Maximal Invariant Sets for Restrictions of Tent and Unimodal Maps

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Abstract

The aim of this paper is to study the restriction of a tent map $f\colon I \to I$ to a subinterval $J \subset I$, and to characterize when the maximal invariant of $f|_J$ is a countable set. We shall extend our results for a certain class of unimodal maps, via a semi-conjugacy. In the case of tent maps there is a critical case where the maximal invariant set is a Cantor set with Hausdorff dimension zero. The caracterization of this critical case leads to interesting arithmetical phenomena, particularly in the case of the complete tent map.

1 Introduction

The classical family of tent maps is the family $(T_\lambda)_{1<\lambda\leq 2}$ defined by

$$T_\lambda(x) = \begin{cases} \lambda x, & \text{if} \quad 0 \leq x \leq \dfrac{1}{2} \\ \lambda(1-x), & \text{if} \quad \dfrac{1}{2} \leq x \leq 1 \end{cases} .$$

It is the simplest continuous family of unimodal maps, and gives topological models for many differentiable unimodal maps (for instance, the complete tent map T_2 is conjugated to $f(x) = x^2 - 2$ in [-2,2] via the homeomorphism $h\colon [0,1] \to [-2,2], \quad h(x) = 2\cos(\pi x)$).

The aim of this work is to study restrictions of a tent map T_λ to closed subintervals $J \subset [0,1]$, and characterize when the maximal invariant set $K = K_{\lambda,J}$ of $T_\lambda|_J$ is an uncountable set, when it has positive Hausdorff dimension (see definition below) and when it has positive Lebesgue measure (case in which it must be a nondegenerate interval). Our main result is that, given $\lambda \in (1,2]$, there is an explicitly given interval $[a(\lambda), b(\lambda)] \subset [0,1]$ such that K is uncountable if and only if J contains $[a(\lambda), b(\lambda)]$ and it has positive Hausdorff dimension if and only if the interior of J contains $[a(\lambda), b(\lambda)]$. In the case $\lambda = 2$ we relate $a(\lambda)$ and $b(\lambda)$ with the classical Morse-Thue constant and use it to show their transcendence.

We also discuss generalizations of our results to C^2 unimodal maps, via the study of (semi) conjugacies between them and tent maps.

Let us finish this introduction with some definitions:

Given a metric space X and a positive real number α, we define the Hausdorff measure of dimension α associated to a covering $\mathcal{U} = (U_\lambda)_{\lambda \in L}$ of X by bounded sets U_λ by $m_\alpha(\mathcal{U}) = \sum_{\lambda \in L} (\text{diam } U_\lambda)^\alpha$, where diam U_λ denotes the diameter of U_λ, and, if we define the norm of a covering $\mathcal{U}$ by $||\mathcal{U}|| = \sup_{U \in \mathcal{U}} (\text{diam } U)$, then the Hausdorff measure of dimension α of the set X is $m_\alpha(X) = \liminf_{\substack{\mathcal{U} \text{ covering of } X \\ ||\mathcal{U}|| \to 0}} m_\alpha(\mathcal{U})$.

It is not difficult to see that there is a unique $d \in [0, +\infty]$ such that if $\alpha > d$ then $m_\alpha(X) = 0$ and if $\alpha < d$ then $m_\alpha(X) = +\infty$. This number d is called the *Hausdorff dimension* of X. It is easy to see that if $X \subset \mathbb{R}$ then its Hausdorff dimension $d =: HD(X)$ belongs to $[0,1]$.

Given a Cantor set $K \subset \mathbb{R}$, a gap of K is a connected component of the complement of K. Given a bounded gap U, let U_L and U_R the nearest gaps respectively at the left and at the right of U that are bigger than U, and let I_L and I_R be the intervals between U_L and U and between U and U_R, respectively. We define $\tau(U) = \min\{|I_L|/|U|,\ |I_R|/|U|\}$, and we define the *thickness* of K by $\tau(K) = \inf\{\tau(U), U \text{ bounded gap of } K\}$. Its importance follows from Newhouse's gap Lemma (see [P-T]), according to which if K_1 and K_2 are Cantor sets on the real line such that $\tau(K_1) \cdot \tau(K_2) \geq 1$ then either K_1 is contained in a gap of K_2, K_2 is contained in a gap of K_1 or $K_1 \cap K_2 \neq \emptyset$. It is possible to prove that we always have $HD(K) \geq \log 2 / \log\left(2 + \frac{1}{\tau(K)}\right)$. In particular, if $\tau(K) > 0$ then $HD(K) > 0$. The equality holds for the affine Cantor sets K_α with $\alpha \in \left(0, \frac{1}{2}\right)$, which are defined as the maximal invariant

set of $\varphi_\alpha\colon [0,\alpha] \cup [1-\alpha, 1] \to [0,1]$, given by $\varphi_\alpha(x) = \begin{cases} 1 - x/\alpha, & \text{if} \quad x \in [0,\alpha] \\ (x-1+\alpha)/\alpha, & \text{if} \quad x \in [1-\alpha, 1] \end{cases}$,
for which $\tau(K_\alpha) = \frac{\alpha}{1-2\alpha}$ and $HD(K_\alpha) = -\log 2/\log\alpha$.

2 The Case of Tent Maps

Let $1 < \lambda \leq 2$. The tent map $T_\lambda\colon [0,1] \to [0,1]$ is defined by

$$T_\lambda(x) = \begin{cases} \lambda x, & \text{if } x \leq \frac{1}{2} \\ \lambda(1-x), & \text{if } x \geq \frac{1}{2} \end{cases}$$

Since $T_\lambda([0,1]) = [0, \frac{\lambda}{2}]$, if $M_J := \bigcap_{n=0}^{\infty} T^{-n}(J)$ is the maximal invariant of $T_\lambda|_J$, we have $M_J = M_{J\cap[0,\frac{\lambda}{2}]} \cup \Big(J \cap T_\lambda^{-1}(M_{J\cap[0,\frac{\lambda}{2}]})\Big)$. So, we may restrict our attention to the case $J \subset [0, \frac{\lambda}{2}]$. Moreover, if $x \geq \lambda(1-\frac{\lambda}{2}) = T_\lambda(\frac{\lambda}{2})$ and $x \in [0,\frac{\lambda}{2}]$ then $T_\lambda(x) \geq \lambda(1-\frac{\lambda}{2})$, so the interval $L = [\lambda(1-\frac{\lambda}{2}), \frac{\lambda}{2}]$ is T_λ-invariant, and $M_{J\cap[0,\frac{\lambda}{2}]} = J \cap \bigcup_{r=0}^{\infty} \lambda^{-r} M_{J\cap L}$, and so we may restrict our attention to the case $J \subset [\lambda(1-\frac{\lambda}{2}), \frac{\lambda}{2}]$.

