STATISTICAL PROPERTIES OF UNIMODAL MAPS: SMOOTH FAMILIES WITH NEGATIVE SCHWARZIAN DERIVATIVE

ARTUR AVILA AND CARLOS GUSTAVO MOREIRA

Abstract. We prove that there is a residual set of families of smooth or analytic unimodal maps with quadratic critical point and negative Schwarzian derivative such that almost every parameter is either regular or Collet-Eckmann with subexponential recurrence of the critical orbit, both cases allowing a detailed and robust statistical description of the dynamics. This proves a version of Palis conjecture in this setting.

1. Introduction

'The main strategy of the study of all mathematical models is, according to Poincaré, the consideration of each model as a point of a space of different but similar admissible systems'(V. Arnold in [Ar]). One of the main concerns of dynamical systems is to stablish properties valid for typical systems. Since the space of such systems is usually infinite dimension, there are of course many concepts of 'typical'. According to [Ar] again, 'The most physical genericity notion is defined by Kolmogorov (1954), who suggested to call a property of dynamical systems exceptional, if it holds only on Lebesgue measure zero set of values of the parameters in every (topologically) generic family of systems, depending on sufficiently many parameters'.

In the last decade Palis [Pa] described a general program for (dissipative) dynamical systems in any dimension. He conjectured that a typical dynamical system has a finite number of atractors described by physical measures, the union of their basins has full Lebesgue measure, and those physical measures are stochastically stable. Typical was to be interpreted in the Kolmogorov sense: full measure in generic families. Our aim here is to give a proof of this conjecture for an important class of one-dimensional dynamical systems.

Here we consider unimodal maps, that is, continuous maps from an interval to itself which have a unique turning point. More specifically,

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we consider quasiquadratic unimodal maps, that is, we assume that the map is C^3 with negative Schwarzian derivative and that the critical point is non degenerate.

1.1. **The quadratic family.** The basic model for unimodal maps is the quadratic family, $p_a(x) = a - x^2$, where $-1/4 \leq a \leq 2$ is a parameter. Despite its simple appearance, the dynamics of those maps presents many remarkable phenomena. Restricting to the probabilistic point of view, its richness first became apparent with the work of Jakobson [J], where it was shown that for a positive measure of parameters the behavior is stochastic, more precisely, there is an absolutely continuous invariant measure (the physical measure) with positive Lyapunov exponent. On the other hand, it was later shown by Lyubich [L1] and Graczyk-Swiatek [GS] that regular parameters (with a periodic hyperbolic attractor) are (open and) dense. So at least two kinds of very distinct observable behavior are present on the quadratic family, and they alternate in a complicate way.

Besides regular and stochastic behavior, different behavior was shown to exist, including examples with bad statistics, like absence of a physical measure or a physical measure concentrated on a hyperbolic repeller. Those pathologies were shown to be non-observable in [L2] and [MN]. Finally in [L3] it was proved that almost every real quadratic map is either regular or stochastic.

Among stochastic maps, a specific class grabbed lots of attention in the 90's: Collet-Eckmann maps. They are characterized by a positive Lyapunov exponent for the critical value, and gradually they were shown to have 'best possible' near hyperbolic properties: exponential decay of correlations, validity of central limit and large deviations theorems, good spectral properties and zeta functions ([KN], [Y]). Let's call attention to the robustness of the statistical description, with a good understanding of stochastic perturbations: strong stochastic stability ([BV]), rates of convergence to equilibrium ([BBM]).

In [AM1] the regular or stochastic dichotomy was extended to show that almost every stochastic map is actually Collet-Eckmann and satisfy many strong assymptotics, in particular implying the validity of the above mentioned results.

The position of the quadratic family in the borderline of real and complex dynamics made its study a convergence point of many different techniques: most of the deeper results depend on this interaction. It gradually became clear however that studying the quadratic family allows to obtain results on more general unimodal maps.

1.2. **Universality.** Starting with the works of Milnor-Thurston, and also through the discoveries of Feigenbaum and Coullet-Tresser, the quadratic family was shown to be not only a prototype of unimodal maps, but to present universal features both combinatorially and geometrically. More recently, the result of density of hyperbolicity among unimodal maps was obtained in [K] exploiting the validity of this result for quadratic maps.

In [ALM], a general method was devised to transfer information from the quadratic family to real analytic families of unimodal maps. It is shown that the decomposition of spaces of analytic unimodal maps according to combinatorial behavior is essentially a codimension-one lamination.

Thinking of two analytic families as transversals to this lamination, one may try to compare the parameter space of both families via the holonomy map. A straightforward application of this method allowed to conclude that the bifurcation pattern of a general analytic family is locally the same as the quadratic family from the topological point of view (outside of countably many 'bad parameters').

The 'holonomy' method was then succesfully applied to extend the regular or stochastic dichotomy from the quadratic family to a general analytic family. The probabilistic point of view presents new difficulties however. First, the statistical properties of two topologically conjugate maps need not correspond by the (generally not absolutely continuous) conjugacy. Fortunately many properties are preserved, in particular the criterias used by Lyubich in his result.

The second difficulty is that the holonomy map is usually not absolutely continuous, so typical combinatorics for the quadratic family may not be typical for other families: it is needed to show that the class of regular or stochastic maps is still typical after application of the holonomy map.

1.3. **Results and plan of the proof.** Let's call a k-parameter family good if almost every parameter is either regular or Collet-Eckmann (and some additional technical conditions). Our goal will be to prove that good families are generic. This question naturally makes sense in different spaces of unimodal maps (corresponding to different smoothness). We only deal with the last steps of this problem (going from the quadratic family to analytic and then smooth categories), basing ourselves on the building blocks [L2], [L3], [ALM] and [AM1].

We start by describing how the holonomy method of [ALM] can be applyed to generalize the results of [AM1] to general analytic families (to put together those two papers we need to do a non-trivial strengthning of [AM1]). As a consequence we conclude that essentially all analytic families are good.

To get to the smooth setting (at least C^3, since we are assuming negative Schwarzian derivative), our strategy is different: we show a certain robustness of good families, which together with their denseness (from the analytic case) will yield genericity. Our main tool is one of the nice properties of Collet-Eckmann maps: persistence of the Collet-Eckmann condition under generic unfolding (a result of [T1]). Using an abstract scheme we reduce the global result to this local one.

Let's mention that we have obtained the same results without the negative Schwarzian assumption, also allowing us to get to C^2 smoothness in [A]. The techniques are very different however, since we do not use anymore the holonomy method we use here (of global nature), which is replaced by a local holonomy analysis and use of a finite version of the 'infinitesimal perturbation' method of [ALM]. For analytic maps this also allowed us to obtain more refined assymptotic estimates which have interesting consequences [AM2] as pathological measure-theoretical behavior of the lamination.

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2. General definitions

2.1. **Notation.** Let $I = [-1, 1]$, B^k be the closed unit ball in $\mathbb{R}^k$. Let $C^r(I)$ denote the space of C^r maps $f : I \to \mathbb{R}$.

By a unimodal map we will mean a smooth symmetric (even) map $f : I \to I$ with a unique critical point at 0 such that $f(-1) = -1$, $Df(-1) \geq 1$. If f is C^3, we define the Schwarzian derivative as

$$Sf = \frac{D^3 f}{Df} - \frac{3}{2}\left(\frac{D^2 f}{Df}\right)^2$$

defined on $I \setminus \{0\}$.

For $a > 0$, let $\Omega_a \subset \mathbb{C}$ denote an a neighborhood I.

Let $\mathcal{A}_a$ denote the space of holomorphic maps on Ω_a which have a continuous extension to $\partial\Omega_a$, satisfying $\phi(z) = \phi(-z)$, $\phi(-1) = \phi(1) = -1$ and $\phi'(0) = 0$.

Notice that $\mathcal{A}_a$ is a closed affine subspace of the Banach space of bounded holomorphic maps of Ω_a. We endow it with the induced metric and affine structure.

We define $\mathcal{A}_a^{\mathbb{R}} \subset \mathcal{A}_a$ the space of maps which are real symmetric.

2.2. **More on unimodal maps.** A C^3 unimodal map such that $Sf < 0$ on $I \setminus \{0\}$ and such that its critical point is non-degenerate (that is, $D^2 f \neq 0$) will be called a S-unimodal map.

A unimodal map is called **regular** (or hyperbolic) if it has a hyperbolic periodic attractor.

A k-parameter family of unimodal maps is a map $\Gamma : B^k \times I \to I$ such that for $p \in B^k$, $\gamma_p(x) = \Gamma(p, x)$ is a unimodal map. Such a family is said to be C^n or analytic, according to Γ being C^n or analytic. We introduce the natural topology in spaces of smooth families (C^n with $n = 2, ..., \infty$), but do not introduce any topology in the space of analytic families.

An analytic family Γ will be called **non-trivial** if there exists a regular parameter.

