A GENERALIZATION OF SARD'S THEOREM

S.M. BATES AND C.G. MOREIRA

ABSTRACT. Let E be a euclidean space and F any normed vector space. If $A \subset E$ is a t-sigmafinite set of rank r for a C^k function $f: E \to F$, then $f(A)$ is $(r + (t - r)/k)$-null for any $t > r$. We also prove that if $\alpha \in (0, 1]$ and $A \subset E$ is a t-sigmafinite (resp. t-null) set of rank r for a $C^{k+\alpha}$ mapping $f: E \to F$, then $f(A)$ is $(r + (t - r)/k)$-sigmafinite (resp. $(r + (t - r)/k)$-null) for any $t > r$.

If E is an infinite-dimensional Banach space, then the theorems above are true for $k \leq 2$. We give examples of polynomial mappings that show that we cannot improve our results by improving the differentiability hypothesis. In particular, we exhibit a polynomial $P: \ell_2 \to \mathbb{R}$ which maps a 3-finite critical set onto the interval $[0, 1]$.

INTRODUCTION

For $t \in \mathbb{Z}_+$, we recall that a set $A \subset \mathbb{R}^n$ is said to be (countably) t-rectifiable provided all but at most a t-null subset of A is contained in a countable union of C^1-embedded t-dimensional submanifolds of $\mathbb{R}^n$.

Theorem 1. *Fix $t, s > 0$ and $r \in \mathbb{N}$ with $t > r$ and set $d = r + (t - r)/s$. Let E be a euclidean space and F any normed vector space. Let $A \subset E$ be a t-sigmafinite set of rank r for a differentiable map $f: E \to F$.*

1. *If $s \in \mathbb{Z}_+$ and $f \in C^s$, then $f(A)$ is d-null.*
2. *If $s > 1$ and $f \in \Lambda_s$, then $\mathcal{H}^d(f(A)) \leq (2\|f\|_s)^d(1 + r)^{d/2}\mathcal{H}^t(A)$.*
3. *If $f \in \Lambda_s$, $t \in \mathbb{Z}_+$ and A is t-rectifiable, then $f(A)$ is d-null.*

Note that if $k > 1$ and $t < r$, then $t < r + (t - r)/k$, and the conclusion of the theorem in this case follows trivially from the fact that any C^1 function maps t-sigmafinite sets to t-sigmafinite sets. If $t = r$, then the statement of the theorem is false, as illustrated by the identity map on $\mathbb{R}^r$. Finally, we note that if F is finite-dimensional and $r = \dim(F)$, then the assertion follows from then inequality $r < r + (t - r)/k$, since F itself is $(r + (t - r)/k)$-null.

For differentiable mappings from infinite-dimensional domains, similar results hold in the smoothness classes Λ_s for $s \in (1, 3]$. When $t = \dim(E)$, statement (2) subsumes previous results of [1, 3, 5].

Theorem 2. *Fix $s \in [1, 3]$ and define $d = r + (t - r)/s$. Suppose X, Y are Banach spaces and $A \subset X$ is a t-sigmafinite rank-r set for a differentiable map $f: X \to Y$.*

1. *If $s \in \{1, 2\}$ and f is C^s, then $f(A)$ is d-null.*
2. *If $f \in \Lambda_s$, then $\mathcal{H}^d(f(A)) \leq (2\|f\|_s)^d(1 + r)^{d/2}\mathcal{H}^t(A)$.*
3. *If $f \in \Lambda_s$, $t \in \mathbb{Z}_+$ and A is t-rectifiable, then $f(A)$ is d-null.*

Supported by an NSF Postdoctoral Fellowship in Mathematics.
Supported by CNPq, Brasil.

In particular, Theorem 2 asserts that for any Banach space X, a C^3 real-valued function maps any t-dimensional critical set to a set of dimension ≤ 3.

Theorem 3. *For any infinite-dimensional Banach space X, there exists a cubic polynomial $P\colon X \to \mathbb{R}$ and a 3-finite critical set $A \subset X$ such that $P(A) = [0,1]$.*

Notations and Definitions.

For $s \in (0,1)$, we denote by Λ_s the Lipschitz space of those continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ for which

$$\|f\|_s = \sup_{x\in\mathbb{R}} |f(x)| + \sup_{x,h\in\mathbb{R}} \frac{|f(x+h) - f(x)|}{|h|^s} < \infty.$$

A function $f\colon \mathbb{R} \to \mathbb{R}$ belongs to Λ_1 if f is continuous and

$$\|f\|_1 = \sup_{x\in\mathbb{R}} |f(x)| + \sup_{x,h\in\mathbb{R}} \frac{|f(x+h) + f(x-h) - 2f(x)|}{|h|^s} < \infty.$$

For $s > 1$, let $[s]$ denote the greatest integer *strictly less than* s. The class Λ_s is then defined inductively as the space of $C^{[s]}$ functions $f\colon \mathbb{R} \to \mathbb{R}$ for which

$$\|f\|_s = \|f\|_{s-1} + \|f'\|_{s-1} < \infty.$$

When $s > 1$ is not an integer, each element of the Hölder class C^s coincides locally with some member of Λ_s. For $k \in \mathbb{Z}_+$, the Lipschitz space Λ_k strictly contains C^k. A function $f \in C^{k-1}$ coincides locally with some member of Λ_k provided that the $(k-1)$-st derivative of f satisfies a local Zygmund condition (see [8]).