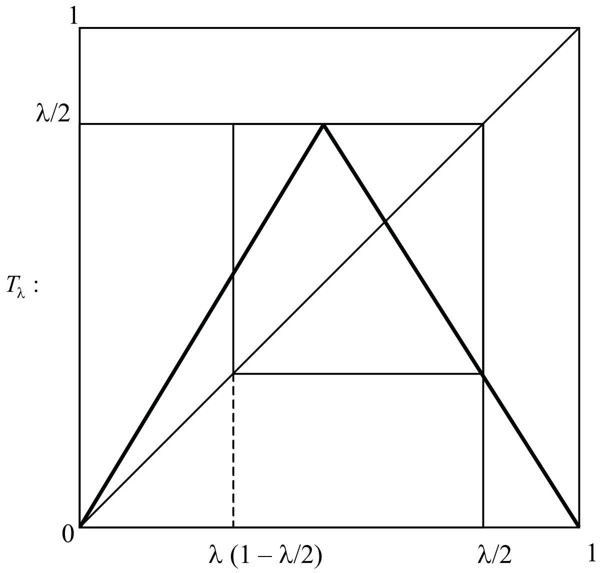

Figure 1

Figure 1 above shows the dynamics of T_λ and the interval L.

Let $J = [\lambda(1-\frac{\lambda}{2})+\varepsilon, \frac{\lambda}{2}]$. If $x \in M_J$ then $x \leq \frac{\lambda}{2} - \frac{\varepsilon}{\lambda}$, since if $x > \frac{\lambda}{2} - \frac{\varepsilon}{\lambda}$ then $T_\lambda(x) < \lambda(1-\frac{\lambda}{2})+\varepsilon \Rightarrow M_J = M_{[\lambda(1-\frac{\lambda}{2})+\varepsilon, \frac{\lambda}{2}-\frac{\varepsilon}{\lambda}]}$. On the other hand, if $J = [\lambda(1-\frac{\lambda}{2}), \frac{\lambda}{2}-\varepsilon]$ then, since $[\lambda(1-\frac{\lambda}{2})+\varepsilon\lambda, \frac{\lambda}{2}-\varepsilon] = J \cap T_\lambda(J)$, we have $M_J \subseteq \bigcup_{r=0}^{\infty} \lambda^{-r} M_{[\lambda(1-\frac{\lambda}{2})+\varepsilon\lambda, \frac{\lambda}{2}-\varepsilon]}$, and so we may restrict our attention to the intervals $J_\varepsilon = [\lambda(1-\frac{\lambda}{2})+\varepsilon\lambda, \frac{\lambda}{2}-\varepsilon] =: [a_\varepsilon, b_\varepsilon]$. Indeed, the discussion above implies that if $J = [\lambda(1-\frac{\lambda}{2})+\delta, \frac{\lambda}{2}-\eta]$, then $J \supset J_\varepsilon$, where $\varepsilon = \max\{\frac{\delta}{\lambda}, \eta\}$ and $M_J \subseteq \bigcup_{r=0}^{\infty} \lambda^{-r} M_{J_\varepsilon}$ (see Figure 2). $T_\lambda|_{J_\varepsilon}$ is conjugated to the function $\psi_{\lambda,c}\colon [0, \frac{1}{\lambda}] \cup [1-\frac{c}{\lambda}, 1]$ defined by

$$\psi_{\lambda,c}(x) = \begin{cases} 1-\lambda x, & \text{if } x \in [0, \frac{1}{\lambda}] \\ \lambda(x-1+\frac{c}{\lambda}), & \text{if } x \in [1-\frac{c}{\lambda}, 1], \end{cases}$$

where

$$\begin{aligned} c = \frac{(\frac{b_\varepsilon}{\lambda} - a_\varepsilon)\cdot\lambda}{b_\varepsilon - a_\varepsilon} &= \frac{b_\varepsilon - \lambda a_\varepsilon}{b_\varepsilon - a_\varepsilon} = \frac{\frac{\lambda}{2} - \varepsilon - \lambda^2(1+\varepsilon-\frac{\lambda}{2})}{\frac{\lambda}{2} - \varepsilon - \lambda(1+\varepsilon-\frac{\lambda}{2})} \\ &= \frac{\lambda(\lambda-1)^2 - 2(\lambda^2+1)\varepsilon}{\lambda(\lambda-1) - 2(\lambda+1)\varepsilon} \\ &= \lambda - 1 - \frac{4\varepsilon}{\lambda(\lambda-1) - 2(\lambda+1)\varepsilon} < \lambda - 1 \quad \text{for} \quad \varepsilon > 0. \end{aligned}$$

by the affine conjugacy $h(x) = \frac{b_\varepsilon - x}{b_\varepsilon - a_\varepsilon}$ (see Figure 3).