A unimodal map f is called **Collet-Eckmann** (CE) if there exists constants $C > 0$, $\lambda > 1$ such that for every $n > 0$,

$$|Df^n(f(0))| > C\lambda^n.$$

This means that the map is strongly hyperbolic along the critical orbit. It is also useful to study the hyperbolicity of backward iterates of the critical point, so we say that f is **Backwards Collet-Eckmann** (BCE) if there exists $C > 0$, $\lambda > 1$ such that for any $n > 0$ and any x with $f^n(x) = 0$, we have

$$|Df^n(x)| > C\lambda^n.$$

By a result of Nowicki (see [MvS]), for S-unimodal maps CE implies BCE, so we will mostly discuss the Collet-Eckmann condition (except for the last section where we consider C^2 unimodal maps as well).

Very often it is useful to estimate how fast is the recurrence of the critical orbit. We will be mainly interested in two kinds of control: **Polynomial Recurrence** (P) if there exists $\alpha > 0$ such that

$$|f^n(0)| > n^{-\alpha}$$

for big enough n and **Subexponential Recurrence** (SE) if for all $\alpha > 0$,

$$|f^n(0)| > e^{-\alpha n}$$

for n big enough.

We will say that f is **Weakly Regular** (WR) if

$$\lim_{\delta \to 0} \liminf_{n \to \infty} \frac{1}{n} \sum_{\substack{1 \leq k \leq n \\ f^k(0) \in (-\delta, \delta)}} \ln(|Df(f^k(0))|) = 0.$$

This condition is related to stochastic stability for C^2 maps, see [T2].

We will consider spaces of S-unimodal maps: we define $\mathcal{U}^r \subset C^r(I)$ the set of S-unimodal maps. Spaces of analytic unimodal maps are now easily defined: $\mathcal{U}_a = \mathcal{U}^3 \cap \mathcal{A}_a^{\mathbb{R}}$.

2.3. **The quadratic family.** The quadratic family is the most studied family of unimodal maps. It is usually defined by

$$q_t(x) = t - x^2,$$

so that for $-1/4 \leq t \leq 2$, there exists a unique symmetric interval $I_t = [-\beta_t, \beta_t]$ such that $p_t(I_t) = I_t$ and $p_t(-\beta_t) = -\beta_t$, so p_t can be seen as a unimodal map of I_t (which depends on t). It is also not difficult to see that $Sp_t(x) < 0$ if $x \neq 0$.

By an affine reparametrization of the parameter t and of each interval I_t, we obtain a canonical one-parameter family of S-unimodal maps in the interval I, which we denote p_t, $t \in B^1$, which will be called the quadratic family as well.

2.4. **Quasisymmetric maps.** Let $\gamma \geq 1$ be given. We say that a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is **quasisymmetric** (qs) if there exists a constant $k > 1$ such that for all $x \in \mathbb{R}$ and any $h > 0$

$$\frac{1}{k} \leq \frac{f(x+h) - f(x)}{f(x) - f(x-h)} \leq k.$$

A homeomorphism h is quasisymmetric if and only if it admits a real-symmetric extension to a quasiconformal map $\tilde{h} : \mathbb{C} \to \mathbb{C}$ (Ahlfors-Beurling). We will say that h is γ-qs (or that γ is a qs constant for h) if the dilatation of $\tilde{h}$ is bounded by γ. This definition of the quasisymmetric constant is convenient since the composition of quasisymmetric maps g and f is readily seen to be quasisymmetric and the qs constant of $g \circ f$ is bounded by the product of the qs constants of g and f.

If $X \subset \mathbb{R}$ and $h : X \to \mathbb{R}$ has a γ-quasisymmetric extension to $\mathbb{R}$ we will also say that h is γ-qs.

3. Statement of the results

We would like to classify the typical behavior in generic families of unimodal maps. This classification should reveal refined information on the stochastic description of the dynamics of those typical parameter.

We will therefore consider a smooth enough family of unimodal maps Γ. The techniques of the present paper will need the fact that Γ is a family of S-unimodal maps. This includes two main restrictions: the negative Schwarzian and the quadratic critical point. The first one is serious, since this condition is not dense, but can be removed with more refined techniques (see [A]). The second one is no serious loss of generality, since quadratic critical point is certainly typical among unimodal maps.

We first consider the analytic case. In this case, the generic families we will consider will be families satisfying a non-degeneracy condition, that is, we assume they contain a regular map (non-trivial families). This condition is easy to verify, both in practice (if the family is non-degenerate this can be verified in finite time) as in theory (they are indeed generic for meaningful concepts, see Proposition 4.4). Our main result is a dichotomy on the typical dynamical behavior in any such family.

Theorem A. *Let Γ be a non-trivial k-parameter family of S-unimodal maps. Then for almost every non-regular parameter $p \in B^k$, f_p satisfy the Collet-Eckmann and Polynomial Recurrence conditions.*

Our second result about non-trivial families is the robustness of a slightly weaker dichotomy under C^2 perturbations of the family.

Theorem B. *Let Γ be a non-trivial k-parameter family of S-unimodal maps. Let Γ^n be a sequence of C^2 families such that $\Gamma^n \to \Gamma$ in the C^2 topology. For each n, let X_n be the set of parameters $p \in B^k$ where Γ^n is either regular or satisfy simoultaneously the Backwards Collet-Eckmann, Collet-Eckmann, Subexponential Recurrence and Weak Regularity conditions. Then $|X_n| \to 1$. In particular, almost every parameter of Γ is Weakly Regular.*

As a consequence, we can use a Baire argument to conclude that a dichotomy is still valid among topologically generic (residual set) smooth families, so the following is just an easy Corollary of Theorems A and B.

Theorem C (Smooth Dichotomy). *In topologically generic k-parameter C^r, $r = 3, 4, ..., \infty$ families of S-unimodal maps, almost every non-regular parameter satisfy the Backwards Collet-Eckmann, Collet-Eckmann, Subexponential Recurrence and Weak Regularity conditions.*

It is good to recall that both types of behavior stablished on the dichothomy are indeed observable for open sets of families of unimodal maps ([J], [BC]).

3.1. **Ergodic meaning.** The importance of the above dichotomy is the fact that each of the two possibilities has very well defined stochastic properties. We quickly recall those (we assume that maps are S-unimodal).

Regular maps have a periodic attractor whose basin is big both topologically (open and dense set) as in the measure-theoretical sense (full measure). Moreover the attractor and its basin are stable under C^1 perturbations. The dynamics of such maps can be described in deterministic terms.

Maps satisfying CE and SE have non-deterministic dynamics. They can however describe them through their stochastic properties, and it turns out that such maps have the main good properties usually found in hyperbolic maps. First, there is a physical measure, that is an invariant probability which describes assymptotic behavior of orbits: for almost every x and for every continuous $\phi : I \to \mathbb{R}$,

$$\lim \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k(x)) = \int \phi d\mu.$$

This physical measure has a positive Lyapunov exponent and is indeed absolutely continuous and supported on a cycle of intervals, so the assymptotic behavior is non-deterministic. The convergence to the assymptotic stochastic model is exponential, see the results on decay of correlations ([KN], [Y]) and convergence to equilibrium. Those properties are beautifully related to a spectral gap of a transfer operator and to zeta functions, see [KN]. Notice finally that exponential decay of correlations is actually equivalent to the Collet-Eckmann condition (see [NS]).

While the dynamics is highly unstable under deterministic perturbations (nearby maps can be regular for instance), the stochastic description given by the physical measure μ is robust under stochastic perturbations: the perturbed system has a stationary measure which is close to μ in the sense of the L^1 distance between their densities ([BV]). For studies of decay of correlations for the perturbed systems, see [BBM].

4. Analytic families

4.1. **Hybrid conjugacy and holonomy maps.** A hybrid conjugacy between two S-unimodal maps f and $\tilde{f}$ is a topologic conjugacy which, if there exists an attracting periodic orbit, is locally smooth in the basin of attraction of this periodic orbit.

Two S-unimodal maps $f, \tilde{f}$ are said to be hybrid equivalent if there exists a hybrid conjugacy between them. The set of all maps which are hybrid equivalent to some f is called hybrid class of f.

It follows from a result of Guckenheimer (see [MvS]) that any S-unimodal map f is hybrid conjugate to some quadratic map. The problem of uniqueness is much harder. The following result is due to

Lyubich [L1] and Graczyk-Swiatek [GS], and is a consequence of (the proof of) the equivalent rigidity result for quadratic maps:

Theorem 4.1. *Let h be a hybrid conjugacy between two analytic S-unimodal maps f and $\tilde{f}$. Then h is quasisymmetric.*

This rigidity result (in the case of the quadratic family) implies that any S-unimodal map f is hybrid conjugate to a unique quadratic map $\chi(f)$. The map χ is called the straightening.

Lemma 4.2. *If f is an analytic S-unimodal map and $\chi(f)$ is either regular or CE and P then f is too.*

Proof. Since regularity is clearly preserved by hybrid conjugacy, we only have to analyze invariance of the conditions CE and P. By Theorem 4.1, we just have to prove invariance under quasisymmetric conjugacy.

