepsilon_1$ for all $\lambda \in \Lambda$, then

$$\sum_\lambda p^\lambda |V_{\Delta\rho}(\widehat{E}_\lambda)| \leq \kappa_4 \sum_\lambda p^\lambda |V_{\Delta\rho}(E_\lambda)|,$$

where $\widehat{E}_\lambda = \{\hat{n}_\lambda > \tau\}$.

Proof: We first observe that, for any bounded subset E,

$$|V_{(\Delta+\Delta_1+1)\rho}(E)| \leq (1 + \Delta^{-1}(\Delta_1 + 1))|V_{\Delta\rho}(E)|.$$

On the other hand, if $[x-\Delta_1\rho, x+\Delta_1\rho]$ intersects $E_{\lambda'} - a_\lambda^{\lambda'}$ and $|y-x| < \Delta\rho$, then $[y-\rho, y+\rho]$ is contained in $V_{(\Delta+\Delta_1+1)\rho}(E_{\lambda'}) - a_\lambda^{\lambda'}$.

We thus take Δ large enough to have

$$\kappa_4 = (1+\Delta^{-1}(\Delta_1+1))\kappa_1 < 1, \quad \varepsilon_1 = \varepsilon(1+\Delta^{-1}(\Delta_1+1))^{-1},$$

and apply the proposition to the $V_{(\Delta+\Delta_1+1)\rho}(E_\lambda)$. $\square$

6.5 After these general statements, we come back to the setting of the Scale recurrence Lemma.

Recall that $\Sigma(\rho)$ is the set of words $\underline{a}$ in Σ such that

$$c_0^{-1}\,\rho \le |I(\underline{a})| \le c_0\,\rho$$

and that we write

$$J_R = [-R, -R^{-1}] \cup [R^{-1}, R].$$

We will need in the proof to distinguish two cases:

A) g and g' are orientation preserving (after perhaps modifying the orientation of some of the $I(a)$, $I(a')$, $a \in \mathcal{A}$, $a' \in \mathcal{A}'$). Equivalently, all periodic points of g *and* g' have a positive eigenvalue.

B) There exists at least one periodic point of g or g' with negative eigenvalue.

We will do the proof in case B). Case A) is simpler and we will rapidly indicate at the end how to modify the proof.

In relation to the abstract setting of §6.1, we set

$$A = \mathbb{A} \times \mathbb{A}' \times \{-1,+1\},$$

$$\Lambda = \Sigma(\rho) \times \Sigma'(\rho) \times \{-1,+1\}.$$

The map $(\alpha,\omega)\colon \Lambda \to A \times A$ is defined as follows: for $\underline{a} \in \Sigma(\rho)$, $\underline{a}' \in \Sigma'(\rho)$, $u \in \{-1,+1\}$

$$\alpha(\underline{a}, \underline{a}', u) = (a_0, a_0', uw)$$

$$\omega(\underline{a}, \underline{a}', u) = (a_n, a_{n'}', u),$$

where a_n , $a'_{n'}$ are the last letters of $\underline{a}$, $\underline{a}'$ and $w = +1$ (resp. -1) if $f_{\underline{a}}$, $f'_{\underline{a}'}$ have the same (resp. opposite) orientation.

It follows from the hypothesis of case B), and the hypothesis that Σ, Σ' are topologically mixing, that if c_0 has been chosen large enough we have, for all $i, j \in A$

$$c^{-1}\,\rho^{-(d+d')} \leq \#\,\Lambda_i^j = N_i^j \leq c\rho^{-(d+d')}.$$

We are given a family $E(\underline{a}, \underline{a}')$ of subsets of J_R , for $\underline{a} \in \Sigma(\rho)$, $\underline{a}' \in \Sigma'(\rho)$. Define, for $(\underline{a}, \underline{a}', u) \in \Lambda$

$$F(\underline{a}, \underline{a}', u) = \{x, ue^x \in E(\underline{a}, \underline{a}')\},$$

which is a subset of $[-r, r]$, $r = \text{ Log } R$. To say that all $J_R - E(\underline{a}, \underline{a}')$ have small measure amounts to say that all $[-r, r] - F(\lambda)$, $\lambda \in \Lambda$, have small measure:

$$m([-r, r] - F(\lambda)) \leq \bar{c}_1 \,.$$

Finally, we define the family of translations $(a_\lambda^{\lambda'})_{(\lambda,\lambda')\in\Lambda^{(2)}}$ of §6.2. To do this, we choose, for each $\underline{a} \in \Sigma(\rho)$ (resp. $\underline{a}' \in \Sigma'(\rho)$) an element $\underline{\theta} \in \Sigma^-$ (resp. $\underline{\theta}' \in \Sigma'^-$) ending with $\underline{a}$ (resp. $\underline{a}'$). Then, if $\lambda = (\underline{a}, \underline{a}', u)$, $\lambda' = (\underline{b}, \underline{b}', v)$ satisfy $\omega(\lambda) = \alpha(\lambda')$, we set

$$a_\lambda^{\lambda'} = \text{ Log } |I^{\underline{\theta}'}(\underline{b}')| - \text{ Log } |I^{\underline{\theta}}(\underline{b})|.$$

We will prove later the following

Lemma: The hypothesis of the Scale recurrence Lemma implies that the hypothesis of Proposition 6.2 holds, namely there exist $\Delta_0 > 0$, $0 < \kappa_0 < 1$ such that for $\xi \in [1, \Delta_0\rho^{-1}]$ the operator defined by

$$w_\lambda = \sum p_\lambda^{\lambda'}\, e^{ia_\lambda^{\lambda'}\,\xi}\, z_{\lambda'}$$

has norm $\leq \kappa_0$.

6.6 Let $\Delta_1 > 0$, and let then Δ be given by the corollary in 6.4. For $\lambda \in \Lambda$, we set

$$E_0(\lambda) = [-r, r] - V_{\Delta\rho}(F(\lambda)).$$

Starting from E_0, we define for $n \geq 0$ sets $E_n(\lambda)$ in the following way; if $E_n(\lambda)$ has already been defined, we define, with $c_3 > 0$ fixed sufficiently small, $\hat{\hat{E}}_n(\lambda)$ as the set of $x \in [-r, r]$ such that

$$\#\{\lambda', \omega(\lambda) = \alpha(\lambda'), [x + a_\lambda^{\lambda'} - \Delta_1\rho, x + a_\lambda^{\lambda'} + \Delta_1\rho] \subset [-r, r] - E_n(\lambda')\}$$

is less than $c_3\,\rho^{-(d+d')}$. We then set

$$E_{n+1}(\lambda) = E_0(\lambda) \cup \hat{\hat{E}}_n(\lambda).$$

We will see that, provided $\bar{c}_1$, c_3 are sufficiently small, we have

Lemma: $\sum\limits_\lambda p^\lambda |V_{\Delta\rho}(\hat{\hat{E}}_n(\lambda))| \leq \kappa_5 \sum\limits_\lambda p^\lambda |V_{\Delta\rho}(E_n(\lambda))|$, with $0 < \kappa_5 < 1$.

Before proving the lemma we finish the proof of the Scale recurrence Lemma. It is clear that for each $\lambda \in \Lambda$, the sequence $(E_n(\lambda))_{n\geq 0}$ is increasing. On the other hand, by the lemma, we have

$$\sum p^\lambda |V_{\Delta\rho}(E_{n+1}(\lambda))| \leq \kappa_5 \sum p^\lambda |V_{\Delta\rho}(E_n(\lambda))| + \operatorname*{Max}_\lambda |V_{\Delta\rho}(E_0(\lambda))|.$$

Defining $E_\infty(\lambda) = \bigcup\limits_{n\geq 0} E_n(\lambda)$, we thus obtain

$$\sum p^\lambda |V_{\Delta\rho}(E_\infty(\lambda))| \leq \frac{1}{1-\kappa_5} \operatorname*{Max}_\lambda |V_{\Delta\rho}(E_0(\lambda))|.$$