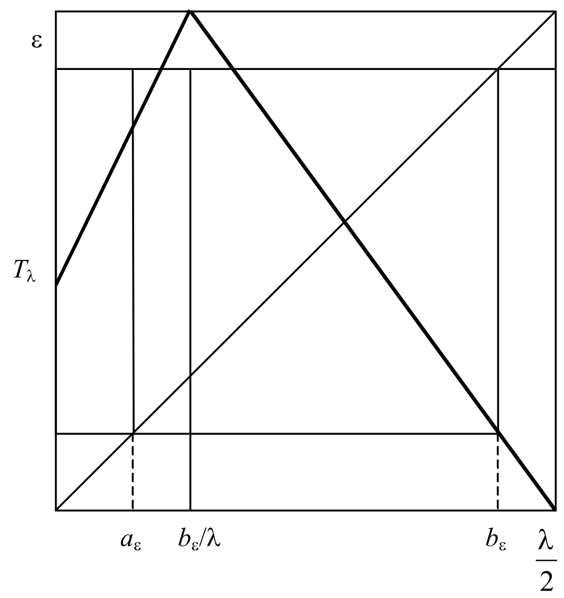

Figure 2

We shall study the maximal invariant of $\psi_{\lambda,c}$ in $[0,1]$. Let

$$\tilde{\psi}_{\lambda,c} = \begin{cases} c - \lambda^2 x, & \text{if } x \in [0, \frac{c}{\lambda^2}] \\ 1 - \lambda + \lambda^2 x, & \text{if } x \in [\frac{\lambda-1}{\lambda^2}, \frac{1}{\lambda}] \\ \lambda(x - 1 + \frac{c}{\lambda}), & \text{if } x \in [1 - \frac{c}{\lambda}, 1] \end{cases} = \begin{cases} \psi^2_{\lambda,c}, & \text{if } x \in [0, \frac{1}{\lambda}] \cap \psi^{-2}_{\lambda,c}([0,1]) \\ \psi_{\lambda,c}, & \text{if } x \in [1 - \frac{c}{\lambda}, 1] \end{cases}.$$

(See Figure 4).

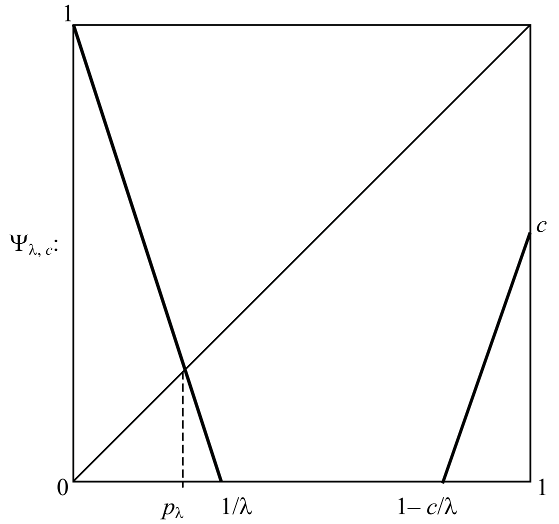

Figure 3

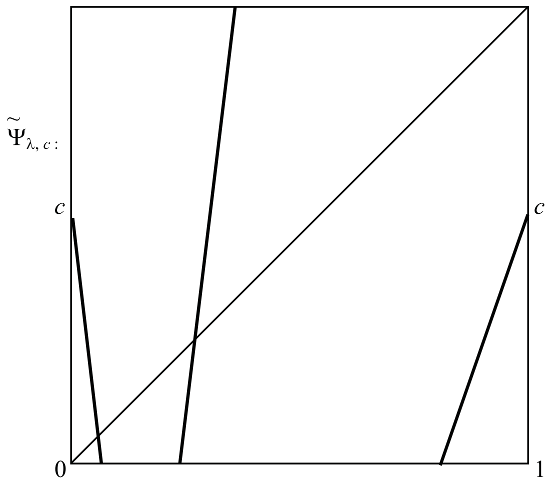

Figure 4

Clearly the maximal invariants of $\tilde{\psi}_{\lambda,c}$ and $\psi_{\lambda,c}$ in $[0,1]$ are equal.

Let $\lambda - 1 > p_\lambda = \frac{1}{\lambda+1}$ be the fixed point of $\psi_{\lambda,c}$. We have the following possibilities:

i) $c \geq p_\lambda$: In this case the maximal invariant of $\psi_{\lambda,c}$ contains (indeed is equal to) a Cantor set. Indeed, if $\tilde{I} = \left[\frac{c-p_\lambda}{\lambda^2}, p_\lambda\right]$ then the maximal invariant of $\tilde{\psi}_{\lambda,c}|_{\tilde{I}}$ is a regular Cantor set, with positive Hausdorff dimension and positive thickness (see Figure 5).

ii) $c < \frac{\lambda-1}{\lambda^2}$: In this case the maximal invariant of $\psi_{\lambda,c}$ is countable. Indeed, in this case $[0,c]$ is $\tilde{\psi}_{\lambda,c}$-invariant, and the maximal invariant $M_{[0,1]}$ is equal to $\bigcup_{r=0}^{\infty} \tilde{\psi}^{-r}_{\lambda,c}(M_{[0,c]})$, but $M_{[0,c]} = \left\{\frac{c}{\lambda^2+1}\right\}$, since $\tilde{\psi}_{\lambda,c}|_{[0,c]} = c - \lambda^2 x$, if $x \in [0, \frac{c}{\lambda^2}]$, and is not defined otherwise (see Figure 6).