1. Decompositions of Critical Sets

Lemma 1.1. *Fix $\eta > 0$. Let E be a euclidean space, F a normed vector space, and $A \subset E$ a set of rank-r points for a differentiable map $f\colon E \to F$.*

1. *If $s \in \mathbb{Z}_+$, $s \geq 2$, and $f \in C^s$, then there exists a decomposition $\{A_i\}$ of A such that for each i, there is a constant $M_i > 0$ and a splitting of E such that for any set $S \subset E$,*

$$|f(S \cap A_i)| \leq (\|f\|_1 + \eta)|S'| + M_i|S'||S''| + \eta|S|^s.$$

2. *If $s > 2$ and $f \in \Lambda_s$, then there exists a decomposition $\{A_i\}$ of A such that for each i, there is a constant $M_i > 0$ and a splitting of E such that for any set $S \subset E$,*

$$|f(S \cap A_i)| \leq (\|f\|_1 + \eta)|S'| + M_i|S'||S''| + M_i|S|^s.$$

3. *If A is t-rectifiable for some $t \in \mathbb{Z}_+$, then there exists a decomposition $\{A_i\}$ of A such that A_0 is t-null, $\sum_i |A_i|^t \leq \mathcal{H}^t(A_i) + 1$, and for each $i \in \mathbb{Z}_+$, there is a constant $M_i > 0$ and a splitting of E such that for any set $S \subset E$,*

$$|f(S \cap A_i)| \leq (\|f\|_1 + \eta)|S'| + M_i|S'||A_i| + M_i|A_i|^s.$$

To prove the lemma, we will need the following generalization of the Morse decomposition lemma:

Lemma 1.2. *Let E be a euclidean space, F any normed vector space, and $h\colon E \to F$ a C^k map. If $h = 0$ on $A \subset E$, then for each $\eta > 0$ there exists a decomposition $\{A_i\}$ of A such that A_0 is countable, and for each $i \in \mathbb{Z}_+$, there is a closed ball $B_i \subset \mathbb{R}^{n_i}$ and a C^1 map $\psi_i\colon B_i \to E$ such that*

1. *$A_i \subset \psi_i(B_i)$ and $|x-y| \le |\psi_i(x) - \psi_i(y)| \le (1+\eta)|x-y|$ for all $x, y \in B_i$.*
2. *For all $x \in \psi_i^{-1}(A_i)$ and $y \in B_i$,*

$$h(\psi_i(y)) - h(\psi_i(x))| \le \eta |x-y|^k.$$

Proof. Let (n,k) denote the statement of the lemma for all $\eta > 0$ and any C^k mapping from n-dimensional euclidean space into any Banach space.

We first prove $(1,k)$ for all $k \in \mathbb{Z}_+$. Fix $\eta > 0$ and let A_0 be the set of isolated points of A. If $\dim(E) = 1$ and F is any Banach space, then $D^i h(x) = 0$ for each $i = 1, \cdots, k$ and $x \in A \setminus A_0$. (This follows easily from the mean-value theorem when $F = \mathbb{R}$ and can in general be deduced from the equation $D^i(y^* \circ f) = y^* \circ D^i f$, valid for each $y^* \in F^*$). By continuity of $D^k f$, we can choose a covering of $A \setminus A_0$ by closed intervals B_i such that $|D^k h| < \eta$ on each B_i. From Taylor's theorem, it follows that for each $x \in A_i$ and $y \in B_i$,

$$|h(y)| = |h(y) - h(x)| \le \eta |x-y|^k,$$

which proves statement $(1,k)$.

To prove case (n,k) of the lemma, we proceed inductively, assuming that statements $(n-1,k)$ and $(n,k-1)$ have been proven (if $k=1$, we only assume $(n-1,1)$). Given a subset A of an n-dimensional euclidean space E, define

$$U_A = \{x \in A : \text{every } C^k \text{ function } g\colon E \to \mathbb{R} \text{ with } g(A) = 0 \text{ satisfies } Dg(x) = 0\},$$

$V_A = A \setminus U_A$. Fix $\eta > 0$. To decompose A, we find appropriate decompositions of V_A and U_A.

¿From the definition of V_A, it follows that for each $\eta > 0$, there exists a countable collection of C^1 embeddings $\varphi_i\colon B'_i \to E$ such that each B'_i is a closed ball in $\mathbb{R}^{n-1}$, $V_A \subset \bigcup_i \varphi_i(B'_i)$, and

$$|x-y| \le |\varphi_i(x) - \varphi_i(y)| \le (1+\eta)^{1/2}|x-y|.$$

By applying the $(n-1,k)$ statement of the lemma to the mapping $h \circ \varphi_i$, we obtain a decomposition $\{A'_i\}$ of $\varphi_i^{-1}(V_A)$ and a sequence of C^1 maps $\psi_{ij}\colon B_{ij} \to B_i$ such that:

1. $A'_j \subset \psi_{ij}(B_{ij})$ and $|x-y| \le |\psi_{ij}(x) - \psi_{ij}(y)| \le (1+\eta)^{1/2}|x-y|$ for all $x, y \in B_{ij}$.
2. For all $y \in B_{ij}$,

$$|h(\varphi_i(\psi_{ij}(y)))| \le \eta d(y, \psi_{ij}^{-1}(A'_j))^k.$$

Setting $V_{ij} = \varphi_i(A'_j)$ and $\tilde{\psi}_{ij} = \varphi_i \circ \psi_{ij}$ then gives the desired decomposition of V_A.

Next, we note that $Dh = 0$ on U_A. If $k = 1$, then we can use the continuity of Dh to find a countable covering of U_A by balls B_i on which $\|Dh\| \le \eta$, so that $\{B_i \cap U_A\}$ gives the desired decomposition of U_A. For $k > 1$, we apply the $(n, k-1)$ hypothesis to the map $Dh\colon E \to L(E,F)$ in order to find a decomposition $\{U_i\}$ of U_A and mappings $\psi_i\colon B_i \to E$ such that

1. $U_i \subset \psi_i(B_i)$ and $|x-y| \le |\psi_i(x) - \psi_i(y)| \le (1+\eta)|x-y|$ for all $x, y \in B_i$.

2. For all $x \in \psi_i^{-1}(U_i)$ and $y \in B_i$,

$$|Dh(\psi_i(y)) - Dh(\psi_i(x))| \le \eta |x-y|^{k-1}.$$

Given $x \in \psi_i^{-1}(U_i)$ and $y \in B_i$, we set $\gamma(t) = ty + (1-t)x$, so that

$$\begin{aligned} |h(\psi_i(y))| &= \left| \int_0^1 D(h \circ \psi_i)(\gamma(t))\gamma'(t)\,dt \right| \\ &\le \eta |x-y|^{k-1}|x-y|. \end{aligned}$$

Thus, $|h(\psi_i(y))| \le \eta d(y, \psi_i^{-1}(U_i))^k$ for all $y \in B_i$, and this completes the proof.

