Bifurcations of the Essential Dynamics of Lorenz maps on the real line and the bifurcation scenario for the Linear Family *

Rafael Labarca and Carlos Moreira

Dedicated to the memory of Professor Camilo Blazquez

1 Introduction

In a remarkable contribution a meteorologist, E.N. Lorenz [1], showed numerical evidence of the existence of a strange attractor for a quadratic system of ordinary differential equations in three variables. Some time later J. Guckenheimer, [5], produced a work where he introduced symbolic dynamics in order to understand the topological equivalence classes for nearly similar attractors. At that time R.F. Williams,[7], introduced a geometrical model in order to understand the dynamics of these Lorenz attractors. Using this geometrical model the dynamical behavior of the three dimensional vector field can be reduce to the dynamical behavior of a one-dimensional map with one discontinuity and Guckenheimer and Williams,[6], used this fact to shown uncountable many classes of non-equivalent geometric Lorenz attractors. The evidence of the non-equivalence where the kneading sequences associated to these one-dimensional maps. The class of maps introduced in this way is contained in the class of maps DM_0 that we work with(see section 2.1).Associated to any $f \in DM_0$ we have two kneading sequences $(a_f\,, b_f) = I(f)$ (see section 2.3 for the definition of these sequences) that satisfies $a_f = \inf\{\sigma^k(a_f)\,, k \in \mathbb{N}\}$, $b_f = \sup\{\sigma^k(b_f)\,, k \in \mathbb{N}\}$ and $\{a_f\,, b_f\} \subset \Sigma_{a_f\,, b_f}$ (here $\Sigma_{a,b}$ denote the set $\cap_{i=0}^{\infty}\sigma^{-i}([a,b])$, see section 2.3 for details). These properties inspired ([23])the following definitions. A sequence of two symbols $a = (0\,, \ldots) \in \Sigma_2$ (resp. $b = (1\,, \ldots) \in \Sigma_2$) is called *minimal* (resp. *maximal*) if $a = \inf\{\sigma^k(a)\,, k \in \mathbb{N}\}$ (resp. $b = \sup\{\sigma^k(b)\,, k \in \mathbb{N}\}$. Here $\sigma : \Sigma_2 \to \Sigma_2$ denotes the usual shift map. We will denote by Min_2 (resp. Max_2) the set of minimal(resp. maximal) sequences in Σ_2 . The properties that satisfies the kneading sequences allow us to define the *Lexicographical World* as $LW = \{(a\,, b) \in Min_2 \times Max_2\,, \{a\,, b\} \subset \Sigma_{a\,,b}\}$. The itinerary $I : DM_0 \to LW\,, f \to (a_f\,, b_f)$ defines a continuous and surjective map(see section 2.4). We will say that the map $f \in DM_0$ has *essentially* the same dynamics as the map $g \in DM_0$ if $I(f) = I(g)$.

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It is clear that two topologically equivalent maps are essentially equivalent. Therefore, the lexicographical world provide a *universal model* for (essentially) equivalent dynamics in this context. This mean the following: given $(a\,,b) \in LW$ then there is $f \in DM_0$ such that $\Sigma_{a\,,b} = \Sigma_{a_f,b_f}$ and a surjective map $I_f : \Gamma_f \to \Sigma_{a,b}$ such that $I_f \circ f = \sigma \circ I_f$ and reciprocally (see section 2.3 for the definition of the set Γ_f and the map I_f). So, all the possible essential dynamics produced by elements in DM_0 are modeled by elements in LW and it is natural to ask if there is a parameterized family $\alpha : U \subset \mathbb{R}^k \to DM_0$ with the following property : given any $(a,b) \in LW$ the set $\alpha^{-1}(\{(a,b)\})$ is non empty. We believe that the correct answer to this question is no.