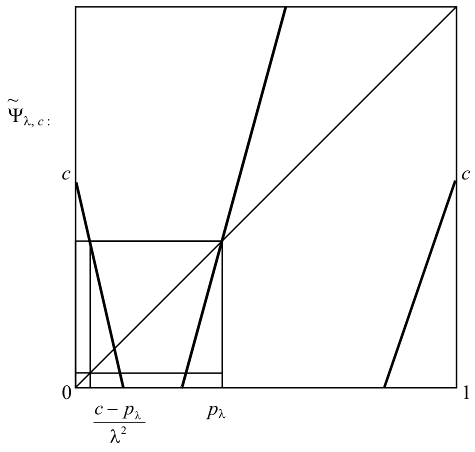

Figure 5

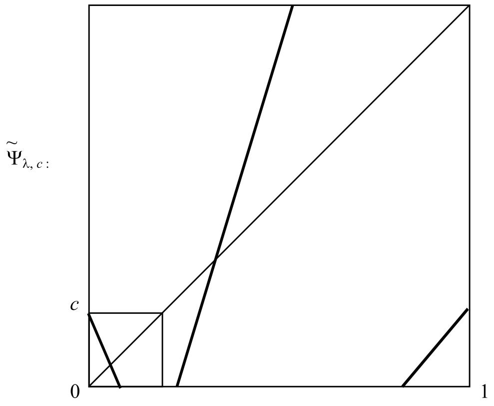

Figure 6

iii) $\frac{\lambda-1}{\lambda^2} \leq c \leq \frac{1}{\lambda+1} = p_\lambda$: In this case $[0,c]$ is still $\tilde{\psi}_{\lambda,c}$-invariant, and $M_{[0,1]} = \bigcup_{r=0}^{\infty} \tilde{\psi}_{\lambda,c}^{-r}(M_{[0,c]})$. Therefore, $M_{[0,1]}$ is a Cantor set (resp. countable) iff $M_{[0,c]}$ is a Cantor set (resp. countable). On the other hand, $\tilde{\psi}_{\lambda,c}|_{[0,c]}$ is conjugated by $h(x) = \frac{x}{c}$ to $\psi_{\lambda^2, \lambda^2 - \frac{\lambda-1}{c}}$. We have now a renormalization operator $F(\lambda, c) = (\lambda^2, \lambda^2 - \frac{\lambda-1}{c})$ (see Figure 7).

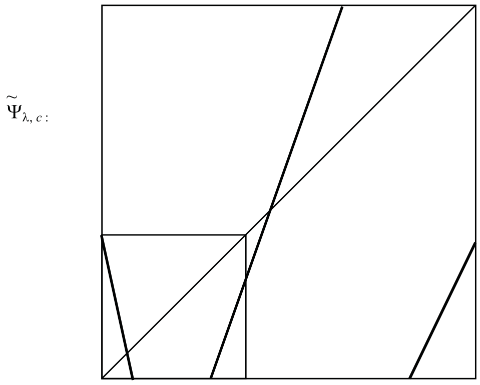

Figure 7

Lemma 2.1. *Given λ, there is a unique value of c, denoted by $c(\lambda)$, such that we can iterate infinitely many times the renormalization operator F beginning with (λ, c). This value is given by the expression $c(\lambda) = (\lambda - 1)\frac{\alpha(\lambda)}{1+\alpha(\lambda)}$, where $\alpha(\lambda) = \sum_{r=1}^{\infty} \frac{1}{\prod_{j=1}^{r}(\lambda^{2^j}-1)}$.*

Remark. The characterization of the maximal invariant set of $\psi_{\lambda,c}$ will be as follows:

- $\lambda - 1 > c > c(\lambda) \Rightarrow M_{[0,1]}$ is a Cantor set with positive Hausdorff dimension. (Since the case i) will occur after a certain number of renormalizations)

- $c < c(\lambda) \Rightarrow M_{[0,1]}$ is countable. (Since the case ii) will occur after a certain number of renormalizations)

We shall also prove that if $c = c(\lambda)$ then $M_{[0,1]}$ is a Cantor set with Hausdorff dimension zero (and so with thickness 0).

Proof. Suppose that we can iterate F infinitely many times. Let

$$F^n(\lambda, c) = (\lambda_n, c_n) \Rightarrow \lambda_{n+1} = \lambda_n^2, c_{n+1} = \lambda_n^2 - \frac{\lambda_n - 1}{c_n} \Rightarrow$$
$$\lambda_n = \lambda_0^{2^n}, c_n = \frac{\lambda_n - 1}{\lambda_n^2 - c_{n+1}} \Rightarrow c_0 = \cfrac{\lambda_0 - 1}{\lambda_0^2 - \cfrac{\lambda_0^2 - 1}{\lambda_0^4 - \cfrac{\lambda_0^4 - 1}{\ddots \lambda_0^{2^n} - \cfrac{\lambda_0^{2^n} - 1}{\lambda_0^{2^{n+1}} - c_{n+1}}}}}$$

Since $c_n \in [0,1]$, $\forall n$, $\lambda_0^{2^n} \to \infty$ and, if $\lambda_n \geq 2$, $q_n(c) := \frac{\lambda_n - 1}{\lambda_n^2 - c}$ is $\frac{1}{2}$ contracting in $[0,1]$, c_0 must be equal to the generalized continued fraction

$$\cfrac{\lambda_0 - 1}{\lambda_0^2 - \cfrac{\lambda_0^2 - 1}{\lambda_0^4 - \cfrac{\lambda_0^4 - 1}{\ddots}}}.$$

We have

$$x_n := \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{\ddots + \cfrac{b_{n-1}}{a_n}}}} = \frac{p_n}{q_n},$$

where p_n and q_n are defined by

$$\begin{cases} p_{-1} = 1, p_0 = 0, p_{n+2} = a_{n+2}p_{n+1} + b_{n+1}p_n \\ q_{-1} = 0, q_0 = 1, q_{n+2} = a_{n+2}q_{n+1} + b_{n+1}q_n \end{cases}$$

This can be proved by induction:

$$\begin{aligned} x_{n+1} &= \frac{\left(a_n + \frac{b_n}{a_{n+1}}\right) p_{n-1} + b_{n-1}p_{n-2}}{\left(a_n + \frac{b_n}{a_{n+1}}\right) q_{n-1} + b_{n-1}q_{n-2}} \\ &= \frac{a_{n+1}(a_n p_{n-1} + b_{n-1}p_{n-2}) + b_n p_{n-1}}{a_{n+1}(a_n q_{n-1} + b_{n-1}q_{n-2}) + b_n q_{n-1}} = \frac{a_{n+1}p_n + b_n p_{n-1}}{a_{n+1}q_n + b_n q_{n-1}} = \frac{p_{n+1}}{q_{n+1}}. \end{aligned}$$