By [NP] the Collet-Eckmann condition is invariant under quasisymmetric conjugacy (and even under topological conjugacy). Since quasisymmetric maps are Hölder, it is easy to see that P is also invariant. ☐

4.2. **Hybrid laminations.** It is natural to study the hybrid class of some map f. This is what is done in Theorem A of [ALM] in the analytic setting, where it is shown that in $\mathcal{U}_a$, every hybrid class is a codimension-one analytic submanifold. Moreover, different hybrid class fit together in some nice structure, called hybrid lamination.

If Γ is a one-parameter family of S-unimodal maps, it can be naturally thought of as an analytic curve in some $\mathcal{U}_a$.

A consequence of the nice structure of the hybrid lamination is the following

Lemma 4.3 (see Theorem C of [ALM])**.** *If Γ is not contained in a hybrid class then there is an open set of parameters, with countable complement, where Γ is transverse to the hybrid lamination.*

Lemma 4.4 (Non-trivial families are strongly generic among analytic families)**.** *An analytic family of S-unimodal maps which fails to be non-trivial must be contained in some non-regular hybrid class. In particular, non-trivial families are dense in the space of C^n families of S-unimodal maps, $n = 3, ..., \infty$.*

Proof. Indeed, if an analytic family is not contained in some non-regular hybrid class, then its image by the straightening has to be a non-trivial parameter interval of quadratic maps. Since regular maps are dense among quadratic maps, the first statement follows. For the second statement, given a C^r family, approximate it by an analytic family. If such an analytic family is contained in a hybrid class, perturb it further in order to intersect two hybrid classes. $\square$

Definine the map χ_Γ on B^1 defined by $\chi_\Gamma(t) = \chi(\gamma_t)$. In [ALM] the map χ_Γ is considered as the holonomy map from Γ to the quadratic family along the hybrid lamination in some $\mathcal{U}_a$. Using this interpretation, they obtain

Theorem 4.5. *Let Γ be a non-trivial one-parameter family of unimodal maps. If Γ contains a non-regular map, then there is an open set $U \subset B^1$ with countable complement such that the straightening χ_Γ is quasisymmetric in any compact interval $J \subset U$.*

4.3. **Dichotomy in the quadratic family.** The main result of [AM1] is that almost every parameter in the quadratic family is either regular or Collet-Eckmann with a polynomial recurrence of the critical orbit. To obtain the same result for a non-trivial family using Theorem 4.5, we will need a stronger estimate, since quasisymmetric maps are not in general absolutely continuous.

Let's say that a set $X \subset B^1$ has total qs-probability if the image of $B^1 \setminus X$ by any quasisymmetric map $h : B^1 \to B^1$ has 0 Lebesgue measure.

By a slight modification of the proofs in [AM1] (see appendix), it is possible to obtain

Theorem 4.6. *The set of quadratic maps which are either regular or CE and P has total qs-probability.*

Remark 4.1. In [AM1] it is obtained a better result than polynomial recurrence, namely it is shown that the assymptotyc exponent of the recurrence
$$\limsup_{n\to\infty} \frac{-\ln(f^n(0))}{\ln(n)}$$
is exactly 1 for almost every non regular map. However, for a set of total qs-probability, we are only able to show that the assymptotic exponent is bounded.

4.4. **Proof of Theorem A.** Let Γ be a non-trivial family. If all parameters are regular, there is nothing to prove, so assume that there is a non-regular parameter.

First assume Γ is one-parameter. By Theorems 4.6 and 4.5, for almost every $t \in B^1$, $\chi_\Gamma(t)$ is either regular or satisfies CE and P. By Lemma 4.2, this implies that γ_t is either regular or CE and P.

Assume now that Γ is a k-parameter family. Let $p \in B^k$ be a regular parameter. Let $L : B^1 \to B^k$ be an affine map such $p \in L(B^1)$. Let Γ^L be the one-parameter family defined by $\gamma_t^L = \gamma_{L(t)}$. Then Γ^L is a non-trivial one-parameter family and for almost every t, γ_t^L is either regular or CE and P. The result follows by application of Fubini's Theorem.

5. Robustness of the dichotomy

To obtain the robustness claimed on Theorem B our approach will be to exploit an important result of Tsujii, whose core is a strong generalization of Benedicks-Carleson result and techniques. This result establishes that the CE and SE conditions are infinitesimally persistent in one-parameter families unfolding generically: they are density points of CE and SE parameters. The connection with our robustness result, which has a global nature, is done using an abstract (and technical) scheme.

5.1. **Tsujii's theorem.** Let Γ be a C^2 k-parameter family of unimodal maps. Assume that p_0 is a parameter such that γ_{p_0} satisfies CE, BCE, SE and has a quadratic critical point. Tsujii's Theorem considers the case where Γ is a generic unfolding at p_0. For one-parameter families, generic unfolding means precisely

$$\sum_{j=0}^{\infty} \frac{(\partial_p \gamma_p)(\gamma_p^j(0))}{D\gamma_p^j(\gamma_p(0))} \neq 0,$$

at $p = p_0$, where ∂_p stands for the parameter derivative. This transversality condition will be called Tsujii transversality.

If Γ is a one-parameter family, we will say that (Γ, p_0) satisfies the **Tsujii conditions** if all above requirements are satisfied.

The following is an immediate consequence of the main theorem of Tsujii in [T1].

Theorem 5.1. *Let Γ be a C^2 one-parameter family of unimodal maps. Assume (Γ, t_0) satisfies the Tsujii conditions. Then t_0 is a density point of parameters t for which (Γ, t) satisfies the Tsujii conditions and for which γ_t is WR.*

5.2. **A higher dimensional version.** In order to pass from one-parameter to k-parameters, we will need the following easy proposition. Let us say that $p \in B^k$ is a density point of a set X along a line l through p if p is a density point of $l \cap X$ in l (endowed with the linear Lebesgue measure).

Proposition 5.2. *If $p \in B^k$ is a density point of X along almost every line, then p is a density point of X in B^k.*

Proof. Let E be the characteristic function of X. For each line l through p, let $A_l : \mathbb{R} \to l$ be an isometric parametrization of l taking 0 into p. Let P^{k-1} be the space of such lines with the natural probability measure (obtained by identification with the $k-1$ dimensional projective space). Let

$$\rho_\epsilon(l) = \int_{-1}^{1} |r| E(A_l(\epsilon r)) dr.$$

Assuming that p is a density point of X along almost every l we have, for almost every l

$$\lim_{\epsilon \to 0} \rho_\epsilon(l) = 1.$$

Using polar coordinates, the relative measure of X in an ϵ ball around p is given by

$$\int_{P^{k-1}} \rho_\epsilon(l) dl.$$

By Fatou's Lemma

$$\liminf_{\epsilon \to 0} \int \rho_\epsilon(l) dl \geq \int \liminf_{\epsilon \to 0} \rho_\epsilon(l) dl = 1.$$

This shows that p is a density point of X. □

The k-parameter analog of the Tsujii transversality is the existence of a line along which the one-parameter transversality condition is satisfied. In other words, there exists a an affine map $L : B^1 \to B^k$ passing through p_0 such that the induced one-parameter family $\Gamma \circ L$ is Tsujii transverse at the parameter t_0 such that $L(t_0) = p_0$.

By linearity, if (Γ, p_0) is Tsujii transverse, then all lines passing through p_0 are Tsujii transverse except the lines parallel to a certain codimension-one space of $\mathbb{R}^k$.

Lemma 5.3. *Let Γ be a C^2 k-parameter family of unimodal maps. Assume (Γ, p_0) satisfies the Tsujii conditions. Then p_0 is a density point of parameters p for which (Γ, p) satisfies the Tsujii conditions and for which γ_p is WR.*

Proof. If Γ is Tsujii transverse at p_0 then it is Tsujii transverse along almost every line through p_0. Along such a line it is a density point of parameters satisfying the Tsujii conditions and which are WR. The result follows from proposition 5.2. □

5.2.1. *Tsujii transversality and hybrid lamination.* Let's take a closer look at the Tsujii transversality for an analytic Γ. Let $\gamma_p = f$.

Assuming

$$\sum_{k=0}^{\infty} \frac{1}{Df^k(f(0))} < \infty \tag{5.1}$$

(in particular if f is CE), let

$$\nu_f(v) = \sum_{k=0}^{\infty} \frac{v(f^k(0))}{Df^k(f(0))}$$

be a functional defined on continuous vectorfields v on the interval.