We observe that $V_{\Delta\rho}(E_0(\lambda))$ is contained in $[-r - \Delta\rho, r + \Delta\rho] - F(\lambda)$, hence its measure is less than $2\Delta\rho + \bar{c}_1$. Consider

$$F_\infty(\lambda) = [-r, +r] - E_\infty(\lambda).$$

Let $x \in F_\infty(\lambda)$; by definition of $\hat{\hat{E}}_n(\lambda) \subset E_{n+1}(\lambda)$, we have, for every $n \geq 0$

$$\#\{\lambda', \omega(\lambda) = \alpha(\lambda'), [x + a_\lambda^{\lambda'} - \Delta_1\rho, x + a_\lambda^{\lambda'} + \Delta_1\rho] \subset [-r, r] - E_n(\lambda')\} \geq c_3\rho^{-(d+d')}$$

and therefore

$$\#\{\lambda', \omega(\lambda) = \alpha(\lambda'), [x + a_\lambda^{\lambda'} - \Delta_1\rho, x + a_\lambda^{\lambda'} + \Delta_1\rho] \subset F_\infty(\lambda')\} \geq c_3\rho^{-(d+d')}.$$

Now we come back to J_R, setting, for $\underline{a} \in \Sigma(\rho)$, $\underline{a}' \in \Sigma'(\rho)$

$$E^*(\underline{a}, \underline{a}') = \{e^x, x \in F_\infty(\underline{a}, \underline{a}', +1)\} \cup \{-e^x, x \in F_\infty(\underline{a}, \underline{a}', -1)\}.$$

Observe that $F_\infty(\lambda)$ is contained in $V_{\Delta\rho}(F(\lambda))$; this implies conclusion (i) of the Scale recurrence Lemma.

For conclusion (ii), as $\underline{\theta} \to k^{\underline{\theta}}$ is Lipschitz, there exists $C > 0$ such that if $\lambda = (\underline{a}, \underline{a}', u)$, $\lambda' = (\underline{b}, \underline{b}', v)$ satisfy $\omega(\lambda) = \alpha(\lambda')$ and $\underline{\theta}$, $\underline{\theta}'$ are *any* elements in Σ^-, Σ'^- ending with $\underline{a}$, $\underline{a}'$ respectively, then

$$\left| \operatorname{Log} |I^{\underline{\theta}'}(\underline{b}')| - \operatorname{Log} |I^{\underline{\theta}}(\underline{b})| - a_\lambda^{\lambda'} \right| \leq C\rho.$$

Taking $\Delta_1 > 2C$, for $x \in F_\infty(\lambda)$, we will have

$$x + \operatorname{Log} |I^{\underline{\theta}'}(\underline{b}')| - \operatorname{Log} |I^{\underline{\theta}}(\underline{b})| + z \in F_\infty(\lambda')$$

for $|z| \leq \dfrac{\Delta_1}{2}\rho$ and at least $c_3\rho^{-(d+d')}$ elements λ' (with $\alpha(\lambda') = \omega(\lambda)$), for *all* $\underline{\theta}$, $\underline{\theta}'$ ending with $\underline{a}$, $\underline{a}'$ respectively. This immediately implies conclusion (ii) of the Scale recurrence Lemma.

6.7 We now prove the lemma at the end of §6.5.

Let $\eta_0 > 0$ be extremely small, and $\xi \in [1, \Delta_0\,\rho^{-1}]$. If the norm of the operator in the lemma is $\geq (1 - \eta_0)^{1/2}$, we can find $z \in \mathbb{C}^\wedge$ with

$$||z||^2 = \Sigma\, p^\lambda |z_\lambda|^2 = 1,$$

$$||w||^2 \geq 1 - \eta_0\,,$$

where

$$w_\lambda = \sum_{\lambda'} p_\lambda^{\lambda'} \exp(i\xi\, a_\lambda^{\lambda'}) z_{\lambda'}\,.$$

Now, we have

$$|w_\lambda|^2 \leq \sum_{\lambda'} p_\lambda^{\lambda'} |z_{\lambda'}|^2,$$

and all p^λ have order $\rho^{d+d'}$. Therefore there exists $\eta_1 > 0$ very small, and $\widetilde{\Lambda} \subset \Lambda$, with

$$\# (\Lambda - \widetilde{\Lambda}) \leq \eta_1 \, \rho^{-(d+d')},$$

$$|w_\lambda|^2 \geq (1 - \eta_1) \sum_{\lambda'} p_\lambda^{\lambda'} |z_{\lambda'}|^2, \text{for all } \lambda \in \widetilde{\Lambda}.$$