In our case, p_n and q_n satisfy the recurrence $Z_{n+2} = \lambda_0^{2^{n+2}} Z_{n+1} - (\lambda_0^{2^{n+1}} - 1)Z_n$, $\forall n \geq 0$. Then we have $Z_{n+2} - (\lambda_0^{2^{n+2}} - 1)Z_{n+1} = Z_{n+1} - (\lambda_0^{2^{n+1}} - 1)Z_n \Rightarrow Z_n - (\lambda_0^{2^n} - 1)Z_{n-1}$ is constant $\Rightarrow Z_{n+2} = (\lambda_0^{2^{n+2}} - 1)Z_{n+1} + c$, $\forall n \geq 0 \Rightarrow Z_n = \prod_{j=1}^n (\lambda_0^{2^j} - 1) \cdot w_n$, where w_n satisfies $w_{n+2} = w_{n+1} + \frac{c}{\prod_{j=1}^{n+2}(\lambda_0^{2^j}-1)}$, $\forall n \geq 0 \Rightarrow w_n = w_1 + c\sum_{r=2}^n \prod_{j=1}^r (\frac{1}{\lambda_0^{2^j}-1})$, where $c = Z_1 - (\lambda_0^2 - 1)Z_0$, $w_1 = \frac{Z_1}{\lambda_0^2-1}$.

Applying this results for p_n and q_n we obtain $p_n = \prod_{j=1}^n (\lambda_0^{2^j} - 1) \cdot w_n$ and $q_n = \prod_{j=1}^n (\lambda_0^{2^j} - 1)\tilde{w}_n$, where $w_n = w_1 + c\sum_{r=2}^n \prod_{j=1}^r (\frac{1}{\lambda_0^{2^j}-1})$ and $\tilde{w}_n = \tilde{w}_1 + \tilde{c}\sum_{r=2}^n \prod_{j=1}^r (\frac{1}{\lambda_0^{2^j}-1})$. We have $c = p_1 - (\lambda_0^2 - 1)p_0 = (\lambda_0 - 1) - (\lambda_0^2 - 1) \cdot 0 = \lambda_0 - 1$, and $w_1 = \frac{p_1}{\lambda_0^2-1} = \frac{1}{\lambda_0+1} \Rightarrow w_n = (\lambda_0 - 1)\sum_{r=1}^n \prod_{j=1}^r (\frac{1}{\lambda_0^{2^j}-1})$. On the other hand, $\tilde{c} = q_1 - (\lambda_0^2 - 1)q_0 = \lambda_0^2 - (\lambda_0^2 - 1) = 1$, and $w_1 = \frac{q_1}{\lambda_0^2-1} = \frac{\lambda_0^2}{\lambda_0^2-1} \Rightarrow \tilde{w}_n = 1 + \sum_{r=1}^n \prod_{j=1}^r (\frac{1}{\lambda_0^{2^j}-1})$. Since $\frac{p_n}{q_n} = \frac{w_n}{\tilde{w}_n}$, if we define $\alpha(\lambda) = \sum_{r=1}^\infty \prod_{j=1}^r (\frac{1}{\lambda^{2^j}-1})$, we have $c(\lambda) = \lim_{n\to\infty} \frac{p_n}{q_n} = \frac{(\lambda-1)\alpha(\lambda)}{\alpha(\lambda)+1}$. $\square$

Let us consider the case $c = c(\lambda)$ in more detail: Let $c_{-1} = 1$, $c_0 = c(\lambda)$, $\lambda_n = \lambda_0^{2^n}$, $c_{n+1} = \lambda_n^2 - \frac{\lambda_n - 1}{c_n}$, $\forall n \geq 0$ and $d_n = \prod_{r=0}^n c_r$. Then, for each $n \geq 0$, the interval $[0, d_{n-1}]$ is invariant by $\psi_{\lambda,c}^{2^n}$, which is conjugated by the diffeomorphism $h_n(x) = \frac{x}{d_n}$ to ψ_{λ_n,c_n}. We can decompose $\psi_{\lambda,c}^{2^n}$ in his two components $\varphi_n^{(1)} := \psi_{\lambda,c}^{2^n}|_{[0,\frac{d_{n-1}}{\lambda_n}]}$ and $\varphi_n^{(2)} := \psi_{\lambda,c}^{2^n}|_{[d_{n-1}(1-\frac{c_n}{\lambda_n}),d_{n-1}]}$, which are expansive affine maps.

Let K be the maximal invariant set of $\psi_{\lambda,c}$ in $[0,1]$. Clearly K is a closed set. To prove that K is a Cantor set, it is enough to prove that K has no isolated points. Let $\delta = \max_{x\in K}(\min_{y\in K}|y - x|)$. We want to prove that $\delta = 0$.

Let $K_n = K \cap [0, d_{n-1}]$. We have $K_0 = K$, and $K_n = (\varphi_n^{(2)})^{-1}(K_{n+1}) \cup \bigcup_{j=0}^\infty (\varphi_n^{(1)})^{-j}(K_{n+1})$, that is, K_n is the union of countable many congruent or reduced copies of K_{n+1}, which shows that $\delta_n \leq \delta_{n+1}$, where $\delta_n := \max_{x\in K_n}(\min_{y\in K_n}|y - x|)$. Since $\delta_n \leq d_{n-1}$, we have $\lim_{n\to\infty}\delta_n = 0 \Rightarrow \delta = \delta_0 = 0$, because $\delta_0 \leq \delta_n$, $\forall n \in \mathbf{N}$.