Lemma 5.4. *If f satisfies the summability condition then there exists an even polynomial vector field v, with $v(-1) = v(1) = 0$ and such that $\nu_f(v) \neq 0$.*

Proof. Let $S = \sum |Df^k(f(0))|^{-1}$. Let ϵ be so small that

$$\sum_{k>0, f^k(0)\in(-\epsilon,\epsilon)} \frac{1}{|Df^k(f(0))|} < 1/3.$$

Let v be an even polynomial vector field satisfying $v(-1) = v(1) = 0$, $|v(x)| < 2$, $x \in I$, $1 < v(x)$, $x \in (-\epsilon/2, \epsilon/2)$, $|v(x)| < 1/(10S)$, $x \in I \setminus (-\epsilon, \epsilon)$. Then $\nu_f(v) > 1 - 2/3 - 1/10 > 0$. $\square$

Lemma 5.5. *The kernel of ν_f intersected with $T\mathcal{A}_a^{\mathbb{R}}$ is the tangent space to the hybrid class of f.*

Proof. By the previous lemma, ν_f is non-trivial over $T\mathcal{A}_a^{\mathbb{R}}$, so the above intersection is a closed codimension-one subspace of $T\mathcal{A}_a^{\mathbb{R}}$. It is enough to show that if v is tangent then $\nu_f(v) = 0$. Assuming that v is tangent, consider an analytic family γ_t contained in the hybrid class of f, such that $\gamma_0 = f$ and

$$\left.\frac{d}{dt}\gamma_t\right|_{t=0} = v.$$

It is remarked in [ALM] that

$$\alpha_{n+1} = Df^n(f(0)) \sum_{k=0}^{n} \frac{v(f^k(0))}{Df^k(f(0))}$$

is precisely

$$\left.\frac{d}{dt}(\gamma_t)^{n+1}(0)\right|_{t=0}.$$

Moreover, $t \to (\gamma_t)^{n+1}(0)$ are holomorphic functions of the complex parameter t, taking values in Ω_a, and whose domain is some definite neighborhood of 0. It follows by Cauchy estimates on the derivative that this sequence is bounded independent of n. By the summability condition (5.1), $|Df^n(f(0))| \to \infty$, so we have necessarily $\nu_f(v) = 0$. $\square$

Remark 5.1. In [ALM] it is shown that the sequence α_n is not only bounded (for tangent vector fields v), but the vector field defined on the orbit of the critical value by $f^k(0) \to \alpha_k$, $k > 0$, extends to a quasiconformal vector field of $\mathbb{C}$.

So Tsujii transversality can be interpreted for such a map (satisfying the summability condition (5.1)) as transversality of the family to the hybrid class of γ_p.

Since for maps with negative Schwarzian derivative CE implies the BCE, we can conclude from Theorem A, Lemma 4.3 and this discussion the following

Lemma 5.6. *If Γ is a non-trivial k-parameter family then almost every parameter is regular or satisfies the Tsujii conditions.*

5.3. **Generalities about density points.**

5.3.1. *Density points in abstract.* Let $\mathcal{U}^2$ be the space of C^2 unimodal maps (without, naturally, the hypothesis of negative Schwarzian derivative), and K be the space of C^2 k-parameter families of unimodal maps.

Let $X \subset K \times B^k$ be the set of (Γ, p) such that either γ_p is regular or satisfies the Tsujii conditions and WR. For $\Gamma \in K$, let $X_\Gamma = \{p \in B^k | (\Gamma, p) \in X\}$.

Let $Y \subset B^k$ be measurable with $|Y| > 0$. We define the density of X along Γ on Y as

$$d(\Gamma, Y) = \frac{|Y \cap X_\Gamma|}{|Y|}.$$

Instead of defining the classical infinitesimal density:

$$\liminf_{\epsilon \to 0} d(\Gamma, B_\epsilon(p))$$

we will need to consider the stability of the density with respect to perturbations of Γ. With this in mind we introduce two parameters. Let

$$D^-(\Gamma^0, p) = \liminf_{\substack{\Gamma \to \Gamma^0 \\ \gamma_p = \gamma_p^0}} \liminf_{\epsilon \to 0} d(\Gamma, B_\epsilon(p)),$$

$$D^+(\Gamma^0, p) = \liminf_{\epsilon \to 0} \liminf_{\Gamma \to \Gamma^0} d(\Gamma, B_\epsilon(p)).$$

Lemma 5.6 and Tsujii's result gives a direct way to estimate D^-

Lemma 5.7. *Let Γ be a non-trivial family. Then for almost every $p \in B^k$, $D^-(\Gamma, p) = 1$.*

However, for computations of stability D^+ is more relevant. We proceed to discuss the effect of the interchange of limits in the definition of D^- and D^+.

5.3.2. *Approximation condition.* Let's say that (Γ^0, p_0) is well approximable if for any $\Gamma^n \to \Gamma^0$, $\epsilon_n \to 0$ and any neighborhood $\mathcal{V}$ of Γ^0, there exists sequences k_n and l_n such that for any sequence $m_n \geq l_n$ we can find a family $\tilde{\Gamma} \in \mathcal{V}$ such that

$$\lim_{n\to\infty} \frac{|\{p \in B_{\epsilon_{k_n}}(p_0) | \tilde{\gamma}_p \neq \gamma_p^{m_n}\}|}{|B_{\epsilon_{k_n}}|} = 0$$

and furthermore $\tilde{\gamma}_{p_0} = \gamma^0_{p_0}$ (this actually follows from the last property).

Lemma 5.8. *For any $\Gamma \in K$ and $p \in B^k$, (Γ, p) is well approximable.*

Proof. We use the following easy interpolation result, which is a consequence of existence of C^∞ partitions of unity (this interpolation obviously fails for analytic maps):

Let $U \subset B^k$ be open. For each compact $K \subset U$, for all $\epsilon > 0$ there exists a $\delta > 0$ such that if Γ' is δ-close to Γ then there exists Γ'' ϵ-close to Γ coinciding with Γ out of U and coinciding with Γ' in K.

We can now obtain $\tilde{\Gamma}$ in the following way. Consider k_n such that $\epsilon_{k_{n+1}}/\epsilon_{k_n} \to 0$, and let $U_n = \mathrm{int}(B_{\epsilon_{k_n}} \setminus B_{\epsilon_{k_{n+1}}})$, $K_n \subset U_n$ compact such that $|K_n|/|U_n| \to 1$.

If we now choose l_n growing fast enough, for $m_n \geq l_n$ then Γ^{m_n} is very close to Γ and the above interpolation result shows that we can interpolate Γ and Γ^{m_n} inside each U_n, obtaining a family $\tilde{\Gamma}$, ϵ-close to Γ, such that for each n, $\tilde{\Gamma}$ and Γ^{m_n} coincide in K_n. $\square$

Lemma 5.9. *In this setting,*

$$D^+(\Gamma^0, p_0) \geq D^-(\Gamma^0, p_0).$$

Proof. Take sequences Γ^n and ϵ_j satisfying

$$\lim_{j\to\infty} \liminf_{n\to\infty} d(\Gamma^n, B_{\epsilon_j}(p_0)) = D^+(\Gamma^0, p_0).$$

and fix a neighborhood $\mathcal{V}$ of Γ^0. Let k_n, l_n be the sequences from the approximation condition. Let $m_n > l_n$ be a sequence such that

$$\lim_{n\to\infty} d(\Gamma^{m_n}, B_{\epsilon_{k_n}}(p_0)) = D^+(\Gamma^0, p_0).$$

and let $\tilde{\Gamma}$ be as in the approximation condition. Then it is clear that

$$\lim_{n\to\infty} d(\tilde{\Gamma}, B_{\epsilon_{k_n}}(p_0)) = D^+(\Gamma^0, p_0).$$

Since we can get $\tilde{\Gamma}$ to approximate Γ^0 by shrinking $\mathcal{V}$, we get $D^+ \geq D^-$. $\square$

5.3.3. *Proof of Theorem B.* Let Γ be a non-trivial family. Then almost every parameter satisfy $D^-(\Gamma, p) = 1$. In particular, for almost every p we have $D^+(\Gamma, p) = 1$.

Fix $\epsilon > 0$. Let $p \in B^k$ be such that $D^+(\Gamma, p) = 1$. By definition of D^+ there exists a sequence of balls $U^n(p)$ centered on p and converging to p, and neighborhoods $\mathcal{V}^n(p) \subset K$ of Γ such that if $\tilde{\Gamma} \in \mathcal{V}^n(p)$ then

$$d(\tilde{\Gamma}, U^n(p)) > 1 - \epsilon/2.$$

By Vitali's Lemma, there exists sequences p_j, n_j such that $U^{n_j}(p_j)$ are disjoint and $|\cup U^{n_j}(p_j)| = 1$. Let m such that $\cup_{j=1}^m U^{n_j}(p_j) > 1-\epsilon/2$. Let $\mathcal{V} = \cap_{j=1}^m \mathcal{V}^{n_j}(p_j)$. Then if $\Gamma \in \mathcal{V}$, $d(\tilde{\Gamma}, B^k) > 1 - \epsilon$.

The last statement is obvious.