□

Lemma 1.3. *Let E be a euclidean space, F any normed vector space, and $h\colon E \to F$ a Λ_s map with $s > 2$. If $h = 0$ on $A \subset E$, then for each $\eta > 0$ there exists a decomposition $\{A_i\}$ of A such that A_0 is countable, and for each $i \in \mathbb{Z}_+$, there is a closed ball $B_i \subset \mathbb{R}^{n_i}$ and a C^1 map $\psi_i\colon B_i \to E$ such that*

1. *$A_i \subset \psi_i(B_i)$ and $|x-y| \le |\psi_i(x) - \psi_i(y)| \le (1+\eta)|x-y|$ for all $x, y \in B_i$.*
2. *For all $x, x' \in \psi_i^{-1}(A_i)$ and y in the line segment xx',*

$$|h(\psi_i(y)) - h(\psi_i(x))| \le M_i |x-y|^{s-1}|x-x'|.$$

Proof of Lemma 1.1. We first prove the lemma assuming that with respect to some splitting $E = E' \oplus E''$, the map f is of the form

$$f(z,y) = (h(z), g(z,y))$$

for some h, g such that $D_2 g = 0$ on A. From Lemma 1.2 we obtain a decomposition $\{A_i\}$ of A and C^1 mappings $\psi_i\colon B_i \to E$ such that

1. $A_i \subset \psi_i(B_i)$ and $|x-y| \le |\psi_i(x) - \psi_i(y)| \le (1+\eta)|x-y|$ for all $x, y \in B_i$.
2. For all $x, x' \in \psi_i^{-1}(A_i)$ and $y \in xx'$,

$$|D_2 g(\psi_i(y))| \le \eta |x-x'|^{s-1}.$$

Fix $i \in \mathbb{Z}_+$ and suppose that $(z,y), (z',y') \in S \cap A_i$. Then

$$\begin{aligned} |f(z,y) - f(z',y')| &\le |f(z,y) - f(z',y)| + |f(z',y) - f(z',y')| \\ &\le \|f\|_1 |S'| + |g(z',y) - g(z',y')|. \end{aligned}$$

To bound the term on the right, let $\gamma\colon [0,1] \to E$ be the composition of ψ_i with the affine path between $\psi_i^{-1}(z,y)$ and $\psi_i^{-1}(z',y')$. If $\tilde{\gamma}\colon [0,1] \to E$ is the composition of γ with the projection onto $z' \times E''$, then

$$\begin{aligned} |g(z,y) - g(z',y')| &= \left| \int_0^1 D_2 g(\tilde{\gamma}(t))\tilde{\gamma}'(t)\,dt \right| \\ &\le \int_0^1 |D_2 g(\tilde{\gamma}(t)) - D_2 g(\tilde{\gamma}(t))||\tilde{\gamma}'(t)|\,dt + \int_0^1 |D_2 g(\gamma(t))||\tilde{\gamma}'(t)|\,dt. \end{aligned}$$

Since $|\tilde{\gamma}'(t)| \leq (1+\eta)|S''|$ for all t, we have

$$|D_2 g(\bar{\gamma}(t)) - D_2 g(\tilde{\gamma}(t))||\tilde{\gamma}'(t)| \leq (1+\eta)\|f\|_2 |S'||S''|.$$

Finally, if $x, y \in B_i$ satisfy $\psi_i(x), \psi_i(y) \in S \cap A_i$ and $\psi_i(y) \in Q$, then property (1) of the map ψ_i implies that $|x-y| \leq |S|$. From (2), it follows that $\|D_2 g\|$ is bounded on $S \cap \psi_i(B_i)$ by $\eta |S|^{s-1}$, and so

$$\int_0^1 \|D_2 g(\gamma(t))\| |\tilde{\gamma}'(t)|\, dt \leq (1+\eta)\eta |S|^s.$$

¿From these estimates, part (1) of the lemma follows.

Part (2) of the lemma follows by exactly the same argument, where Lemma 1.3 is applied in place of Lemma 1.2 to decompose A.

Turning to part (3) of the lemma, we suppose that A is t-rectifiable. By combining Lemma 1.3 with the definition of rectifiability, we then obtain a countable decomposition $\{A_i\}$ of A such that A_0 is t-null, and for each $i \in \mathbb{Z}_+$, there exists a ball $B_i \subset \mathbb{R}^t$ and a C^1 embedding $\psi_i \colon B_i \to E$ such that

1. $A_i \subset \psi_i(B_i)$ and each point of $\psi_i^{-1}(A_i)$ is a Lebesgue density point.
2. $|x-y| \leq |\psi_i(x) - \psi_i(y)| \leq 2|x-y|$ for all $x, y \in B_i$.
3. For all $x, x' \in \psi_i^{-1}(A_i)$ and $y \in xx'$,
$$|D_2 g(\psi_i(y))| \leq M_i |x-x'|^{s-1}.$$

To obtain the desired decomposition of A, we will further decompose each A_i. Given $\eta > 0$, we fix a positive integer P such that $2/P < \eta$. Since each $x \in \psi_i^{-1}(A_i)$ is a density point, there exists $\varepsilon_x > 0$ such that if $Q(x,\varepsilon)$ is a cube of edge $\varepsilon < \varepsilon_x$ centered at $x \in \psi_i^{-1}(A_i)$, then

$$\frac{\mathcal{H}^t(Q \cap \psi_i^{-1}(A_i))}{\mathcal{H}^t(Q)} \geq 1 - P^{-t}. \tag{$*$}$$

The collection

$$\mathcal{Q} = \{Q(x,\varepsilon) : x \in \psi_i^{-1}(A_i), \varepsilon < \varepsilon_x\}$$

is a Vitali family for $\psi_i^{-1}(A_i)$, and so there exists a sequence of pairwise disjoint cubes $\{Q_j\} \subset \mathcal{Q}$ such that $\psi_i^{-1}(A_i) \setminus \bigcup_j Q(x_j, \varepsilon_j)$ is t-null and $\sum_j \varepsilon_j^t < \mathcal{H}^t(\psi_i^{-1}(A_i)) + 1$.