Let us prove now that the Hausdorff dimension of K is 0. Let $n \in \mathbf{N}$ and define

$$\tilde{c}_\lambda(n) = \cfrac{\lambda - 1}{\lambda^2 - \cfrac{\lambda^2-1}{\ddots \, \cfrac{}{\lambda^{2^n}-1}}}.$$

Then, if $c = \tilde{c}_\lambda(n)$, $\psi_{\lambda,c}$ is n times renormalizable, and after n renormalizations we obtain the set K_n which is the maximal invariant set of $\psi_{\lambda^{2^n},1}$, such that $\tilde{K}$ is the union of countable many diffeomorphic images of K_n, and so has the same Hausdorff dimension of K_n, where $\tilde{K}$

is the maximal invariant set of $\psi_{\lambda,c}$ in $[0,1]$. On the other hand, K_n is an affine Cantor set and his Hausdorff dimension h satisfies $(\lambda^{2^n})^{-h}+(\lambda^{2^n})^{-h}=1 \Rightarrow h=\frac{\log 2}{2^n\log\lambda}$ (see [P-T]). Since $c(\lambda)<\tilde{c}_\lambda(n)$, $\forall n\in\mathbf{N}$, we have $HD(K)\leq -\frac{\log 2}{2^n\log\lambda}$, $\forall n\in\mathbf{N}\Rightarrow HD(K)=0$. Our last claims follow from the fact that if $c_1<c_2$ then there is an affine image of K_{λ,c_1} contained in K_{λ,c_2}, where $K_{\lambda,c}$ is the maximal invariant of $\psi_{\lambda,c}$ in $[0,1]$. Indeed, the restriction of ψ_{λ,c_2} to $[\varepsilon,1-\lambda\varepsilon]$ is conjugated to $\psi_{\lambda,\frac{c_2-\lambda^2\varepsilon-\varepsilon}{1-(1+\lambda)\varepsilon}}$ (see Figure 8). Taking $\varepsilon=\frac{c_2-c_1}{1+\lambda^2-c_1(1+\lambda)}$ we are done.

Remark. If $c>c(\lambda)$ then $c>\tilde{c}_\lambda(n)$ for some n, and so $HD(K)\geq HD(K_n)=\frac{\log 2}{2^n\log\lambda}>0$

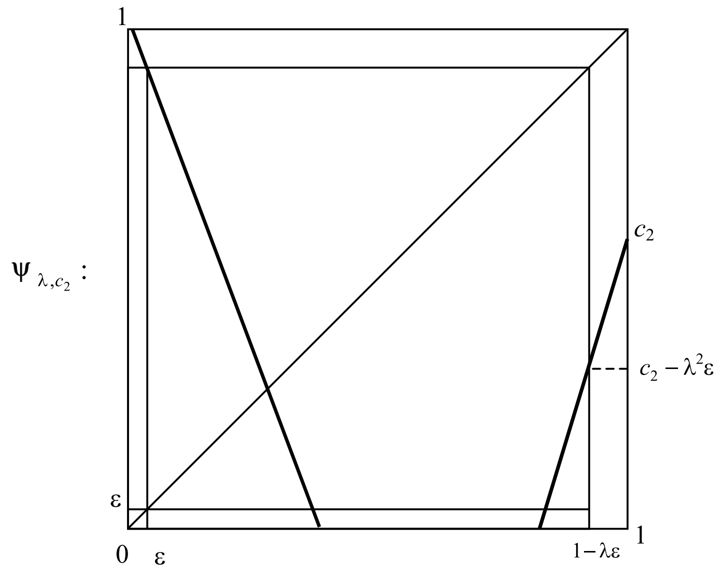

Figure 8

We have proved the following

Theorem 2.2. *Let $\lambda\in(1,2]$ and $[a,b]\subset[0,1]$. Let $\varepsilon=\max\{0,\frac{\lambda}{2}-b,\frac{a}{\lambda}-(1-\frac{\lambda}{2})\}$, and let* $c=\frac{\frac{\lambda}{2}-\varepsilon-\lambda^2(1+\varepsilon-\frac{\lambda}{2})}{\frac{\lambda}{2}-\varepsilon-\lambda(1+\varepsilon-\frac{\lambda}{2})}=\lambda-1-\frac{4\varepsilon}{\lambda(\lambda-1)-2(\lambda+1)\varepsilon}$.

If $c(\lambda)=\frac{(\lambda-1)\alpha(\lambda)}{1+\alpha(\lambda)}$, where $\alpha(\lambda)=\sum_{r=1}^{\infty}\frac{1}{\prod_{j=1}^{r}(\lambda^{2^j}-1)}$, and if K is the maximal invariant of T_λ in $[a,b]$, then:

i) $\lambda-1>c>c(\lambda)\Rightarrow K$ is a Cantor set with positive Hausdorff dimension;

ii) $c=c(\lambda)\Rightarrow K$ is a Cantor set with Hausdorff dimension 0;

iii) $c < c(\lambda) \Rightarrow K$ is countable.

Remark 0: If $c = \lambda - 1$, which is equivalent to $\varepsilon = 0$, then K is an interval.

Remark 1: Conditions i), ii) and iii) are respectively equivalent to

i') $[a(\lambda), b(\lambda)] \subset (a, b)$;

ii') $[a(\lambda), b(\lambda)] \subset [a, b]$ and $a = a(\lambda)$ or $b = b(\lambda)$;

iii') $[a(\lambda), b(\lambda)] \not\subset [a, b]$

where, given $\varepsilon > 0$ for which $c = c(\lambda)$, $a(\lambda) = a_\varepsilon = \frac{\lambda(1-c(\lambda))}{1+\lambda^2-c(\lambda)(1+\lambda)}$ and $b(\lambda) = b_\varepsilon = \frac{\lambda(\lambda-c(\lambda))}{1+\lambda^2-c(\lambda)(1+\lambda)}$.

Remark 2: $c = \lambda - 1 - \frac{4\varepsilon}{\lambda(\lambda-1)-2(\lambda+1)\varepsilon}$ is a strictly decreasing function of ε, whose largest value is $\lambda - 1 > c(\lambda)$, so for each $\lambda \in (1, 2]$, there is an unique $\varepsilon > 0$ such that $c = c(\lambda)$.