5.3.4. *Proof of Theorem C (Smooth Dichotomy).* By Proposition 4.4 non-trivial families are dense among C^n families of S-unimodal maps, $n = 3, ..., \infty$. Theorem B implies that for all ϵ the set D_ϵ of C^n families of S-unimodal maps for which the set of bad parameters (not regular or BCE, CE, SE and WR) has measure less then ϵ, contains a neighborhood of all non-trivial families, that is, an open and dense set. Therefore $\cap D_{1/2^n}$ is a residual set. Clearly any family in $\cap D_{1/2^n}$ satisfies the stated dichotomy.

Appendix A. Quasisymmetric robustness of Collet-Eckmann and polynomial recurrence

The aim of this Appendix is to sketch a proof of Theorem 4.6. This proof is similar in strategy to the one of the main results of [AM1], however non-trivial modifications are needed. To avoid too much intersection, this will be a concise exposition concentrated mainly on the new steps needed for this improvement: *the concepts and notations of* [AM1] *will be considered background for this Appendix*, the reader can find a self-contained proof of this result in [AM3].

A.1. **Quasisymmetric maps.**

A.1.1. *Quasisymmetric reparametrization.* Let now H be an arbitrary but fixed k-quasisymmetric map from B^1 to the parameter space of the quadratic family. To prove Theorem 4.6, it will be enough to show that almost every $x \in B^1$ correspond under H to a parameter which is either regular or satisfies the Collet-Eckmann and Polynomial Recurrence conditions.

From now on, all mentions to parameter space will (unless explicitly stated otherwise) refer to the above reparametrization.

A.1.2. *Quasisymmetric capacities.* The γ-capacity of a subset X of an interval I is defined as follows:

$$p_\gamma(X|I) = \sup \frac{|h(X)|}{|h(I)|}$$

where the supremum is taken over all γ-qs maps $h : \mathbb{R} \to \mathbb{R}$.

Notice that if I^j are disjoint subintervals of I and $X \subset \cup I_j$ then

$$p_\gamma(X|I) \leq p_\gamma(\cup I_j|I) \sup p_\gamma(X \cap I^j|I^j).$$

A.2. **Sequence of first return maps.** The main objects of the statistical analysis of [AM1] concern the following objects: we are given a unimodal map (which we will assume finitely renormalizable and with a recurrent critical point) $f : I \to I$ and a sequence of nested intervals $I_n \subset I$. The inductive relation between the I_n is as follows: the first return map R_n to I_n consists of countably many intervals $\{I_n^j\}_{j \in \mathbb{Z}}$, with the convention that $0 \in I_n^0$ (the central component), and we let $I_n^0 = I_{n+1}$.

The special sequence of intervals I_n that we consider is called the principal nest, see [L1]. Since we assume f to be finitely renormalizable, there exists a smallest interval $T \subset I$ which is periodic (of period m). For the pricipal nest, $I_1 = [-p, p]$, where p is the orientation reversing fixed point of $f^m : T \to T$. A level of the principal nest is called central if $R_n(0) \in I_{n+1}$.

Each non-central branch of R_n is a diffeomorphism onto I_n. Let us introduce some convenient notations related to the iteration of the non-central branches of R_n. Let Ω be the set of finite sequences of non-zero integers, an element of Ω is denoted $\underline{d} = (j_1, ..., j_m)$. If $\underline{d} \in \Omega$ has length $|\underline{d}| = m$, we denote $R_n^{\underline{d}}$ the branch of $R_n^{|\underline{d}|}$ with combinatorics $\underline{d}$, that is, the domain of $R_n^{\underline{d}}$ is the set $I_n^{\underline{d}} = \{x \in I | R_n^{k-1}(x) \in I_n^{j_k}\}$. We let $C_n^{\underline{d}} = (R_n^{\underline{d}})^{-1}(I_{n+1})$.

Let us denote by L_n the first landing map from I_n to I_{n+1}. This map relates easily to R_n using the above description: the domain of L_n is $\cup C_n^{\underline{d}}$, and $L_n | C_n^{\underline{d}} = R_n^{\underline{d}}$. The reader should think of L_n as a high iterate of R_n. This leads to the following inductive relation between return maps $R_{n+1} = L_n \circ R_n$.

The return time of a point x belongint to an interval I_n^j is denoted by $r_n(x)$ (or $r_n(j)$, since it does not depend on $x \in I_n^j$), that is, $R_n | I_n^j = f^{r_n(j)}$. The landing time is denoted by l_n. The combinatorics at level n of a point x is denoted $\underline{d}_n(x)$, so that $x \in C_n^{\underline{d}_n(x)}(x)$. We let τ_n be the unique integer such that $R_n(0) \in I_n^{\tau_n}$. The return time of the critical point is denoted $v_n = r_n(0)$. Let us say that f is a **simple map** if its principal nest has at most finitely many central levels.

Notice that $I_{n+1} = R_{n-1}^{-1}(C_{n-1}^{\underline{d}})$ for some $\underline{d}$. The interval $\tilde{I}_{n+1} = R_{n-1}^{-1}(I_{n-1}^{\underline{d}}) \subset I_n$ is a big neighborhood of I_{n+1} which will be useful later. This choice of neighborhood is particularly good for simple maps, and it turns out that in this case $\tilde{I}_{n+1}$ is still much smaller then I_n for big n.

A.2.1. *Phase-parameter relation.* The starting point of [AM1] are two theorems of Lyubich describing the (non-reparametrized) parameter space of the quadratic family: infinitely renormalizable maps have zero Lebesgue measure ([L3]) and almost every finitely renormalizable non-regular map is simple [L2]. We will need the following improvement of [ALM]: Lyubich's proof actually allow to conclude that the set of ragular or simple maps has full measure after quasisymmetric reparametrization.

In view of those results, Theorem 4.6 is reduced to proving that the set of parameters which are Collet-Eckmann and polynomially recurrent have full measure among simple maps. From now on we exclude non-simple maps from measure-theoretical considerations, and we will use "with total probability" to refer to a set of parameters with full measure among simple maps.

To estimate the probability in the parameter corresponding to a certain behavior of the n-th stage of the principal nest, we make use of the Phase-Parameter Lemmas of [AM1]. They describe how the partition of the phase space induced by return and landing maps R_n and L_n induce parameter partitions of certain parameter windows J_n, which correspond topologically to the phase partition.

More precisely, we have:

Theorem A.1. *To each non-renormalizable map quadratic map f with a recurrent critical point, there exists a sequence of parameter intervals $\{J_n\}$ such that*

1. *J_n is the maximal interval containing f such that for all $g \in J_n$, there exists a continuation $I_{n+1}[g]$ of I_{n+1} such that $g|I \setminus I_{n+1}[g]$ is topologically conjugate to $f|I \setminus I_{n+1}[f]$, and this conjugacy (denoted $h_n[g]$) changes in a continuous way;*
2. *There exists a homeomorphism $\Xi_n : I_n \to J_n$ such that $\Xi_n(C_n^{\underline{d}})$ is the set of all $g \in J_n$ such that $R_n[g](0) \in h_n(C_n^{\underline{d}})$.*

This result follows immediately from [AM1] since the reparametrization is a homeomorphism.

We let $J_n^\tau = \Xi_n(I_n^{\tau_n})$. The correspondence Ξ_n is uniquely defined if restricted to $K_n = I_n \setminus \cup C_n^{\underline{d}}$. More importantly, it is quasisymmetric if restricted to certain subsets of K_n. To make this precise, let $K_n^\tau = K_n \cap I_n^{\tau_n}$ (forgetting information outside $I_n^{\tau_n}$) and $\tilde{K}_n = K_n \setminus \cup I_n^j \setminus \tilde{I}_{n+1}$ (forgetting information inside each I_n^j and also inside $\tilde{I}_{n+1}$).

Theorem A.2. *Let f be a simple map. Then for all $\gamma > k$, there exist $n_0 > 0$ such that for all $n > n_0$,*

- **PhPa1:** $\Xi_n | K_n^\tau$ *is γ-qs;*
- **PhPa2:** $\Xi_n | \tilde{K}_n$ *is γ-qs;*
- **PaPa1:** $h_n[g] | K_n$ *is γ/k-qs for all $g \in J_n^\tau$;*
- **PaPa1:** $h_n[g] | \tilde{K}_n$ *is γ/k-qs for all $g \in J_n$;*

While in [AM1] the quasisymmetric constant γ of the phase-parameter relation could be taken arbitrarily close to 1 for deeper levels of the principal nest, this does not hold here due to introduction of reparametrization, which multiply all constants by k. This will be the source of many difficulties addressed in this Appendix.

For the remaining of this Appendix we fix some constant $\gamma > k$, and will start our consideration with levels of the principal nest where the reparametrized phase-parameter relation is already γ-qs. We will also need some very large constants $\tilde{b} < b$ which depend on γ. We let $a = b^{-1}$ and $\tilde{a} = \tilde{b}^{-1}$.

From now on we will estimate always the γ-capacity of bad sets in phase space. To conclude results for the parameter we will use the following variation of the Borel-Cantelli Lemma.