In these directions are the main results of the present paper: we describe, in a precise way, which part of the lexicographical world is realized by the piecewise linear family

$F : Q \to DM_0$, $F_{\lambda,\mu} = F_{(\lambda,\mu)}$ given by

$$F_{\lambda,\mu}(x) = \left\{ \begin{array}{ll} (1+\lambda)x+1, & x<0 \\ (1+\mu)x-1, & x>0 \end{array} \right\}$$

answering a question posed by Hubbard and Sparrow in [3]. Here $Q = \{(\lambda,\mu) \in \mathbb{R}^2;\ \ -1 \le \mu \le 1 \text{ and } -1 \le \lambda \le 1\}$. Our main results are that there is an uncountable set of sequences $(a,b) \in LW$ such that $(I \circ F)^{-1}(\{(a,b)\})$ is empty(it is interesting to note that, in doing this, we prove that period doubling solenoid do not exists for this family of maps in an independent way from [12] and [15]) and the exact description of these sets when they are non empty.

This paper is organized as follows: in section 2 we state our results, in section 3 we introduce some symbolic dynamics and the lexicographical world and in section 4 we prove the results.

Some comments are in order : de Melo and Martens announced, in an unpublished paper [22], the existence of parametrized families of elements in DM_0 that are topologically universal in the sense that given any geometric Lorenz flow then its dynamics is " essentially " the same as the dynamics of some element of the family. The exact way in which these families realized the dynamics is given in [29].

Finally, we have to mention that there are many authors which have investigated these families from the same and from another points of view (combinatorial, differentiable, geometrical or topological)(see the list of references). In particular, compare our results with those of Glendinning in [30].

2 Statement of the results

2.1 The Set DM_0.

In the sequel DM_0 will denote the set of maps $f : (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$ such that:

(1) The restriction maps $f|_{(-\infty,0)} : (-\infty, 0) \mapsto \mathbb{R}$ and $f|_{(0,\infty)} : (0, \infty) \to \mathbb{R}$ are continuous and non-decreasing maps.

(2)
$$f(0^+) = \lim_{x\downarrow 0} f(x) \in (-\infty, 0]$$
and
$$f(0^-) = \lim_{x\uparrow 0} f(x) \in [0, \infty[$$

An element in DM_0 will be called *injective* if its restriction to the interval $]f(0^+)\,,\, f(0^-)[$ is an injective map.

We will say that $f \in DM_0$ is *increasing* if the restriction maps $f|_{(-\infty,0)} : (-\infty, 0) \mapsto \mathbb{R}$ and $f|_{(0,\infty)} : (0, \infty) \to \mathbb{R}$ are increasing.

We call the elements in DM_0 *Lorenz maps.*

2.2 The Lexicographical Order

Let Σ_2 denote the set of sequences $\theta : \mathbb{N} \to \{0, 1\}$ endowed with the topology given by the metric
$$d(\alpha\,, \beta) = \sum_{i=0}^{\infty} \frac{\overline{d}(\alpha_i\,, \beta_i)}{2^i},$$
where
$$\overline{d}(\alpha_i\,, \beta_i) = \begin{cases} 0\,, \alpha_i = \beta_i \\ 1\,, \alpha_i \neq \beta_i \end{cases}$$
Let $\sigma : \Sigma_2 \to \Sigma_2$ be the shift map $\sigma(\theta_0\,, \theta_1\,, \theta_2\,, \ldots) = (\theta_1\,, \theta_2\,, \ldots)$. Let Σ_0 and Σ_1 denote the sets $\{\theta \in \Sigma_2\ ;\ \theta_0 = 0\}$ and $\{\theta \in \Sigma_2\ ;\ \theta_0 = 1\}$ respectively. It is clear that $\Sigma_2 = \Sigma_0 \cup \Sigma_1$.

In Σ_2 we consider the *lexicographical order*: $\theta < \alpha$ for any $\theta \in \Sigma_0$ and $\alpha \in \Sigma_1$ or $\theta < \alpha$ if there is $n \in \mathbb{N}$ such that $\theta_i = \alpha_i$ for $i = 0, 1, 2, \ldots, n-1$ and $\theta_n = 0$ and $\alpha_n = 1$.