Let $\lambda \in \widetilde{\Lambda},\ \omega(\lambda) = i$; for $\lambda' \in \Lambda_i$, set

$$Z_\lambda^{\lambda'} = \exp(i \, a_\lambda^{\lambda'} \, \xi) z_{\lambda'} \, .$$

We have

$$-|w_\lambda|^2 + \sum_{\Lambda_i} p_\lambda^{\lambda'} |z_{\lambda'}|^2 = \frac{1}{2} \sum_{\Lambda_i} \sum_{\Lambda_i} p_\lambda^{\lambda'_0} \, p_\lambda^{\lambda'_1} |Z_\lambda^{\lambda'_0} - Z_\lambda^{\lambda'_1}|^2,$$

hence

$$\sum_{\Lambda_i} \sum_{\Lambda_i} |Z_\lambda^{\lambda'_0} - Z_\lambda^{\lambda'_1}|^2 \leq \eta_2 \, \rho^{-2(d+d')}$$

with very small $\eta_2 > 0$, as $p_\lambda^{\lambda'}$ has order $\rho^{d+d'}$. On the other hand $\sum\limits_{\Lambda_i} |Z_\lambda^{\lambda'}|^2$ has order $\rho^{-(d+d')}$.

Thus, if we write

$$Z_\lambda = N_i^{-1} \sum_{\Lambda_i} Z_\lambda^{\lambda'} \, ,$$

$$z_{\lambda'} = \exp(-i \, a_\lambda^{\lambda'} \, \xi) Z_\lambda + \widetilde{Z}_\lambda^{\lambda'} \, ,$$

we will have $|Z_\lambda|$ of order 1 and

$$\sum_{\Lambda_i} |\widetilde{Z}_\lambda^{\lambda'}|^2 \leq \eta_3 \, \rho^{-(d+d')},$$

with very small $\eta_3 > 0$.

If we take now $\lambda_0, \lambda_1 \in \widetilde{\Lambda} \cap \Lambda^i$, and compare the two expressions for $z_{\lambda'}$ that we get, we obtain

$$\sum_{\Lambda_i} |\exp(i\, a_{\lambda_0}^{\lambda'}\, \xi) - \exp(i\, a_{\lambda_1}^{\lambda'}\, \xi)|^2 \leq \eta_4\, \rho^{-(d+d')},$$

meaning that

$$\left|\sin\left[\frac{1}{2}(a_{\lambda_0}^{\lambda'} - a_{\lambda_1}^{\lambda'})\xi\right]\right| < \eta_5$$

holds for all $\lambda' \in \Lambda_i$ but at most $\eta_5\, \rho^{-(d+d')}$ elements. The constants η_4, η_5 here are still small.

To derive a contradiction, we will choose λ_0, λ_1 of the form

$$\lambda_0 = (\underline{a}^0, \underline{a}', u), \qquad \lambda_1 = (\underline{a}^1, \underline{a}', u)$$

which gives, with $\lambda' = (\underline{b}, \underline{b}', v)$:

$$\begin{aligned} a_{\lambda_0}^{\lambda'} &= \mathrm{Log}\, |I^{\widetilde{\underline{\theta}}^1}(\underline{b})| - \mathrm{Log}\, |I^{\widetilde{\underline{\theta}}^0}(\underline{b})| \\ &= \mathrm{Log}\, D\big[k^{\widetilde{\underline{\theta}}^1} \circ (k^{\widetilde{\underline{\theta}}^0})^{-1}\big](y) \end{aligned}$$

where $\widetilde{\underline{\theta}}$ ends with $\underline{a}^i$ and $y \in I^{\widetilde{\underline{\theta}}^0}(\underline{b})$.