3 The Case $\lambda = 2$:

For $\lambda = 2$ we have an interesting arithmetic phenomenum involving the binary representation of $b(\lambda)$. Let $x \in [0, 1]$ such that $T_2^n(x) \neq 0$, $\forall n \in \mathbf{N}$ (this is equivalent to $x \neq \frac{K}{2^n}$, $\forall K, n \in \mathbf{N}$). Then we can associate to x the sequence $(\sigma_i)_{i \geq 0}$, where

$$\sigma_i = \begin{cases} 0, & \text{if } T_2^i(x) \in (0, 1/2) \\ 1, & \text{if } T_2^i(x) \in (1/2, 1) \end{cases}.$$

If the representation of x in basis two is $x = \sum_{j=0}^{\infty} \frac{\mu_j}{2^{j+1}}$ then $\sigma_0 = \mu_0$ and, for $n \geq 1$,

$$\sigma_n = \begin{cases} \mu_n, & \text{if } \mu_{n-1} = 0 \\ 1 - \mu_n, & \text{if } \mu_{n-1} = 1. \end{cases}$$

and so we can reobtain the μ_i's from the σ_i's. The restriction of T_2 to $[a(2), b(2)]$ is conjugated to $\psi_{2,c(2)}$, and $b(2)$ correspond to 0 by this conjugacy. Let us study the orbit of 0 by $\psi_{2,c(2)}$.

Let

$$\tau_n := \begin{cases} 0, & \text{if } \psi_{2,c(2)}^n(0) \in [0, \frac{1}{2}] \\ 1, & \text{if } \psi_{2,c(2)}^n(0) \in (\frac{1}{2}, 1] \end{cases}$$

$$\tilde{\tau}_n := \begin{cases} 0, & \text{if } (\psi^2_{2,c(2)})^n(0) \in [0, \frac{1}{4}) \\ 1, & \text{if } (\psi^2_{2,c(2)})^n(0) \in [\frac{1}{4}, \frac{1}{2}] \end{cases}$$

then:

i) $\tilde{\tau}_n$ is well-defined, because $[0, c(2)]$ is invariant by $\psi^2_{2,c(2)} \Rightarrow (\psi^2_{2,c(2)})^n(0) \in [0, c(2)] \subset [0, \frac{1}{2}]$, $\forall n \geq 0$;

ii) $\tilde{\tau}_n = 0 \Leftrightarrow (\tau_{2n} = 0$ and $\tau_{2n+1} = 1)$, and $\tilde{\tau}_n = 1 \Leftrightarrow (\tau_{2n} = 0$ and $\tau_{2n+1} = 0)$;

iii) $\tilde{\tau}_n = \tau_n$, $\forall n \geq 0$, because, by the construction, $\psi^2_{2,c(2)}|_{[0,c(2)]}$ is conjugated to $\psi_{4,c(4)}$, and the combinatorial behavior of the orbit of 0 by $\psi_{\lambda,c(\lambda)}$ does not depend on λ.

The properties i), ii) and iii) allow us to obtain the sequence τ_n recursively by the rules:

$$\tau_0 = 0; \quad \tau_n = 0 \Rightarrow \tau_{2n} = 0 \text{ and } \tau_{2n+1} = 1; \quad \tau_n = 1 \Rightarrow \tau_{2n} = 0 \text{ and } \tau_{2n+1} = 0.$$

We have $(\tau_n) = 0100010101000100\ldots$.

Since the conjugacy reverses orientation, the sequence (σ_i) associated to $b(2)$ is given by $\sigma_i = 1 - \tau_i$, and the sequence μ_i is given by $\mu_0 = \sigma_0 = 1$,

$$\mu_{n+1} = \begin{cases} \sigma_{n+1}, & \text{if } \mu_n = 0 \\ 1 - \sigma_{n+1}, & \text{if } \mu_n = 1. \end{cases}$$

So, $(\sigma_n) = 1011101010111011\ldots$, and $(\mu_n) = 1101001100101101\ldots$, that is, $b(2) = 0,1101001100101101\ldots$ in basis 2, and $a(2) = 2 - 2b(2) = 0,010110011010010\cdots = \sum_{i=0}^{\infty} \frac{\lambda_i}{2^{i+1}}$, where $\lambda_i = 1 - \mu_{i+1}$, $\forall i$.

This shows, in particular, that $a(2)$ and $b(2)$ have both 50% of zeros and 50% of ones in their binary representations, but do not have three consecutive zeros nor three consecutive ones.

It is easy to prove by induction that $b(2) = 2m$, where

$$m = 0,0110100110010110\cdots = \sum_{i=1}^{\infty} \frac{m_i}{2^i},$$

and the digits m_i of the binary representation of m form the Morse-Thue sequence defined by $m_1 = 0$,

$$m_k = 0 \Rightarrow m_{2k-1} = 0, \quad m_{2k} = 1$$

$$m_k = 1 \Rightarrow m_{2k-1} = 1, \quad m_{2k} = 0.$$

This number m is called the Morse-Thue constant, and is equal to the external angle in the Mandelbrot set corresponding to the Feigenbaum map of the quadratic family (see [S] pp. 279, 299; [D]). We have $b(2) = 2m = f(\frac{1}{2})$, where $f(x) = \sum_{i=0}^{\infty} m_{i+1}x^i$ satisfies $f(x) = (1-x)f(x^2) + \frac{x}{1-x^2}$, and so, by the results of [L-Po], we know that $b(2)$ (and so $a(2)$) is a transcendental number. Since we have $b(2) = \frac{4-2c(2)}{5-3c(2)}$, where

$$c(2) = \frac{\alpha(2)}{1+\alpha(2)}, \qquad \alpha(2) = \sum_{r=1}^{\infty} \frac{1}{\prod_{j=1}^{r}(2^{2^j}-1)},$$

we obtain

$$m = \frac{1}{2}b(2) = \frac{2-c(2)}{5-3c(2)} = \frac{2+\alpha(2)}{5+2\alpha(2)},$$

which gives an explicit expression for m in terms of a series.

4 The Case of Unimodal Maps

In this section we consider C^2 unimodal maps with non-flat critical point. These maps have no wandering intervals, by Theorem A of Chapter IV of [M-S]. Theorem B of the same chapter implies that these maps have only a finite number of periodic attractors. In what follows a pre-periodic point is a point with finite positive orbit, i.e., a point that lands on a periodic orbit.