For $a \leq b$ in Σ_2 let $[a\,, b]$ denote the interval $\{\theta \in \Sigma_2 |\, a \leq \theta \leq b\}$. $\Sigma_{a\,,b}$ will denote the set $\bigcap_{n=0}^{\infty} \sigma^{-n}([a\,, b])$.

2.3 The Set Σ_{a_f, b_f} and the Essential Dynamics.

For $f \in DM_0$ let $\Gamma_f = (\mathbb{R} \setminus \bigcup_{j=0}^{\infty} f^{-j}(0))$ denote the set of "continuity" of the map f.

For $x \in \Gamma_f$ we define $I_f(x) \in \Sigma_2$ by
$$I_f(x)(i) = 0 \quad \text{if } f^i(x) < 0 \text{ and } I_f(x)(i) = 1 \quad \text{if } f^i(x) > 1.$$

For $x = 0$ we define:
$$I_f(0^+) = \lim_{x\downarrow 0\,, x\in\Gamma_f} I_f(x)$$

and

$$I_f(0^-) = \lim_{x \uparrow 0\,, x \in \Gamma_f} I_f(x).$$

In the same way to any $x \in \bigcup_{j=0}^{\infty} f^{-j}(0)$ such that $f^i(x) \neq 0,\ 0 \leq i < n$; $f^n(x) = 0$ we associate the sequences :

$$I_f(x^+) = (I_f(x)(0), \ldots, I_f(x)(n-1),\ I_f(0^+))$$

and

$$I_f(x^-) = (I_f(x)(0), \ldots, I_f(x)(n-1),\ I_f(0^-)).$$

For $x \in \Gamma_f$ we define $I_f(x^+) = I_f(x^-) = I_f(x)$.

Let $I_f = \{I_f(x^+); \quad x \in [f(0^+), f(0^-)[\} \cup \{I_f(x^-); \quad x \in]f(0^+), f(0^-)]\}$.

Clearly $\sigma(I_f) \subset I_f$. Let us denote $a_f = I_f((f(0^+))^+)$ and $b_f = I_f((f(0^-))^-)$

Lemma 1 $I_f = \bigcap_{n=0}^{\infty} \sigma^{-n}([a_f, b_f]) = \Sigma_{a_f, b_f}$.

We observe that associated to any $f \in DM_0$ we can define a continuous map

$$h : [f(0^+)\,, f(0^-)] \cap \Gamma_f \to \Sigma_{a_f\,, b_f} \subset \Sigma_2,$$

such that $h \circ f = \sigma \circ h$. The map h is given by $h(x) = I_f(x)$and collapses intervals into points. This map cannot be extended, continuously, to the set $\cup_{i=0}^{\infty} f^{-i}(0)$. There are two kinds of intervals that the map h can collapses: The wandering intervals and the intervals that are contained in the stable manifold of periodic sinks. An interval $I \subset [f(0^+)\,, f(0^-)]$ is called a <u>*wandering interval*</u> , for the map f, if for any $x \in I$ we have that x is a wandering point. We will call a point x a <u>*nonwandering*</u> point if for any neighborhood U_x of x and any positive integer N we can find $n \geq N$ such that $f^n(U_x) \cap U_x \neq \emptyset$. The set of nonwandering points of the map f is denoted by Ω_f . A point $x \notin \Omega_f$ is called a <u>*wandering point*</u>. Given any interval ,I, <u>*the orbit*</u> of this interval is the sequence of iterations $(f^n(I)\,, n \in \mathbb{N})$. Concerning the existence of wandering intervals we have the following:

Lemma 2 *([28]) Let $\{\varphi_\lambda\,, \lambda \in \mathbb{R}\} \subset DM_0$ be a one parameter family of C^2 increasing maps such that for each λ there are sequences $\lambda_n \to \lambda$ and $\mu_n \to \lambda$ with $\varphi_{\lambda_n}(x) > \varphi_\lambda(x)$ and $\varphi_{\mu_n}(x) < \varphi_\lambda(x)\,, \forall x$ then there is a residual set of parameters λ for which φ_λ has no wandering intervals.*

We observe that our two parameter family of linear maps satisfy this property.