Lemma A.3. *Let $X \subset \mathbb{R}$ be a measurable set such that for each $x \in X$ is defined a sequence $D_n(x)$ of nested intervals converging to x such that for all $x_1, x_2 \in X$ and any i, $D_n(x_1)$ is either equal or disjoint to $J_i(x_2)$. Let Q_n be measurable subsets of $\mathbb{R}$ and $q_n(x) = |Q_n \cap D_n(x)|/|D_n(x)|$. Let Y be the set of x in X which belong to finitely many Q_n. If $\sum q_n(x)$ is finite for almost any $x \in X$ then $|Y| = |X|$.*

In practice, the D_n will be the parameter windows defined before (either J_n or J_n^τ), and Q_n will be certain subsets of J_n or J_n^τ corresponding (under the phase-parameter map) to branches of the return map (in the case of J_n) or landings (in the case of J_n^τ), whose behavior we want to avoid. We will then show that such bad events have summable γ-capacity in the phase space, which will yield the conclusion for Lebesgue measure of the parameter using PhPa1 (for landings) or PhPa2 (for returns).

A.3. **Decay of geometry.** We proceed with an analysis of the scaling factor $c_n = |I_{n+1}|/|I_n|$. This quantity is the most fundamental one of all results to follow.

One of the main results we exploit has a relatively simple proof: with total probability, the I_n decrease torrentially to 0, more precisely, c_{n+1}^{-1} is an exponential of c_n^{-1}. This very fast decay imply that the landing map to I_{n+1} is essentially a very high iterate of the return map to I_n (since it takes a long time to hit a very small interval). This very high iteration time will allow us to conclude that the characteristics (say, return time) of each level tend to be more well behaved than in the previous one due to fast convergence to some average (some kind of Law of Large Numbers). The fact that we must deal with qs-capacity instead of Lebesgue measure will essentially reflect in exponential errors (whose size depend on k) in the above description. The estimates on iteration time are summarized below:

Lemma A.4. *With total probability, for all n sufficiently big we have*

$$p_{2\gamma-1}(|\underline{d}^{(n)}(x)| \leq k | x \in I_n) < k c_n^{\tilde{a}}, \tag{1}$$

$$p_{2\gamma-1}(|\underline{d}^{(n)}(x)| \geq k | x \in I_n) < e^{-k c_n^{\tilde{b}}}. \tag{2}$$

We also have

$$p_{2\gamma-1}(|\underline{d}^{(n)}(x)| \leq k | x \in I_n^{\tau_n}) < k c_n^{\tilde{a}}, \tag{3}$$

$$p_{2\gamma-1}(|\underline{d}^{(n)}(x)| \geq k | x \in I_n^{\tau_n}) < e^{-k c_n^{\tilde{b}}}. \tag{4}$$

The phase parameter lemmas (specially PhPa1) allow us to conclude:

Lemma A.5. *With total probability,*

$$c_n^{-\tilde{a}/2} < \frac{\ln(s_n)}{\ln(c_n^{-1})} < c_n^{-2\tilde{b}}.$$

This result implies easily the stated torrential decay of geometry.

A.3.1. *Derivatives.* We then proceed to estimate derivatives of branches of the return map. All Lemmas in this section can be proved using the same arguments of [AM1].

The first step is to exclude the possibility of a 'too recurrent' or 'too low' return.

Lemma A.6. *With total probability, the distance between $R_n(0)$ and $\partial I_n \cup \{0\}$ is at least $|I_n| n^{-\tilde{b}}$. In particular $R_n(0) \notin \tilde{I}_{n+1}$ for all n large enough.*

We are then able to estimate distortion:

Lemma A.7. *With total probability, for n big enough and $j \neq 0$*

$$\operatorname{dist}(f|_{I_n^j}) \leq n^{\tilde{b}}.$$

Since return maps are torrentially expansive on average (from the decay of geometry), we get:

Lemma A.8. *With total probability, for n big enough and for all $\underline{d}$*

$$\operatorname{dist}(R_n^{\underline{d}}) \leq n^{\tilde{b}}.$$

In particular, for n big enough, $\sup_{\underline{d} \neq \emptyset} \operatorname{dist}(R_n^{\underline{d}}) \leq 2^n$ and $\inf |R_n'|_{\cup_{j\neq 0} I_n^j}| > 2$.

The above estimates allow a direct estimate of derivatives under iterates of f.

Lemma A.9. *With total probability, if n is sufficiently big and if $x \in \cup_{j\neq 0} I_n^j$ and $R_n|_{I_n^j} = f^r$, then for $1 \leq k \leq r$, $|(Df^k(x))| > |x| c_{n-1}^3$.*

A.4. **Distribution of return times.** We proceed to estimate the distribution of return times, that is, the "random variables" r_n and l_n. The main tool is to prove first a Large Deviation Estimate for r_n using only the torrential decay, and then show that such estimate leads to much more precise control of the subsequent levels. Since the transition between different levels introduces some distortion (although torrentially small), we are forced to deal with a sequence of quasisymmetric constants in our estimates: instead of just dealing with γ, we must consider $\gamma_n = \gamma(n+1)/n$ and $\tilde{\gamma}_n = \gamma(2n+3)/(2n+1)$. This part is very similar to [AM1], and we only state the final estimate:

Lemma A.10. *With total probability, for all n sufficiently large we have*

1. $p_{\tilde{\gamma}_n}(l_n(x) < c_n^{-s} | x \in I_n) < c_n^{\tilde{a}^2 - s} < c_n^{a-s}$, *with* $s > 0$,
2. $p_{\tilde{\gamma}_n}(l_n(x) < c_n^{-s} | x \in I_n^{\tau_n}) < c_n^{a-s}$, *with* $s > 0$,
3. $p_{\tilde{\gamma}_n}(l_n(x) > c_n^{-s} | x \in I_n) < e^{-c_n^{b-s}}$, *with* $s > b$,
4. $p_{\tilde{\gamma}_n}(l_n(x) > c_n^{-s} | x \in I_n^{\tau_n}) < e^{-c_n^{b-s}}$, *with* $s > b$,
5. $p_{\gamma_n}(r_n(x) < c_{n-1}^{-s} | x \in I_n) < c_{n-1}^{\tilde{a}^2 - s} < c_{n-1}^{a-s}$, *with* $s > 0$,
6. $p_{\gamma_n}(r_n(x) > c_{n-1}^{-s} | x \in I_n) < e^{-c_{n-1}^{\sqrt{b}-s}} < e^{-c_{n-1}^{b-s}}$ *with* $s > b$.
7. $c_{n-1}^{-a} < r_n(\tau_n) < c_{n-1}^{-b}$.
8. $c_{n-1}^{-a} < v_n < c_{n-1}^{-b}$.

A.5. **Dealing with hyperbolicity.** In this section we show by an inductive proccess that the great majority of branches are reasonably hyperbolic. In order to do that, in the following subsection, we define some classes of branches with 'good' distribution of times and which are not too close to the critical point. The definition of 'good' distribution of times has an inductive component: they are composition of many 'good' branches of the previous level. The fact that most branches are good is related to the validity of some kind of Law of Large Numbers estimate. The inductive definition will guarantee that the good distribution of times holds in all scales and allows to preserve hyperbolicity from one step to the other.

A.5.1. *Some kinds of branches and landings.* The several classes of branches we will define do not correspond exactly to the same classes in [AM1], although classes with the same name have essentially the same function in the proof. The adaptation is non-trivial, since the previous proof uses strongly small quasisymmetric constants. This will lead to consideration of extra classes below (bad returns and fast landings).

A.5.2. *Standard landings.* Let us define the set of standard landings at time n, $LS(n) \subset \Omega$ as the set of all $\underline{d} = (j_1, ..., j_m)$ satisfying the following:

LS1: (m not too small or large) $c_n^{-a/2} < m < c_n^{-2b}$,
LS2: (No very large times) $r_n(j_i) < c_{n-1}^{-3b}$ for all i.
LS3: (Short times are sparse in not too small initial segments) For $c_{n-1}^{-2b} \leq k \leq m$

$$\#\{r_n(j_i) < c_{n-1}^{-a/2}\} \cap \{1, ..., k\} < (6 \cdot 2^n) c_{n-1}^{a/2} k.$$

We also define the set of fast landings at time n, $LF(n) \subset \Omega$ by the following conditions

LF1: (m small) $m < c_n^{-a/2}$.
LS2: (No very large times) $r_n(j_i) < c_{n-1}^{-3b}$ for all i.

The phase-estimates below are obtained from the estimate of distribution of return times following the general lines of the estimates of [AM1] for standard landings.

Lemma A.11. *With total probability, for all n sufficiently big,*

(1) $p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LS(n) | x \in I_n) < c_n^{a/3}/2$,
(2) $p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LS(n) \cup LF(n) | x \in I_n) < c_n^{n^2}/2$,
(3) $p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LS(n) | x \in I_n^{\tau_n}) < c_n^{a/3}/2$,
(4) $p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LS(n) \cup LF(n) | x \in I_n^{\tau_n}) < c_n^{n^2}/2$.