Fix j and consider $(z,y), (z',y') \in Q_j \cap \psi_i^{-1}(A_i)$. By condition $(*)$, it follows that there are $x_0, \cdots, x_P$ in $Q \cap \psi_i^{-1}(A_i)$ such that $x = x_0$, $y = x_P$, and $|x_i - x_{i+1}| < 2|Q|P^{-1}$ for all i. Let $\gamma \colon [0,1] \to E$ be the composition of ψ_i with the piecewise linear path connecting the points x_i. If $\tilde{\gamma} \colon [0,1] \to E$ is the composition of γ with the projection onto $z' \times E''$, then

$$\begin{aligned} |g(z,y) - g(z',y')| &= \left| \int_0^1 D_2 g(\tilde{\gamma}(t)) \tilde{\gamma}'(t)\, dt \right| \\ &\leq \int_0^1 |D_2 g(\tilde{\gamma}(t)) - D_2 g(\gamma(t))||\tilde{\gamma}'(t)|\, dt + \int_0^1 |D_2 g(\gamma(t))||\tilde{\gamma}'(t)|\, dt. \end{aligned}$$

Since $|\tilde{\gamma}(t)| \leq 2|Q_j|$ for all t, we have

$$|D_2 g(\gamma(\tilde{t})) - D_2(\gamma(t))||\tilde{\gamma}(t)| \leq 2\|f\|_2 |S'||Q_j|.$$

Finally, property (1) of the map ψ_i implies that for each i, we have $|x_i - x_{i+1}| \leq 2|Q|/P$, and so from (3) it follows that

$$\int_0^1 \|D_2 g(\gamma(t))\| |\tilde{\gamma}'(t)| \, dt \leq 2|Q_j| M_i \sum_{i=0}^{P-1} |x_i - x_{i+1}|^{s-1} \leq 2^s M_i P^{2-s} |Q_j|^s.$$

Thus, proceeding as above, and applying the inequality $2^s M_i P^{2-s} < \eta$, we obtain

$$|f(S \cap \psi_i(Q_j))| \leq \|f\|_1 |S'| + M_i |S'||Q_j| + \eta |Q_j|^s.$$

For each $i \in \mathbb{Z}_+$, we now choose a collection of cubes Q_j associated to A_i as above. The decomposition $\{A_i\}$ given by a denumeration of the sets $\psi_i(Q_j) \cap A$, and $A_0 = A \setminus \bigcup A_i$ then satisfies the lemma.

For the general case, we apply the implicit function theorem locally to find a diffeomorphism $\varphi \colon E' \oplus E'' \to E$ such that $f \circ \varphi$ is of the split form above. If $\psi \colon B \to E$ is a map such that

$$|f(\psi_i(y))| \leq \eta d(y, \psi_i^{-1}(A_i))^k$$

and $\varphi(x), \varphi(y) \in \psi(B)$, then....

Note that if we compose the map f in Lemma 1.3 with a C^1 diffeomorphism $\varphi \colon E \to E$, then we obtain the estimate

$$|(f \circ \varphi)(x) - (f \circ \varphi)(y)| \leq (\|f\|_s + \eta)|\varphi(x) - \varphi(y)|^s \leq (\|f\|_s + \eta)\|\varphi\|_1^s |x - y|^s.$$

In other words, we can compose Morse decompositions with diffeomorphism φ and only increase the constants by the C^1 norm of φ.

□

Lemma 1.4. *Fix $\eta > 0$. Let E, F be Banach spaces and $A \subset E$ a set of rank-r points for a differentiable map $f \colon E \to F$.*

1. *If $s \in \{1, 2\}$ and $f \in C^s$, then there exists a decomposition $\{A_i\}$ of A such that for each i, there is a splitting of E such that for any set of the form $Q = Q' \times S''$*
$$|f(Q \cap A_i)| \leq \|f\|_1 |Q'| + \eta |Q|^s.$$
2. *If $s \in (1, 3]$ and $f \in \Lambda_s$, then there exists a decomposition $\{A_i\}$ of A such that for each i, there is a splitting of E such that for any set of the form $Q = Q' \times S''$*
$$|f(Q \cap A_i)| \leq \|f\|_1 |Q'| + \|f\|_s |Q|^s.$$

Proof. We first prove the lemma assuming that with respect to some fixed splitting $E' \oplus E''$ of E, the map f assumes the form

$$f(z, y) = (h(z), g(z, y))$$

and $D_2 g = 0$ on A. Fix $(z, y), (z', y') \in A$ and define

$$\gamma(t) = (t(z' - z) + z, t(y' - y) + y).$$

By the mean-value theorem, there exists a $c \in (0, 1)$ such that

$$\begin{aligned} |f(z, y) - f(z', y')| &= |D_1 f(\gamma(c))(z' - z) + D_2 f(\gamma(c))(y' - y)| \\ &\leq \|f\|_1 |Q'| + |D_2 g(\gamma(c))||Q|, \end{aligned}$$

since $D_2 f = D_2 g$.

1. Let $\{B_i\}$ be a countable covering of A by balls such that $|D_2 g| < \eta$ on each B_i, and set $A_i = A \cap B_i$. For sets of the form $Q = Q' \times S''$, the preceding inequality then implies that

$$|f(Q \cap A_i)| \leq \|f\|_1 |Q'| + \eta |Q|,$$

thus proving statement (1) of the lemma for the C^1 case. If $f \in C^2$, then the mean-value theorem implies that for some $c' \in (0, c)$,

$$|D_2 g(\gamma(c)) - D_2 g(\gamma(0))| \leq |D(D_2 g)(\gamma(c'))(z - z', y - y')|.$$

Since $D_2 g(\gamma(0)) = D_2 g(\gamma(1)) = 0$, there exists $c'' \in (0, 1)$ such that $D(D_2 g)(\gamma(c''))(z - z', y - y') = 0$, so that

$$|D_2 g(\gamma(c))| \leq |D(D_2 g)(\gamma(c')) - D(D_2 g)(\gamma(c'))||(z - z', y - y')|.$$

Let $\{B_i\}$ be a covering of A by balls B_i such that $|D(D_2 g)(p) - D(D_2 g)(p')| < \eta$ for all $p, p' \in B_i$, and set $A_i = A \cap B_i$. Combining $(*)$ with the preceding inequalities, we find that for any set of the form $Q = Q' \times S''$,

$$|f(Q \cap A_i)| \leq \|f\|_1 + \eta |Q|^2.$$

2. Turning to statement (2) of the lemma, we recall that in the notation above, $D_2 g(\gamma(0)) = D_2 g(\gamma(1)) = 0$. If $f \in \Lambda_s$ for $s \in (1, 2]$, then Norton's lemma implies that

$$|D_2 g(\gamma(c))| \leq \|f\|_s |Q|^{s-1},$$

by Lemma 1.5. Combined with the preceding inequality, this proves the lemma for $s \in (1, 2]$.