Definition 1 *Given $f, g \in DM_0$. We will say that f has* <u>essentially</u> *the same dynamics as g if $I_f = I_g$.*

We note that in this situation, up to the existence of some intervals where the itineraries of the points are the same, the dynamics of the maps f and g are topologically equivalent(see [31]).

2.4 The Lexicographical World

Let $Min_2 = \{a \in \Sigma_0 \ ; \ \sigma^k(a) \geq a \ , k \in \mathbb{N}\}$ and $Max_2 = \{b \in \Sigma_b \ ; \ \sigma^k(b) \leq b \ , k \in \mathbb{N}\}$.

Definition 2 *The set* $LW = \{(a\,,b) \in Min_2 \times Max_2 \ ; \{a\,,b\} \subset \Sigma_{a\,,b}\}$ *will be called the* lexicographical world.

For $a \in Min_2$ its LW-fiber is the set $LW_0(a) = \{b \in Max_2 \ ; \ (a\,,b) \in LW\}$. For $b \in Max_2$ its LW-fiber is the set $LW_1(b) = \{a \in Min_2 \ ; \ (a\,,b) \in LW\}$.

Remark 1 *It is clear that given* $(a\,,b) \in LW$ *then* $\Sigma_{a,b} \neq \emptyset$.

Let us now consider $(a, b) \in LW$.

Lemma 3 *There is* $f \in DM_0$ *such that* $I_f = \Sigma_{a,b}$.

Proof: Let us consider the map $g : (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$ given by

$$g(x) = \begin{cases} 2x - 1, & x > 0 \\ 2x + 1, & x < 0 \end{cases}$$

In this case $I_g = \Sigma_2$. Let $x_a < 0$ and $x_b > 0$ be the points such that $I_g(x_a^+) = a$, $I_g(x_b^-) = b$.

Let $f : (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$ be the map defined by:

$$f(x) = \begin{cases} g(x), & x \leq \overline{x_b} \\ x_b, & \overline{x_b} \leq x < 0 \\ x_a, & 0 < x \leq \overline{x_a} \\ g(x), & \overline{x_a} \leq x \end{cases}$$

here $\overline{x_b} < 0 < \overline{x_a}$ are given by $g(\overline{x_b}) = x_b$ and $g(\overline{x_a}) = x_a$.

We will call this result the *realization lemma*. It is clear that the maps, in this construction, are non-decreasing but we want a realization lemma with increasing maps. In section 3.2 we prove a realization lemma with increasing maps.

Therefore, we have a surjective map $I : DM_0 \to LW$, $I(f) = (a_f, b_f)$ and $DM_0 = \bigcup_{(a,b)\in LW} I^{-1}(\{(a,b)\})$. Also, using C^0 proximity on compact sets, this map is continuous.

In this context the next definition is natural.

Definition 3 *Let* $\alpha : U \subset \mathbb{R}^k \to DM_0$ *be a map.*

1) *We will say that* α *is an* a-universal family *if* $\forall a \in Min_2$ *there is a nonempty set ,* $A(a) \subset U$ *, such that* $a_f = a\,, \forall f \in A(a)$ *.*

2) *We will say that* α *is an* b-universal family *if* $\forall a \in Max_2$ *there is a nonempty set ,* $B(b) \subset U$ *, such that* $b_f = b\,, \forall f \in B(b)$ *.*

3) We will say that α is an <u>LW-universal family</u> *if $\forall\ (a,b) \in LW$ there exists a nonempty set $A(a,b) \subset U$ such that $I \circ \alpha(A(a,b)) = (a,b)$.*

It is clear that associated to any map α, as above, we have an $a(b, LW)$-decomposition of its domain. We will call this a(resp. b, LW)-decomposition the a(resp. b, LW) bifurcation theory defined by α.