It is now time to remember the hypothesis in the Scale recurrence Lemma: there exists $\underline{\theta}^0, \underline{\theta}^1 \in \Sigma^-$ (with the same last letter) and $x \in K^{\underline{\theta}_0}(\theta_0^0)$, such that

$$D\ \mathrm{Log}\ D[k^{\underline{\theta}^1} \circ (k^{\underline{\theta}^0})^{-1}](x) = \gamma \neq 0.$$

In a neighbourhood of $\underline{\theta}^0$, $\underline{\theta}^1$, x, this will still belongs to $\big[\frac{1}{2}\gamma, \frac{3}{2}\gamma\big]$. As $\#(\Lambda - \widetilde{\Lambda}) \leq \eta_1\, \rho^{-(d+d')}$, we can find $\underline{a}^0$, $\underline{a}^1$, $\underline{a}'$, u such that $\lambda_i = (\underline{a}^i, \underline{a}', u) \in \widetilde{\Lambda}$ and $\widetilde{\underline{\theta}}^i$ is close to $\underline{\theta}^i$. The number of $\lambda' = (\underline{b}, \underline{b}', v)$ such that $I^{\widetilde{\underline{\theta}}^0}(\underline{b})$ is close to x is still of order $\rho^{-(d+d')}$, and for two such λ_0', λ_1' we will have

$$|(a_{\lambda_0}^{\lambda_0'} - a_{\lambda_1}^{\lambda_0'}) - (a_{\lambda_0}^{\lambda_1'} - a_{\lambda_1}^{\lambda_1'}) - \gamma(y_0 - y_1)| \leq \frac{1}{2}|\gamma|\, |y_0 - y_1|,$$

where $y_i \in I^{\underline{\tilde{\theta}}^0}(\underline{b}^i)$, $\underline{b}^i$ being the first component of λ'_1.

On the other hand, for most λ' we should have

$$|a_{\lambda_0}^{\lambda'} - a_{\lambda_1}^{\lambda'} - 2k\pi\xi^{-1}| \leq \eta_6\,\xi^{-1}$$

for some integer k and small $\eta_6 > 0$. Combining the two inequalities, we should have for most choices of $\underline{b}^0$, $\underline{b}^1$ (with $I^{\underline{\tilde{\theta}}^0}(\underline{b})$ close to x) either $|y_0 - y_1| \geq \pi\xi^{-1}|\gamma|^{-1}$ or $|y_0 - y_1| \leq 4\eta_6\,\xi^{-1}|\gamma|^{-1}$. We have here $1 \leq \xi \leq \Delta_0\rho^{-1}$, hence $\pi\xi^{-1}|\gamma|^{-1} \geq \pi|\gamma|^{-1}\Delta_0^{-1}\rho$. Taking $\Delta_0 \ll |\gamma|$ and η_6 small enough contradicts the bounded geometry of regular Cantor sets.

6.8 We finally will prove the estimate

$$\sum p^\lambda |V_{\Delta\rho}(\hat{\hat{E}}_n(\lambda))| \leq \kappa_5 \sum p^\lambda |V_{\Delta\rho}(E_n(\lambda))|$$

of 6.6.

Recall that $\hat{\hat{E}}_n(\lambda)$ is the set of $x \in [-r, +r]$ such that the number of λ', with $\omega(\lambda) = \alpha(\lambda')$, for which $[x + a_\lambda^{\lambda'} - \Delta_1\rho, x + a_\lambda^{\lambda'} + \Delta_1\rho] \subset [-r, +r] - E_n(\lambda)$ is less than $c_3\,\rho^{-(d+d')}$.

Consider first the set $\widehat{E}_n(\lambda)$ introduced in 6.4: $\widehat{E}_n(\lambda)$ is the set of $x \in \mathbb{R}$ such that the number of λ', with $\omega(\lambda) = \alpha(\lambda')$, for which $[x + a_\lambda^{\lambda'} - \Delta_1\rho, x + a_\lambda^{\lambda'} + \Delta_1\rho] \subset \mathbb{R} - E_n(\lambda')$ is less than $(1-\tau)N_{\omega(\lambda)}$.