For $s \in (2, 3]$, fix $\delta > 0$ and choose $q, q' \in A$ such that $|p - q|, |p' - q'| < \delta$. Let $\overline{\gamma} : [-2\delta, 2\delta] \to E$ be a smooth extension of γ such that $\|\gamma\|_1 < |Q|$ and $\overline{\gamma}(-2\delta) = q$, $\overline{\gamma}(1 + 2\delta) = q'$. Set $h = D_2 g \circ \overline{\gamma}$. Since $D_2 g = 0$ on A, it follows that $h = 0$ at 4 points of $[-2\delta, 1 + 2\delta]$. Since $h \in \Lambda_{s-1}$, Lemma 1.2 implies that

$$\begin{aligned} |h(c) - h(0)| &\leq \|h\|_{s-1} (1 + 4\delta)^{s-1} \\ &\leq \|f\|_s |Q|^{s-1} (1 + 4\delta)^{s-1}. \end{aligned}$$

Since $\delta > 0$ is arbitrary, this completes the proof.

□

To prove a similar decomposition result for functions of class Λ_s, we require the following observation of Norton [5]

Lemma 1.5. *(Norton) If $g \in \Lambda_1(\mathbb{R}, \mathbb{R})$, $a < b$, and $g(a) = g(b) = 0$, then for all $x \in [a, b]$,*

$$|g(x)| \leq \|g\|_1 |b - a|.$$

□

2. Proof of the main theorems

Fix $t, s > 1, r \in \mathbb{N}$. To prove Theorem 1, we consider a Banach space E, a normed vector space F, and a t-finite set $A \subset E$ of rank-r points for a C^1 map $f : E \to F$. Since Df is continuous, the function $x \mapsto \operatorname{rank}(Df_x)$ is lower semi-continuous on E, and so its level sets are Borel sets. Consequently, there exists a Borel set A' of rank-r points of f which contains A and satisfies $\mathcal{H}^t(A') = \mathcal{H}(A)$.

For some $\mu, \eta > 0$, we suppose for the moment that there is a decomposition $\{A_i\}$ of A' such that for each $i \in \mathbb{Z}_+$ and $\delta \in (0, M_i^{-1})$, choose a covering $\{S_j\}$ of A_i such that $|S_j| = \varepsilon_j < \delta$ for each j and

$$\sum_j \varepsilon_j^t \leq \mathcal{H}^t(A_i) + \delta.$$

For each j, let S_j', S_j'' denote the orthogonal projections of S_j in the subspaces E_i' and E_i'', respectively. Evidently, $|S_j'| \leq |S_j|$, and so there exists an r-cube $Q_j' \subset E'$ of diameter $\leq (1+r)^{1/2}\varepsilon_j$ which contains S_j'. The product $Q_j' \times S_j''$ then contains S_j and has diameter $\leq (1+r)^{1/2}\varepsilon_j$. Given $\mu > 0$, we partition Q_j' into $([\mu^{-1}\varepsilon_j^{1-s}] + 1)^r$ equal subcubes of edge $\leq \mu\varepsilon_j^s$. For every such subcube $\tilde{Q}$, we have $|\tilde{Q}| \leq r^{1/2}\varepsilon_j$, so that by $(*)$,

$$|f(A_i \cap (\tilde{Q} \times S_j''))| \leq (P(\eta, \mu) + M_i\delta)\varepsilon_j^s.$$

For $d = r + (t-r)/s$ and $P = P(\eta, \mu)$, we then have

$$\mathcal{H}^d(f(A_i)) \leq \sum_j ([\mu^{-1}\varepsilon_j^{1-s}] + 1)^r ((P + M_i\delta)\varepsilon_j^s)^d \leq (P + M_i\delta)^d \mu^{-r}(\mathcal{H}^t(A_i) + \delta),$$

so that tending δ to 0 gives $\mathcal{H}^d(f(A_i)) \leq P^d \mu^{-r} \mathcal{H}^t(A_i)$. Since the A_i are disjoint Borel sets, we have

$$\mathcal{H}^d(f(A)) \leq \sum_{i=1}^{\infty} \mathcal{H}^d(f(A_i)) \leq P^d \mu^{-r} \mathcal{H}^t(A). \qquad (**)$$

Proof of Theorem 1. To prove statement (1) of Theorem 1, we note that by Lemma 1.1(1), the inequality $(*)$ above holds for all $\eta, \mu > 0$ with

$$P(\eta, \mu) = (\|f\|_1 + \eta + 1)\mu.$$

Since $d > r$, tending η, μ to 0 in the inequality $(**)$ gives $\mathcal{H}^d(f(A)) = 0$.

Similarly, Lemma 1.1(2) implies that when $f \in \Lambda_s$ for $s > 2$, then $(*)$ holds for

$$P(\eta, \mu) = (\|f\|_1 + \eta)\mu + \|f\|_s + \eta).$$

Thus, we obtain statement (2) of Theorem 1 by setting $\eta = 1$ and tending μ to zero in the equality $(**)$.

Finally Lemma 1.1(3) asserts if A is t-rectifiable, then $A \setminus A_0$ admits a decomposition of the form above for all $\eta, \mu > 0$ and

$$P(\eta, \mu) = (\|f\|_1 + \eta + 1)\mu.$$

Since A_0 is t-null, Theorem 1(1) implies that $f(A_0)$ is d-null, so that the inequality $(**)$ is valid for all $\eta, \mu > 0$. Since $d > r$, we conclude that $\mathcal{H}^d(f(A)) = 0$.