Open Problem: There are LW-universal families?.

Certainly, this is a very hard problem and we believe that there is not a finite $k \in \mathbb{N}$ with this property. In the present paper we prove the following

Theorem 1 *The given two parameter family of linear Lorenz maps is a-universal, is b-universal but it is not LW-universal. Moreover, the respective a, b, LW-bifurcation theories are given.*

3 Symbolic Dynamics and the Lexicographical World

Here we introduce some results and notations that are necessary for our results. The results which we do not prove in this section are proved in [28].

3.1 Dynamical properties for sequences in LW

Let $a_1 \leq a_2$ be two periodic sequences in Σ_0. The sequence $m(a_1, a_2) = \underline{a_1\, a_2}$ will be called the <u>*average*</u> of the sequences a_1 and a_2.

<u>Example</u>: For $a_1 = \underline{01}$, $a_2 = \underline{011}$ we have $m(a_1, a_2) = \underline{01011}$.

Let $A_0 = \{\underline{0_n 1}\ , \underline{01_m}\ ; n, m \in \mathbb{N} \setminus \{0\}\ \}$ and $A_{n+1} = A_n \cup \{\ m(a_1, a_2);\ a_1, a_2 \in A_n$ are consecutive sequences $\}$. Set $A_\infty = \bigcup_{n=0}^{n=\infty} A_n$. The elements in A_∞ will be called <u>*primary sequences*</u>. As we will see in section 4.1, primary sequences are associated with primary bifurcations.

Let us denote $A^0_\infty = A_\infty$ and define, for any $a \in A^0_\infty$ the set:

$$A^1_1(a) = \{c \in \Sigma_0\, ;\ c = \underline{a_- b_+ a^n} \text{ or } c = \underline{a_- b^m b_+}\, , \text{for } n, m \in \mathbb{N}\}.$$

Here for $b = \underline{b_0 b_1 \ldots b_k 0}$ we have $b_+ = b_0 b_1 \ldots b_k 1$.

Observe that if $c_n = \underline{a_- b_+ a^n}$ then $c_n \to a_- b_+ \underline{a}$ and if $d_m = \underline{a_- b^m b_+}$ then $d_m \to a_- \underline{b}$.

Set

$$A^1_2(a) = \{m(a_1, a_2)\, ;\ a_1, a_2 \in A^1_1(a) \text{are consecutives sequences}\} \cup A^1_1(a)$$

and in general

$$A^1_{n+1}(a) = \{m(a_1, a_2)\, ;\ a_1, a_2 \in A^1_n(a) \text{are consecutive sequences}\} \cup A^1_n(a)\ , n \geq 2.$$

Let $A^1_\infty(a) = \bigcup_{n=1}^{n=\infty} A^1_n(a)$ and $A^1_\infty = A^0_\infty \cup \bigcup_{a\in A^0_\infty} A^1_\infty(a)$ Inductively, for any $a \in A^n_\infty(a)$,let

$$A^{n+1}_1(a) = \{c \in \Sigma_0\,;\, c = \underline{a_- b_+ a^j} \text{or } c = \underline{a_- b^k b_+} \text{ for } j,k \in \mathbb{N}\}$$

Now , we define

$$A^{n+1}_2(a) = \{m(a_1\,,a_2)\,;\ a_1\,,a_2 \in A^{n+1}_1(a) \text{ are consecutive sequences}\}\bigcup A^{n+1}_1(a)$$

and

$$A^{n+1}_{m+1}(a) = \{m(a_1\,,a_2)\,;\ a_1\,,a_2 \in A^{n+1}_m(a) \text{ are consecutive sequences}\}\bigcup A^{n+1}_m(a)\ ,m \geq 2.$$

As before, define $A^{n+1}_\infty(a) = \bigcup_{m=1}^\infty A^{n+1}_m(a)$; $A^{n+1}_\infty = \bigcup_{a\in A^n_\infty} A^{n+1}_\infty(a) \cup A^n_\infty$ and finally, $A^\infty_\infty = \bigcup_{j=0}^\infty A^j_\infty$.