A.5.3. *Very good returns, bad returns and excellent landings.* Define the set of very good returns, $VG(n_0, n) \subset \mathbb{Z} \setminus \{0\}$, $n_0 \leq n \in \mathbb{N}$ and the set of bad returns, $B(n_0, n) \subset \mathbb{Z} \setminus \{0\}$, $n_0 \leq n \in \mathbb{N}$, by induction as follows. We let $VG(n_0, n_0) = \mathbb{Z} \setminus \{0\}$, $B(n_0, n_0) = \emptyset$ and supposing

$VG(n_0,n)$ and $B(n_0,n)$ defined, define the set of excellent landings $LE(n_0,n) \subset LS(n)$ satisfying the following extra assumptions.

LE1: (Not very good moments are sparse in not too small initial segments) For all $c_{n-1}^{-2b} < k \leq m$

$$\#\{i | j_i \notin VG(n_0,n)\} \cap \{1,...,k\} < 6 \cdot 2^n c_{n-1}^{a^2} k,$$

LE2: (Bad moments are sparse in not too small initial segments) For all $c_n^{-1/n} < k \leq m$

$$\#\{i | j_i \notin B(n_0,n)\} \cap \{1,...,k\} < 6 \cdot 2^n c_{n-1}^{n} k,$$

We define $VG(n_0,n+1)$ as the set of j such that $R_n(I_{n+1}^j) = C_n^{\underline{d}}$ with $\underline{d} \in LE(n_0,n)$ and the extra condition:

VG: (distant from 0) The distance of I_{n+1}^j to 0 is bigger than $c_n^{n^2}|I_{n+1}|$.

And we define $B(n_0,n+1)$ as the set of $j \notin VG(n_0,n+1)$ such that $R_n(I_{n+1}^j) = C_n^{\underline{d}}$ with $\underline{d} \notin LF(n)$.

Very good and bad returns can be estimated in an inductive fashion analogously to the estimate of [AM1] on very good returns.

Lemma A.12. *With total probability, for all n_0 sufficiently big,*

(1) $p_{\gamma_n}(j^{(n)}(x) \notin VG(n_0,n) | x \in I_n) < c_{n-1}^{a^2}$,
(2) $p_{\gamma_n}(j^{(n)}(x) \in B(n_0,n) | x \in I_n) < 2c_{n-1}^{2n}$,
(3) $p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LE(n_0,n) | x \in I_n) < c_n^{2a/5}$,
(4) $p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LE(n_0,n) \cup LF(n) | x \in I_n) < c_n^{bn}$,
(5) $p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LE(n_0,n) | x \in I_n^{\tau_n}) < c_n^{2a/5}$.

This translates immediately to a parameter estimate:

Lemma A.13. *With total probability, for all n_0 big enough, for all n big enough, $\tau_n \in VG(n_0,n)$.*

Before going on we will need two simple estimates, one for the very good return time and another to estimate branches which are neither very good or bad.

Lemma A.14. *With total probability, for all n_0 big enough and for all $n \geq n_0$, if $j \in VG(n_0,n+1)$ then*

$$m < r_{n+1}(j) < m c_{n-1}^{-4b},$$

where as usual, m is such that $R_n(I_{n+1}^j) = C_n^{\underline{d}}$ and $\underline{d} = (j_1,...,j_m)$.

Lemma A.15. *With total probability for all n_0 sufficiently big, if $n > n_0$, if $j \notin VG(n_0,n) \cup B(n_0,n)$ then $r_n(j) < c_{n-1}^{-a/2} c_{n-2}^{-4b}$.*

At this stage we have most of the tools to show that almost every parameter is "Collet-Eckmann at first returns", that is, $|Df^{k_n}(f(0))|$ is exponentially big for the sequence k_n of first landings of $f(0)$ in I_n. To obtain the full Collet-Eckmann condition (exponential growth for all k), we will need to analyze truncations of branches or landings, that is, we will consider iterates of the type $f^k|_{I_n^j}$ (or $f^k|_{C_n^{\underline{d}}}$) for k less then the return time $r_n(j)$ (or $l_n(\underline{d})$).

We now show that very good branches are well behaved when truncated at a reasonably big time. Here well behaved means "spending most of the time in very good branches of the previous level". So if we are able to control the hyperbolicity of very good branches in some level we will have a good possibility of control truncated very good branches in the next level.

Lemma A.16. *With total probability, for all n_0 big enough and for all $n \geq n_0$, the following holds.*

Let $j \in VG(n_0, n+1)$, as usual let $R_n(I_{n+1}^j) = I_n^{\underline{d}}$ and $\underline{d} = (j_1, ..., j_m)$. Let m_k be biggest possible with

$$v_n + \sum_{j=1}^{m_k} r_n(j_i) \leq k$$

(the amount of full returns to level n before time k) and let

$$\beta_k = \sum_{\substack{1 \leq i \leq m_k, \\ j_i \in VG(n_0,n)}} r_n(j_i).$$

(the total time spent in full returns to level n which are very good before time k) Then $1 - \beta_k/k < c_{n-1}^{a^2/3}$ if $k > c_n^{-2/n}$.

Proof. Let us estimate first the time i_k which is not spent on full returns:

$$i_k = k - \sum_{j=1}^{m_k} r_n(j_i).$$

This corresponds exactly to v_n plus some incomplete part of the return $j_{m_{k+1}}$. This part can be bounded by $c_{n-1}^{-b} + c_{n-1}^{-3b}$ (use the estimate of v_n and LS2 to estimate the incomplete part).

Using LS2 we conclude now that

$$m_k > (k - c_{n-1}^{-b}) c_{n-1}^{3b} > c_n^{-1/n}$$

so m_k is not too small.

Let us now estimate the contribution h_k from bad full returns j_i. The number of such returns must be less than $c_{n-1}^{n/2}m_k$, using LS2 their total time is at most $c_{n-1}^{(n/2)-3b}m_k < m_k$.

The non very good full returns on the other hand can be estimated by LE1 (given the estimate on m_k), they are at most $c_{n-1}^{a^2}m_k$. So we can estimate the total time l_k of non very good full returns with time less then $c_{n-1}^{-a/2}c_{n-2}^{-4b}$ by

$$c_{n-1}^{a^2}c_{n-1}^{-a/2}c_{n-2}^{-4b}m_k,$$

while β_k can be estimated from below by

$$(1-c_{n-1}^{a/4})c_{n-1}^{-a/2}m_k.$$

It is easy to see then that $i_k/\beta_k \ll c_{n-1}^{a/5}$, $h_k/\beta_k \ll c_{n-1}^{a/5}$. We also have

$$l_k/\beta_k < 2c_{n-1}^{a^2/2}.$$

So $(i_k+h_k+l_k)/\beta_k$ is less then $c_{n-1}^{a^2/3}$. Since $i_k+h_k+l_k+\beta_k = k$ we have $1-\beta_k/k < (i_k+h_k+l_k)/\beta_k$. $\square$

A.5.4. *Cool landings.* Let us define the set of cool landings $LC(n_0,n) \subset \Omega$, $n_0, n \in \mathbb{N}$, $n \geq n_0$ as the set of all $\underline{d} = (j_1, ..., j_m)$ in $LE(n_0,n)$ satisfying

LC1: (Starts very good) $j_i \in VG(n_0,n)$, $1 \leq i \leq c_{n-1}^{-a^2/2}$.

LC2: (Short times are sparse in not too small initial segments) For $c_{n-1}^{-a/2} \leq k \leq m$

$$\#\{r_n(j_i) < c_{n-1}^{-a/2}\} \cap \{1,...,k\} < (6\cdot 2^n)c_{n-1}^{a/3}k,$$

LC3: (Not very good moments are sparse in not too small initial segments) For all $c_{n-1}^{-a^2/4} < k \leq m$

$$\#\{r_n(j_i) < c_{n-1}^{-a/2}\} \cap \{1,...,k\} < (6\cdot 2^n)c_{n-1}^{a/3}k,$$

LC3: (Not very good moments are sparse in not too small initial segments) For all $c_{n-1}^{-a^2/4} < k \leq m$

$$\#\{i|j_i \notin VG(n_0,n)\} \cap \{1,...,k\} < (6\cdot 2^n)c_{n-1}^{a^2}k,$$

LC4: (Bad times are sparse in not too small initial segments) For $c_{n-1}^{-n/3} \leq k \leq m$

$$\#\{i|j_i \in B(n_0,n)\} \cap \{1,...,k\} < (6\cdot 2^n)c_{n-1}^{n/6}k,$$

LC5: (Starts with no bad times) $j_i \notin B(n_0,n)$, $1 \leq i \leq c_{n-1}^{-n/2}$.