$\square$

Proof of Theorem 2. To prove statement (1) of Theorem 2, we note that by Lemma 1.1(1), the inequality $(*)$ above holds for all $\eta, \mu > 0$ with

$$P(\eta, \mu) = (\|f\|_1 + \eta + 1)\mu.$$

Since $d > r$, tending η, μ to 0 in the inequality $(**)$ gives $\mathcal{H}^d(f(A)) = 0$.

Similarly, Lemma 1.1(2) implies $(*)$ holds for

$$P(\eta, \mu) = (\|f\|_1 + \eta)\mu + \|f\|_s + \eta).$$

Thus, we obtain statement (2) of Theorem 1 by setting $\eta = 1$ and tending μ to zero in the equality $(**)$.

Finally Lemma 1.1(3) asserts if A is t-rectifiable, then $A \setminus A_0$ admits a decomposition of the form above for all $\eta, \mu > 0$ and

$$P(\eta, \mu) = (\|f\|_1 + \eta + 1)\mu.$$

Since A_0 is t-null, Theorem 1(1) implies that $f(A_0)$ is d-null, so that the inequality $(**)$ is valid for all $\eta, \mu > 0$. Since $d > r$, we conclude that $\mathcal{H}^d(f(A)) = 0$.

$\square$

Lemma 2.1. *Let F be any Banach space. Suppose $s > 1$ and $a < b < c < d$ are real numbers, and suppose that each of the intervals $[a, b]$, $[b, c]$, and $[c, d]$ contains at least $[s]+1$ zeros of a function $f \in \Lambda_s(\mathbb{R}, F)$. Then*

$$\max |f|_{[b,c]}| \leq \|f\|_s \cdot (d-a)^s.$$

Proof. If $F = \mathbb{R}$, then by repeated application of the mean-value theorem, our hypotheses imply that the derivatives $f^{(1)}, \cdots, f^{([s])}$ of f all have zeros in $[a, b]$, $[b, c]$, and $[c, d]$. Thus, for $k = 1, \cdots, [s]-1$,

$$\max_{[b,c]} |f^{(k)}| \leq \max_{[b,c]} |f^{(k+1)}|(c-b).$$

Also,

$$\max_{[b,c]} f - \min_{[b,c]} f \leq \max_{[b,c]} |f^{(1)}|(c-b).$$

Hence,

$$\max_{[b,c]} f - \min_{[b,c]} f \leq \max_{[b,c]} |f^{([s])}|(c-b)^{[s]}.$$

For noninteger s, the proof is finished by noting that

$$\max_{[b,c]} |f^{[s]}| \leq \|f\|_s (c-b)^{(s-[s])},$$

which follows from the $(s-[s])$-Hölder condition satisfied by $f^{[s]}$.

When s is an integer, it suffices to establish that

$$\max_{[b,c]} |f^{[s]}| \leq \|f\|_s (d-a).$$

This is implied by Lemma 1.5.

For an arbitrary Banach space F, we can apply the preceding argument to the function $u^* \circ f$, where $u^* \in F^*$ is any unit covector. Since $\|u^* \circ f\|_s = \|f\|_s$, the lemma follows.

$\square$

3. Sharpness of the Theorems

Recall that a *modulus of continuity* $\mathfrak{m}$ is a class of monotone, continuous functions $\eta : [0,1] \to [0,\infty)$ which are subadditive, i.e.,

$$\eta(s+t) \leq \eta(s) + \eta(t)$$

for all s, t in some subinterval $[0,a)$ of $[0,\infty)$. Any two members η_1, η_2 of $\mathfrak{m}$ are equivalent in the sense that

$$C^{-1} \leq \frac{\eta_1(t)}{\eta_2(t)} \leq C$$

for some constant C and all t in some subinterval $[0,a)$ of $[0,1]$. A partial order $\prec$ is defined for moduli of continuity by setting $\mathfrak{m}_1 \prec \mathfrak{m}2$ whenever

$$\lim_{t \to 0^+} \frac{\eta_1(t)}{\eta_2(t)} = \infty$$

for any $\eta_i \in \mathfrak{m}_i$, $i = 1, 2$.

Fix $n, m \in \mathbb{Z}_+$ and let X, Y be Banach spaces. For any nonnegative integer k and modulus of continuity η, we define the subset $C^{k,\mathfrak{m}}$ of $C^k(X,Y)$ to consist of all f such that for each $x \in X$, there exists $\varepsilon_x > 0$ and $\eta \in \mathfrak{m}$ such that

$$\|D^k f(y) - D^k f(z)\| \leq \eta(\|y - z\|)$$

for all $y, z \in B(x, \varepsilon)$. Note that if $\mathfrak{m}_1 \prec \mathfrak{m}_2$, then $C^{k,\mathfrak{m}_2} \subset C^{k,\mathfrak{m}_1}$.

Example 3.1. *For fixed $\alpha \in (0,1]$, the modulus of continuity $\mathfrak{m}_\alpha$ containing the function $\eta_\alpha(t) = t^\alpha$ gives rise to the familiar spaces $C^{k,\mathfrak{m}_\alpha}$ of C^k functions whose k-th derivatives satisfy a local α-Hölder estimate.*

Since $(-\log(t))^s t^\varepsilon \to 0$ as $t \to 0$ for any fixed $\varepsilon, s > 0$, the modulus of continuity $\mathfrak{m}$ containing the function $\eta(t) = (-\log(t))^s t^\alpha$ satisfies $\mathfrak{m}_\beta \prec \mathfrak{m} \prec \mathfrak{m}_\alpha$ for all $\beta < \alpha$.

$\triangle$

Proposition 3.2. *Fix $\alpha \in (0,1]$. If $\mathfrak{m}$ is a modulus of continuity such that $\mathfrak{m} \prec \mathfrak{m}_\alpha$, then for any $k \in \mathbb{Z}_+$ and $t \in (0,1)$, there exists a $C^{k,\mathfrak{m}}$ function $f : \mathbb{R} \to \mathbb{R}$ and a t-null set A of critical points of f such that $f(A)$ is not $(t/(k+\alpha))$-sigmafinite.*

To prove the proposition, we will use the following extension lemma. Given a compact subset $K \subset \mathbb{R}$, we refer to bounded components of $\mathbb{R} \setminus K$ as *gaps* of K.