The elements in $(A^\infty_\infty \setminus A^0_\infty)$ will be called secondary sequences, as we will see in section 4.2 secondary sequences are associated with secondary bifurcations.

Let denote by B^∞_∞ the set $\{\sup\{\sigma^k(a)\,,k \in \mathbb{N}\}\,,a \in A^\infty_\infty\}$. We will denote by $b(a)$ the sequence $\sup\{\sigma^k(a)\,,k \in \mathbb{N}\}$ for $a \in \Sigma_0$ and by $a(b)$, the sequence $\inf\{\sigma^k(b)\,,\,k \in \mathbb{N}\}$ for $b \in \Sigma_1$. Clearly, $b(a) \in Max_2$ and $a(b) \in Min_2$.

Lemma 4 *For any $a \in A^\infty_\infty$ and $d \in B^\infty_\infty \cap LW_0(a)$ we have that the map $\sigma : \Sigma_{a\,,d} \to \Sigma_{a\,,d}$ is topologically transitive.*

As before let $a \in A^\infty_\infty$ and $d \in LW_0(a)$ be a sequence which is not periodic nor preperiodic. Set $a_{last} = \sup\{\sigma^k(a)\,;\,\sigma^k(a) \in [a\,,0\underline{1}]\,,k \in \mathbb{N}\}$ and $d_{first} = \inf\{\sigma^k(d)\,;\,\sigma^k(d) \in [1\underline{0}\,,d]\,,k \in \mathbb{N}\}$.

In the sequel $int(F)$ will denote the interior of the set F .

Lemma 5 *(1) If there is $p \in \mathbb{N}$ such that $a_{last} \in int(\sigma^p([a_{last}\,,0d]))$ then $\sigma_{a\,,d} : \Sigma_{a\,,d} \to \Sigma_{a\,,d}$ is topologically transitive.*

(2) If there is $p \in \mathbb{N}$ such that $d_{first} \in int(\sigma^p([a_{last}\,0d]))$ then $\sigma_{a\,,d} : \Sigma_{a\,,d} \to \Sigma_{a\,,d}$ is topologically transitive.

(3) If there is $p \in \mathbb{N}$ such that $a_{last} \in int(\sigma^p([1a\,,d_{first}]))$ then $\sigma_{a\,,d} : \Sigma_{a\,,d} \to \Sigma_{a\,,d}$ is topologically transitive.

(4) If there is $p \in \mathbb{N}$ such that $d_{first} \in int(\sigma^p([1a\,,d_{first}]))$ then $\sigma_{a\,,d} : \Sigma_{a\,,d} \to \Sigma_{a\,,d}$ is topologically transitive.

Lemma 6 *Assume that $a \in Min_2$, $d \in Max_2$ satisfies*

(1) $a < a(d) \le a_{last}$ *or* $d_{first} \le b(a) < d$ *,*

(2) There are $p > 1$ *and* $q > 1$ *such that* $\sigma^p([a_{last}\,, 0d]) \subset [a_{last}\,, d_{first}]$ *and* $\sigma^q([1a\,, d_{last}) \subset [a_{last}\,, d_{first}]$ *. Then there are :*

- *an invariant set* $\Lambda \subset \Sigma_{a\,,d}$ *,*

- *an open set* $U \subset \Sigma_2$ *and an integer* $n_1 \in \mathbb{N}$ *such that :*

$$\sigma^{n_1}(U \cap \Sigma_{a\,,d}) \subset \Lambda \text{ and } U \cap \Lambda = \Phi$$

In particular, $\sigma_{a\,,d} : \Sigma_{a\,,d} \to \Sigma_{a\,,d}$ *is not a topological transitive map.*