As in [AM1], cool landings are frequent and we get the following parameter estimate

Lemma A.17. *With total probability, for all n_0 big enough, for all n big enough we have $R_n(0) \in LC(n_0, n)$.*

A.6. **Hyperbolicity and Collet-Eckmann.** We now introduce the hyperbolicity random variable λ_n.

For $j \neq 0$, we define

$$\lambda_n^{(j)} = \inf_{x \in I_n^j} \frac{\ln(|R_n'(x)|)}{r_n(j)}.$$

And $\lambda_n = \inf_{j \neq 0} \lambda_n^{(j)}$. As a consequence of the exponential estimate on distortion for returns (which competes with torrential expansion from the decay of geometry), together with hyperbolicity of f in the complement of I_n^0 we immediately have the following

Lemma A.18. *With total probability, for all n sufficiently big, $\lambda_n > 0$.*

The "minimum hyperbolicity" $\liminf \lambda_n$ of the parameters we will obtain will in fact be positive, as it follows from one of the properties of Collet-Eckmann parameters (uniform hyperbolicity on periodic orbits), together with our estimates on distortion.

Our strategy however is not to show that the minimum hyperbolicity is positive, but that the typical value of λ_m^j stays big as m grows (and is in fact bigger than $\lambda_n/2$ for $m > n$ big). Since we also have to estimate the hyperbolicity of truncated branches it will be convenient to introduce a new class of branches.

We define the set of good returns $G(n_0, n) \subset \mathbb{Z} \setminus \{0\}$, $n_0, n \in \mathbb{N}$, $n \geq n_0$ as the set of all j such that

G1: (hyperbolic return)

$$\lambda_n^{(j)} \geq \lambda_{n_0} \frac{1 + 2^{n_0 - n}}{2},$$

G2: (hyperbolicity in partial return) for $c_{n-1}^{-3/(n-1)} \leq k \leq r_n(j)$ we have

$$\inf_{I_n^j} \frac{\ln(|Df^k|)}{k} \geq \lambda_{n_0} \frac{1 + 2^{n_0 - n + 1/2}}{2} - c_n^{2/(n-1)}.$$

Now we conclude that typical branches are good.

Lemma A.19. *With total probability, for n_0 big enough and for all $n > n_0$, $VG(n_0, n) \subset G(n_0, n)$.*

The proof is the same as in [AM1], the two main features of very good branches exploited here are their good distribution of return times and the condition VG which allows to avoid drastic losses of derivative due to too strong recurrence.

Now we show that cool landings inherit hyperbolicity from very good returns. The proof of this fact needs adjustments for big quasisymmetric constants, so we provide it here.

Lemma A.20. *With total probability, if n_0 is sufficiently big, for all n sufficiently big, if $\underline{d} \in LC(n_0, n+1)$ then for all $c_{n-1}^{-4/(n-1)} < k \leq l_n(\underline{d})$,*

$$\inf_{C_n^{\underline{d}}} \frac{\ln(|Df^k|)}{k} \geq \frac{\lambda_{n_0}}{2}.$$

Proof. Fix such $\underline{d} \in LC(n_0, n+1)$.

Let

$$a_k = \inf_{C_n^{\underline{d}}} \frac{\ln(|Df^k|)}{k}.$$

Consider the sequence $r_i = r_n(j_i)$ where as usual $\underline{d} = (j_1, ..., j_m)$.

As in Lemma A.16, we define m_k as the biggest such that

$$\sum_{i=1}^{m_k} r_i \leq k.$$

We define

$$\beta_k = \sum_{\substack{1 \leq i \leq m_k, \\ j_i \in VG(n_0, n+1)}} r_i,$$

(counting the time up to k spent in complete very good returns) and

$$i_k = k - \sum_{i=1}^{m_k} r_i.$$

(counting the time in the incomplete return at k).

Let's then consider two cases: small m_k ($m_k < c_{n-1}^{-a^2/2}$) and otherwise.

The idea of the first case is that all full returns are very good by LC1, and the incomplete time is also part of a very good return.

Since full very good returns are very hyperbolic by G1 and very good returns are good, we just have to worry with possibly losing hyperbolicity in the incomplete time. To control this, we introduce the queue $q_k = \inf_{C_{n-1}^{\underline{d}}} \ln(|Df^{i_k} \circ f^{k-i_k}|)$. We have $-q_k < -\ln(c_{n-1}^{1/3} c_{n-1}^3)$ by Lemma A.9

If the incomplete time is big (more than $c_{n-1}^{-4/(n-1)}$), we can use G2 to estimate the hyperbolicity of the incomplete time (which is part of a very good return). The reader can easily check the estimate in this case.

If the incomplete time is not big, we can not use G2 to estimate q_k, but in this case i_k is much less than k: since $k > c_{n-1}^{-4/(n-1)}$, at least one return was completed ($m_k \geq 1$), and since it must be very good we conclude that $k > c_{n-1}^{-a/2}$ by LS1, so

$$a_k > \lambda_{n_0} \frac{(1+2^{n-n_0})}{2} \cdot \frac{k-i_k}{k} - \frac{-q_k}{k} > \frac{\lambda_{n_0}}{2}.$$

Let's consider now the case $m_k > c_{n-1}^{-a^2/2}$. For an incomplete time we still have $-q_k < -\ln(c_n c_{n-1}^3)$, so $-q_k/k < c_{n-1}^{-a^2/3}$.

Arguing as in Lemma A.16, we split $k - \beta_k - i_k$ (time of full returns which are not very good) in part relative to bad returns h_k and in part relative to fast returns (not very good or bad) l_k. Using LC4 and LC5 we get

$$h_k < c_{n-1}^{-3b} c_{n-1}^{n/7} m_k,$$

and using LC1 and LC3 we have

$$l_k < c_{n-1}^{-a/2} c_{n-2}^{-4b} (6 \cdot 2^n) c_{n-1}^{a^2} m_k,$$

By LC1 and LC3 we have $\beta_k > c_{n-1}^{-a/2} m_k/2$ so we have

$$\frac{h_k + l_k}{\beta_k} < c_{n-1}^{a^2/2}.$$

By LC1, $\beta_k > c_{n-1}^{a/2} m_k$. Now $\beta_k > c_{n-1}^{-a/2} c_{n-1}^{-a^2}/2$ by hypothesis on m_k. If $i_k < c_{n-1}^{-a/2} c_{n-2}^{-4b}$, i_k/β_k is small and we are done. Otherwise by Lemma A.15, i_k must be either very good or bad. If i_k is very good we can reason as before that $G2$ applies for the estimate of the queue and we are done.

If i_k is bad, by LC5 we have that $m_k > c_{n-1}^{n/2}$, but $i_k < c_{n-1}^{3b}$ by LS2, so i_k/β_k is very small again and we are done. $\square$

Since the critical point always fall on cool landings (Lemma A.17), the Collet-Eckmann condition follows easily from the above Lemma (which guarantees gain of derivative after large trucations), together with Lemma A.9, which controls loss of derivative at small truncations (the argument is identical to the one in [AM1]).

A.7. **Recurrence.** To show that the critical point is polynomially recurrent, we can follow the same lines from [AM1]. First we look at the essentially markov process $R_n | I_n \setminus I_{n+1}$, which shows that with total probability, most (in the γ-qs sense) points in I_n have polynomial recurrence (the exponent must be chosen according to γ) until the first time they fall in I_{n+1}. More precisely, we show (after transferring to the parameter)

Lemma A.21. *With total probability, for n big enough and for $1 \leq i \leq s_n$,*

$$\frac{\ln(|R_n^i(0)|)}{\ln(c_{n-1})} < b^2 \left(1 + \frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right).$$

To obtain the polynomial recurrence for f we relate the return times in terms of R_n to return times in terms of f. In other words, letting k_i be such that $R_n^i(0) = f^{k_i}(0)$, we must relate k_i and i. It is enough to do the estimate for a cool landing and we obtain

Lemma A.22. *With total probability, for n big enough and for $1 \leq i \leq s_n$,*

$$\frac{\ln(k_i)}{\ln(c_{n-1}^{-1})} > a/3 \left(1 + \frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right).$$

Let now $v_n \leq k < v_{n+1}$. If $|f^k(0)| < k^{-3b^3}$ we have $f^k(0) \in I_n$ and so $k = k_i$ for some i. It follows from Lemmas A.21 and A.22 that

$$|f^k(0)| > k^{-3b^3}.$$

This concludes the proof of polynomial recurrence. We notice that polynomial lower bounds are easily obtained: considering $|R_n(0)| = |f^{v_n}(0)| < c_{n-1}$ and using $v_n < c_{n-1}^{-b}$ we get

$$\limsup_{n\to\infty} \frac{-\ln|f^n(0)|}{\ln(n)} > a.$$

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Collège de France, 3 Rue d'Ulm, 75005 Paris – France.
E-mail address: avila@impa.br

IMPA – Estr. D. Castorina 110, 22460-320 Rio de Janeiro – Brazil.
E-mail address: gugu@impa.br