Lemma 3.3. *Fix $k \in \mathbb{Z}_+$ and $\alpha \in (0,1]$, and let $\eta : [0,1] \to \mathbb{R}$ represent a modulus of continuity $\mathfrak{m}$ such that $\mathfrak{m} \prec \mathfrak{m}_\alpha$. If $B \subset \mathbb{R}$ is a compact set and $f : B \to \mathbb{R}$ is a monotone embedding such that*

$$|g(J)| \leq |J|^k \eta(|J|)$$

for each gap J of B, then there exists a $C^{k,\mathfrak{m}}$ extension $f : \mathbb{R} \to \mathbb{R}$ for which B is a critical set.

Proof. Let $[a,b]$ denote the convex hull of B and let $J_i = [a_i, b_i]$ denote its gaps. Set $l_i = |J_i|$ and define $t_i = |g(b_i) - g(a_i)|$.

Let $s : [0,1] \to \mathbb{R}$ be any smooth function such that

1. $s > 0$ on $(1/3, 2/3)$ and $s = 0$ otherwise, and

2. $\int_0^1 s(x)\,dx = 1$.

Denote by M_n the C^n norm of s. We construct the derivative r of f by defining

$$r(x) = \begin{cases} 0 & \text{for } x \in \mathbb{R} \setminus \cup_i J_i \\ \frac{t_i}{l_i} s\left(\frac{x-a_i}{l_i}\right) & \text{for } x \in J_i \end{cases}$$

For each $x \in B$, we again let $\mathcal{I}_x$ denote the set of gaps of B lying to the left of x. By the definitions above, we have

$$g(x) = g(a) + \sum_{I \in \mathcal{I}_x} t_i.$$

Define

$$f(x) = g(a) + \int_a^x r(t)\,dt.$$

Then

$$f(x) = g(a) + \sum_{I \in \mathcal{I}_x} t_i \int_0^1 s(t)\,dt = g(a) + \sum_{I \in \mathcal{I}_x} t_i = g(x).$$

Thus, the function f is a monotone extension of the homeomorphism g.

Evidently f is C^∞ on the complement of B and near isolated points of B. If $x \in B$ is not an isolated point, then for any k there exists a neighborhood U of x such that $y \in U \setminus B$ implies that y lies either in a gap with endpoint x, or in a gap of length ℓ less than $|U|$. In the latter case, property (1) of the function s implies that either $f^{(n)}(y) = 0$ or $|x - y| \geq \ell/3$, and so

$$|f^{(n)}(y)| = |r^{(n-1)}(y)| \leq M_n \ell^{k-n} \cdot \eta(\ell)\ell 3^{k+1}|y - x|^{k-n}\eta(|y - x|).$$

Thus, the derivative $f^{(n)}$ tends to zero on approach to A for each $n \leq k$. We conclude that f is $C^{k,\mathfrak{m}}$ from the preceding inequality and the following fact, whose proof is left to the reader.

Lemma 3.4. *Let $K \subset \mathbb{R}$ be compact and $f\colon \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and C^k on $\mathbb{R} \setminus K$. Suppose that*

1. *for $i = 1, \ldots, k$, $|f^{(i)}(x)| \to 0$ uniformly as $x \to K$, and*
2. *$|f(x) - f(y)| \leq o(|x - y|^k)$ uniformly for $x, y \in K$.*

Then f is C^k.

Finally we note that by construction, each connected component of the critical set of f intersects B, and therefore the critical value set of f equals A.

□

Proof of Proposition 3.2. Given a sequence of positive numbers ℓ_n such that

$$\sum_{n=0}^{\infty} 2^n \ell_n \leq 1,$$

we let $B(\mathfrak{s})$ denote the Cantor set obtained by an analog of the "middle thirds" construction in which 2^n intervals of length ℓ_n are removed at the n-th stage.

Fix $t \in (0,1)$ and define sequences by

$$a_n = 2^{-n/t} \qquad b_n = \eta(2^{-n/t})^{-1/2} 2^{-n/t} \qquad c_n = \eta(2^{-n/kt})^{-1/2} 2^{-n/kt} \qquad d_n = 2^{-n/kt},$$

and let A, B, C, D denote the corresponding central Cantor sets. By Lemma 3.3, the obvious homeomorphism $B \to C$ extends to a $C^{k,\mathfrak{m}}$ map $f : \mathbb{R} \to \mathbb{R}$ containing B within its set of critical set.

We claim that B is t-null, whereas $C = f(B)$ is not $(t/(k+\alpha))$-sigmafinite. Indeed, this can be checked either by a direct computation, or by applying Lemma 3.3 to find C^1 maps $g, h \colon \mathbb{R} \to \mathbb{R}$ such that g, h are critical on A, C, respectively, and $g(A) = B, h(C) = D$. The claim then follows by an application of Theorem 1 to the mappings g, h.

$\square$

4. Morse Criticality in Infinite Dimensions

For $1 \leq p < \infty$, we denote by ℓ_p the space of real sequences $x = (x_j)$ equipped with the norm

$$\|x\|_p = \left(\sum_j |x_j|^p\right)^{1/p}.$$

By ℓ_∞ (resp. c_0), we denote the space of bounded sequences (resp. sequences tending to 0) with the usual sup-norm. For $E = \ell_p$ (or $E = c_0$), we will say that a sequence of functions $f_j : \mathbb{R} \to \mathbb{R}$ satisfies condition $(*)_E$ if $(f_j(0)) \in \ell_p$ (or $(f_j(0)) \in c_0$), and the f_j have locally uniformly bounded derivatives, i.e., for each $k \in \mathbb{N}$, $r > 0$, there exists a real constant $M_{k,r}$ such that

$$|f_j^{(k)}(x)| \leq M_{k,r}$$

whenever $|x| < r$.