As $N_{\omega(\lambda)}$ is of the order of $\rho^{-(d+d')}$, we can choose c_3 small enough to have

$$c_3\,\rho^{-(d+d')} < (1-\tau)N_{\omega(\lambda)} \quad (\forall\,\lambda \in \Lambda).$$

Now, if $c_0 > 1$ is such that

$$c_0^{-1}\rho \leq |I^{\underline{\theta}}(\underline{a})| \leq c_0\rho$$

for all $\underline{\theta} \in \Sigma^-$, $\underline{a} \in \Sigma(\rho)$ with $\theta_0 = a_0$, we will have

$$|a_\lambda^{\lambda'}| \leq 2\,\mathrm{Log}\,c_0$$

for all $\lambda, \lambda' \in \Lambda$ with $\omega(\lambda) = \alpha(\lambda')$.

Then we will have

$$\hat{\hat{E}}_n(\lambda) \cap [-r + 2\,\mathrm{Log}\, c_0 + \Delta_1\rho, r - 2\,\mathrm{Log}\, c_0 - \Delta_1\rho] \subset \widehat{E}_n(\lambda).$$

Now the corollary in 6.4 gives us an estimate of the desired form for $\sum_\lambda p^\lambda |V_{\Delta\rho}(\widehat{E}_n(\lambda))|$, provided

$$\operatorname*{Max}_\lambda |V_{\Delta\rho}(E_n(\lambda))|$$

is sufficiently small.

This follows (inductively) from the desired estimate for $m < n$, as we have seen in 6.6, provided $\bar{c}_1$ is small enough.

We have still to control the $V_{\Delta\rho}$ neighbourhood of the part of $\hat{\hat{E}}_n(\lambda)$ which lies in

$$[-r, -r + 2\,\mathrm{Log}\, c_0 + \Delta_1\rho] \cup [r - 2\,\mathrm{Log}\, c_0 - \Delta_1\rho, r].$$

Let us consider for instance the part in $[r - 2\,\mathrm{Log}\, c_0 - \Delta_1\rho, r]$.

The idea here is to take r large enough, to choose a subinterval $J \subset [-r, +r]$ of length $5\,\mathrm{Log}\, c_0$ where the $E_n(\lambda)$ are relatively thin, and to restrict to the set of λ' which will send $[-r - 2\,\mathrm{Log}\, c_0 - \Delta_1\rho, r]$ into J.

To be more precise, given $\delta > 0$, we can find, if r is large enough, a subinterval $J \subset [-r, +r]$ of length $5\,\mathrm{Log}\, c_0$ such that

$$\sum_\lambda p^\lambda |V_{\Delta\rho}(E_n(\lambda) \cap J)| \le \delta \sum_\lambda p^\lambda |V_{\Delta\rho}(E_n(\lambda))|.$$

Consider now only those λ' for which $r - \mathrm{Log}\, c_0 + a_\lambda^{\lambda'}$ belongs to the interval J_1 of same center as J and length $2\,\mathrm{Log}\, c_0$; the number of such λ's is still of order $\rho^{-(d+d')}$, in particular much larger than $c_3\,\rho^{-(d+d')}$ if c_3 is small enough.

As all p^λ have order $\rho^{(d+d')}$, we will obtain an estimate

$$\sum_\lambda p^\lambda |V_{\Delta\rho}(\hat{\hat{E}}_n(\lambda) \cap [r - 2\,\mathrm{Log}\, c_0 - \Delta\rho, r])| \le C\delta \sum_\lambda p^\lambda |V_{\Delta\rho}(E_n(\lambda))|.$$

Taking $\delta > 0$ small enough, we will have $1 > \kappa_5 = 2C\delta + \kappa_4$ (where κ_4 is from 6.4), which concludes the proof.

6.9 In case A) of 6.5, we just set

$$A = \mathbb{A} \times \mathbb{A}'$$

$$\Lambda = \Sigma(\rho) \times \Sigma'(\rho)$$

$$F^{\pm}(\underline{a}, \underline{a}') = \{x, \pm e^x \in E(\underline{a}, \underline{a}')\}$$

and deal separately with the two families $(F^+(\underline{a}, \underline{a}'))$ and $(F^-(\underline{a}, \underline{a}'))$ in the same manner than in case B).

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