Remark 2 *Given* $a \in A_\infty^\infty$ *we have :*

(i) $\sigma_{a\,,b+\underline{a}} : \Sigma_{a\,,b+\underline{a}} \to \Sigma_{a\,,b+\underline{a}}$ *is not topologically transitive,*

(ii) $\sigma_{a-b+\underline{a}\,,b+\underline{a}} : \Sigma_{a-b+\underline{a}\,,b+\underline{a}} \to \Sigma_{a-b+\underline{a}\,,b+\underline{a}}$ *is not topologically transitive and*

(iii) $\sigma_{a-\underline{b}\,,b+a-\underline{b}} : \Sigma_{a-\underline{b}\,,b+\underline{a}} \to \Sigma_{a-\underline{b}\,,b+a-\underline{b}}$ *is not topologically transitive.*

It is clear that $\Sigma_{a\,,\underline{1}} \neq \Phi$ for any $a \in \Sigma_0$. Hence we can define maps $\varphi\,,\psi\,,\chi :$ $\Sigma_0 \to \Sigma_1$ by :

$$\varphi(a) = \inf\{b \in \Sigma_1 |\, \Sigma_{a\,,b} \neq \Phi\},$$
$$\psi(a) = \inf\{b \in \Sigma_1\,;\, \Sigma_{a\,,b} \text{ contains} \infty\text{-elements}\}$$

and

$$\chi(a) = \inf\{b \in \Sigma_1\,;\, \Sigma_{a\,,b} \text{ is uncountable}\}.$$

Clearly, $a_1 \le a_2$ imply $\varphi(a_1) \le \varphi(a_2)$, $\psi(a_1) \le \psi(a_2)$ and $\chi(a_1) \le \chi(a_2)$ and for all $c \in \Sigma_1$ such that $c < \varphi(a)$ we have $\Sigma_{a\,,c} = \Phi$.

Examples : For any $0\underline{01} \le a \le \underline{01}$ we have $\varphi(a) = \underline{10}$, $\psi(\underline{01}) = \chi(\underline{01}) = 1\underline{10}$. Also $\varphi(\underline{0}) = \psi(\underline{0}) = \chi(\underline{0}) = 1\underline{0}\,; \varphi(0\underline{1}) = \psi(0\underline{1}) = \underline{1}\,; \varphi(\underline{0_n 1}) = \underline{10_n}$, $\psi(\underline{0_n1}) = \chi(\underline{0_n1}) = 1\underline{10_n}$ and $\varphi(\underline{01_m}) = \underline{1_m0}$, $\psi(\underline{01_m}) = \chi(\underline{01_m}) = \underline{11_m0}$.

3.2 The Realization Lemma.

In this section we will prove a realization lemma with increasing maps.

Lemma 7 *Let* $a \in A_\infty^0$*. There is an injective* $f \in DM_0$ *such that* $a_f = a$ *and* $b_f = \varphi(a)$*.*

Proof: Let $a = a_0 < a_1 < \ldots < (a_{\text{last}} = a_p) < 0\underline{1} < 1\underline{0} < (b_{\text{last}} = b_{p+1}) < b_{p+2} < \ldots < b_k = b(a)$, denote the consecutive elements in $\{\sigma^j(a); \quad j \in \mathbb{N}\}$. Let $\ell < 0 < r$ and $I = [\ell, r]$. Consider closed intervals $L_0, L_1, \ldots, L_k$ such that $\bigcup_{i=0}^k L_i = [\ell, r]$; $\text{left}(L_0) = \ell$, $\text{right}(L_0) = \text{left}(L_1), \ldots, \text{right}(L_p) = 0 = \text{left}(L_{p+1}), \ldots, \text{right}(L_{k-1}) = \text{left}(L_k)$; $\text{right}(L_k) = r$. Here $\text{left}[x, y] = x$ and $\text{right}[x, y] = y$.