Lemma 4.1. *Fix $1 \leq p < \infty$ and consider a sequence of C^∞ functions $f_j \colon \mathbb{R} \to \mathbb{R}$ satisfying condition $(*)_{\ell_p}$ (or $(*)_{c_0}$). Then the map $f \colon \ell_p \to \ell_p$ (or $f \colon c_0 \to c_0$) given by $f(x) = (f_j(x_j))$ is C^∞ Fréchet differentiable.*

Proof. From condition $(*)_{\ell_p}$ and the mean-value theorem, it follows that

$$\|f(x) - f(y)\|_p \leq M_{1,\|x\|_p + \|y\|_p},$$

i.e., f is a well-defined, locally Lipschitz endomorphism of ℓ_p. For each $k \in \mathbb{N}$, define $f^{\{k\}} \colon \ell_p \to \ell_\infty$ by

$$f^{\{k\}}(x) = (f_j^{\{k\}}(x_j)).$$

A simple computation shows that each $f^{\{k\}}$ is differentiable, and)

$$Df^{\{k\}}(x)(y) = (f_j^{\{k\}}(x)(y) = (f_j^{\{k\}}(x_j)y_j).$$

The linear map $T \colon \ell_\infty \to L(\ell_p, \ell_p)$ given by $T(x)(y) = (x_i y_i)$ satisfies

$$\|T(x)(y)\|_p = \left(\sum_j |x_j y_j|^p\right)^{1/p} \leq \|x\|_\infty \|y\|_p$$

and is therefore continuous. By the preceding remarks, we have

$$Df^{\{k\}} = T \circ f^{\{k+1\}}.$$

¿From this equation, a simple inductive argument shows that each $f^{\{k\}}$ is C^∞ Fréchet differentiable. Specifically, each $f^{\{k\}}$ is continuous. Assuming that each $f^{\{k\}}$ is C^n, the preceding equation implies that $Df^{\{k\}}$ equals the composition of $f^{\{k+1\}}$ with the bounded linear map T, and so $f^{\{k\}}$ is C^{k+1}.

$\square$

By introducing points of infinite singularity, we can improve the target space of f as follows.

Lemma 4.2. *Fix $1 \leq p < \infty$ and consider a sequence of C^∞ functions $f_j : \mathbb{R} \to \mathbb{R}$ with locally uniformly bounded derivatives and $f_j^{\{k\}}(0) = 0$ for all $k \in \mathbb{N}, j \in \mathbb{Z}_+$. Then the map $f \colon \ell_p \to \ell_1$ given by $f(x) = (f_j(x_j))$ is C^∞ Fréchet differentiable.*

Proof. For each $j \in \mathbb{Z}_+$, $k \in \mathbb{N}$, our hypotheses combined with Taylor's theorem imply that

$$|f_j(x_j) - f_j(0)| \leq M_{[p]+1,|x_j|}|x_j|^p,$$

and so

$$\|f(x)\|_1 = \|f(x) - f(0)\|_1 \leq M_{[p]+1,\|x\|_p}\|x\|_p^p$$

for all $x \in \ell_p$. Thus, f is a well-defined map from ℓ_p into ℓ_1. A similar argument shows that for each $k \in \mathbb{N}$, then map $f^{\{k\}}$ defined in Lemma 4.1 is also a well-defined map from ℓ_p into ℓ_1. Moreover, a simple computation again shows that

$$Df^{\{k\}}(x)(y) = (f_j^{\{k+1\}}(x_j)y_j).$$

Evidently the linear map $T \colon \ell_1 \to L(\ell_p, \ell_1)$ given by $T(x)(y)$ satisfies

$$\|T(x)(y)\|_1 \leq \|x\|_1\|y\|_p$$

and is therefore continuous. As in Lemma 4.1, we have $Df^{\{k\}} = T \circ f^{\{k+1\}}$, and the assertion follows by the same argument as before.

$\square$

Example 4.3. *Consider the polynomial map $f \colon \ell_1 \to \ell_1$ given by the component functions*

$$f_j(x) = 3 \cdot 2^{-j\alpha}x^2 - 2x^3.$$

A check of the definitions shows that the critical set of f contains the Cantor set

$$\mathcal{C} = \{(\beta_j) : \beta_j \in \{0, 2^{-j\alpha}\}\}$$

of rank-0 points for f, and

$$f(\mathcal{C}) = \{(\beta_j) : \beta_j \in \{0, 2^{-3j\alpha}\}\}.$$

The Hausdorff dimensions of $\mathcal{C}$ and $f(\mathcal{C})$ are α^{-1} and $(3\alpha)^{-1}$, respectively, and both have positive, finite measure in these dimensions.

Denoting by $s \colon \ell_1 \to \mathbb{R}$ the continuous linear map

$$s(x) = \sum_j x_j,$$

we note that $\mathcal{C}$ is also contained in the critical set of the composition $g = s \circ f$. Thus, the critical value set of g contains

$$g(\mathcal{C}) = \left\{ \sum_j \beta_j : \beta_j \in \{0, 2^{-3j\alpha}\} \right\},$$

which has positive, finite Hausdorff measure in dimension $(3\alpha)^{-1}$. *Finally, we note that in the special case of* $\alpha = 1/3$, *the set* $g(\mathcal{C})$ *equals the unit interval* $[0,1]$.

References

[1] Bates, S.M., Toward a precise smoothness bypothesis in Sard's theorem, *Proc. Amer. Math. Soc.* **117** (1993) 279–283.
[2] Federer, H., *Geometric Measure Theory*, Grundlehren Math. Wiss., vol. 153, Springer, New York, 1969.
[3] Moreira, C.G., Hausdorff measures and the Morse-Sard theorem, preprint, IMPA, 1994.
[4] Morse, A., The behavior of a function on its critical set, *Ann. Math.* **40** (1939) 62–70.
[5] Norton, A., The Zygmund Morse-Sard theorem, *J. Geom. Analysis* **4** (1994) 403–424.
[6] Sard, A., Images of critical sets, *Ann. Math.* **68** (1958) 247–259.
[7] ——— Hausdorff measure of critical images on Banach manifolds, *Amer. J. Math* *************
[8] Stein, E., *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, NJ, 1970.
[9] Yomdin, Y., The geometry of critical and near-critical values of differentiable mappings, *Math. Ann.* **264** (1983) 495–515.

Department of Mathematics, Columbia University, New York, NY, 10027, USA

IMPA, Estrada Dona Castorina 110, 22460-320, Rio de Janeiro, RJ, Brasil
E-mail address: smb@math.columbia.edu
E-mail address: gugu@impa.br