We may suppose by the same initial considerations that we have done for tent maps that our unimodal maps $f\colon [0,1] \to [0,1]$ satisfy $f(c) = 1$, where c is the critical point, and $f(1) = 0$ (the tent map T_λ restricted to $L = [\lambda(1-\frac{\lambda}{2}), \frac{\lambda}{2}]$ satisfies this combinatorial condition after an affine conjugacy).

We have now two cases:

a) The topological entropy of f is 0: In this case f is combinatorially equivalent to $g_a(x) = ax(1-x)$ for some $a \le c$, where $cx(1-x)$ is the Feigenbaum map of the quadratic family (which is a full family of unimodal maps). We have two subcases.

a.i) $a < c$: In this case $g_a(x)$ has a finite number of periodic points which form the non-wandering set of g_a. In this case, if $J \not\supset [0,1]$, the maximal invariant set of $f|_J$

is infinite if and only if J contains a recurrent interval (that is, there is an interval $I_0 \subset I$ and an $m \in \mathbf{N}$ such that $f^m(I_0) \subset I_0$, $f^k(I_0) \subset I$, for $k = 1, 2, \ldots, m$ and I_0 is contained in the basin of attraction of some periodic point of period m or $f^m|_{I_0} = id$).

a.ii) $a = c$: In this case the closure of the orbit of the critical point is a Cantor set K, and the ω-limit of any point x can be a periodical point if x is pre-periodic or else must be equal to K. Here if $J \neq [0,1]$ and J does not contain any recurrent interval then the maximal invariant of $f|_J$ must be finite (see Ex. 5.3 of Chapter II of [M-S]).

b) The topological entropy of f is strictly positive: In this case we can use Proposition 4.3 of Chapter III of [M-S] to obtain a semi-conjugacy between f and $T_{\exp(h_t(f))}$, where $h_t(f)$ is the topological entropy of f. This semi-conjugacy can collapse an interval only if it is contained into X, where X is the set of points which are eventually mapped into a restrictive interval (see Section II.5 of [M-S]) or into the basin of a periodic attractor.

Let $\lambda = h_t(f)$. We have the following commutative diagram:

$$\begin{array}{ccc} [0,1] & \xrightarrow{f} & [0,1] \\ h \downarrow & & \downarrow h \\ \left[\lambda(1-\frac{\lambda}{2}), \frac{\lambda}{2}\right] & \xrightarrow{T_\lambda} & \left[\lambda(1-\frac{\lambda}{2}), \frac{\lambda}{2}\right] \end{array}$$

h is a monotonous semi-conjugacy, and if x is not a pre-periodic point of T_λ then $h^{-1}(x)$ is a single point.

Let I be an interval. We shall prove that the maximal invariant of $f|_I$ is uncountable if and only if I contains the orbit of some recurrent interval or the maximal invariant of $T_\lambda|_{h(I)}$ is uncountable.

For this, let $a(\lambda)$ and $b(\lambda)$ as before, $J = [a(\lambda), b(\lambda)]$ and $\tilde{J} = h^{-1}(J)$. Since $a(\lambda)$ and $b(\lambda)$ are not pre-periodic, we have $\tilde{I} \not\subset \tilde{J} \to h(\tilde{I}) \not\subset h(\tilde{J}) = J$. Thus, if $T_\lambda|_{h(I)}$ is uncountable then $J \subset h(I) \Rightarrow K$ is contained in the maximal invariant of $T_\lambda|_{h(I)}$, where K is the maximal invariant of $T_\lambda|_J \Rightarrow h^{-1}(K)$ is contained in the maximal invariant of $f|_I$ (because $x \in h^{-1}(K) \Rightarrow T_\lambda^n(h(x)) \in J$, $\forall n \in \mathbf{N} \Rightarrow h(f^n(x)) \in J$, $\forall n \in \mathbf{N} \Rightarrow f^n(x) \in h^{-1}(J)$, $\forall n \in \mathbf{N} \Rightarrow f^n(x) \in I$, $\forall n \in \mathbf{N}$). Therefore in this case the maximal invariant of $f|_I$ is uncountable. If I contains the orbit of some recurrent interval then clearly the maximal invariant of $f|_I$ has non-empty interior.

If $h(I) \not\supset J$ then the turning point of T_λ does not belong to the maximal invariant of $T_\lambda|_{h(I)}$, and so the pre-images of a restrictive interval are disjoint of the maximal invariant of $f|_I$. Moreover, the maximal invariant of $T_\lambda|_{h(I)}$ is countable, and consists only of pre-periodic points. Thus the maximal invariant of $f|_I$ is uncountable if and only if there is a periodic point p of period m and a closed interval I_0, with $p \in I_0$ such that $f^m(I_0) \subset I_0$ and $f^k(I_0) \subset I$, for $0 \leq k < m$, I_0 being collapsed by the semi-conjugacy.

b.i) For unimodal maps f without recurrent intervals (for maps of the quadratic family or, more generally, with negative Schwarzian derivative, it means that f is non-renormalizable, and that the critical point does not belong to the immediate basin of attraction of a periodic point), the map h is a conjugacy, so the maximal invariant set of $f|_I$ is uncountable if and only if $I \supset h^{-1}([a(\lambda), b(\lambda)])$. Here $\lambda = h_t(f)$ and $a(\lambda)$ and $b(\lambda)$ were defined in the end of Section 2.

Under these hypotheses, if f is a Collet-Eckmann map (i.e., if there are $C > 0$, $r > 1$ such that $|Df^n(f(C))| \geq Cr^n$ for any $n \in \mathbb{N}$, where C is the critical point of f) then h is a Hölder-continuous conjugacy, and the maximal invariant set of $f|_I$ has positive Hausdorff dimension if and only if the same occurs for $T_\lambda|_{h(I)}$. In particular, the maximal invariant set of $f|_{h^{-1}([a(\lambda),b(\lambda)]}$ is a Cantor set with zero Hausdorff dimension. We do not know if it is true in more general settings, and in particular if we drop the hypothesis of f being Collet-Eckmann. $\square$

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