Now, define a map f that sends $L_{p+1} \to L_0$, $L_{p+2} \to L_1, \ldots, L_k \to L_{k-p-1}$ and $L_0 \to L_{k-p}$; $L_1 \to L_{k-p+1}, \ldots, L_p \to L_k$. This map satisfy the lemma. □

The next result follows immediately from this lemma:

Corollary 1 *Let* $a \in \overline{A^0_\infty} \setminus A^0_\infty$. *There is an injective map* $f \in DM_0$ *such that* $a_f = a$ *and* $b_f = \varphi(a)$ $\Box$.

We note that these maps can be considered as bijective maps on the circle.

Lemma 8 *Let* $a \in A^\infty_\infty$. *There is* $f \in DM_0$ *an increasing map, such that* $a_f = a$ *and* $\varphi(a) = b_f$.

Proof: Let

$$a = a_0 < a_1 < \ldots < a_p < 0\underline{1} < 1\underline{0} < a_{p+1} < \ldots < a_k.$$

denote the consecutive elements in $\{\sigma^j(a)\,,\, j \in \mathbb{N}\}$. Associated with $a_s\,, 0 \leq s \leq k$,consider a closed interval L_s such that

$$L_0 < L_1 < \ldots < L_p \leq L_{p+1} < \ldots < L_k,$$

$L_p \cap L_{p+1} = \{0\}$ and $L_i \cap L_{i+1} = \emptyset$, otherwise. Let $\eta : \{0\,,1\,,\ldots\,,k\} \to \{0\,,1\,,\ldots\,,k\}$ be the permutation defined by $\sigma(a_j) = a_{\eta(j)}$. Clearly, $\eta(p) = k$ and $\eta(p+1) = 0$. Now, define an increasing map $f \in DM_0$ which satisfy $f(L_i) = L_{\eta(i)}$ and is increasing between L_i and L_{i+1} , $i \neq p\,, 0 \leq i \leq k-1$ $\Box$.

Now we prove:

Lemma 9 *For* $a \in A^\infty_\infty$ *and* $d \in B^\infty_\infty \cap LW_0(a)$. *There is an increasing map* $f \in DM_0$ *such that* $a_f = a\,, b_f = d$.

Proof: Assume $\#(per(a)) = n$ and $\#(per(d) = m$. Let $c_0 = a < c_1 < \ldots < c_p < 0\underline{1} < 1\underline{0} < c_{p+1} < \ldots < c_{m+n-1}$, denote the consecutive elements in $\{\sigma^j(a)\,,\sigma^j(d)\,,\, j \in \mathbb{N}\}$. Associated to c_j consider closed intervals L_s ; $0 \leq s \leq m+n-1$, such that

$$L_0 < L_1 < \ldots < L_p < L_{p+1} < \ldots < L_k,$$

$L_P \cap L_{p+1} = \{0\}$ and $L_i \cap L_{i+1} = \emptyset$, otherwise.

Let $\{i_1\,, i_2\,, \ldots\,, i_n\}$ and $\{j_1\,, j_2\,, \ldots\,, j_m\}$, be the indices defined by $c_{i_s} = \sigma^t(a)$; for some t and $c_{j_r} = \sigma^l(d)$ for some l. Let $\eta : \{i_1\,, \ldots\,, i_n\} \to \{i_1\,, \ldots\,, i_n\}$ and $\rho : \{j_1\,, \ldots\,, j_m\} \to \{j_1\,, \ldots\,, j_m\}$ be the permutations defined by $\sigma(a_{i_s}) = a_{\eta(i_s)}$ and $\rho(d_{j_r}) = d_{\rho(j_r)}$.

Now, define an increasing map $f \in DM_0$ by $f(L_{i_r}) = L_{\eta(i_r)}$ and $f(L_{j_r}) = L_{\rho(j_r)}$ $\Box$.

By a limit process we obtain

Lemma 10 *Associated to any* $a \in Min_2$ *there is an increasing map* $f \in DM_0$ *such that* $a_f = a$ *and* $b_f = \varphi(a_f)$.

As a consequence of these lemmas we get

Proposition 1 *Given* $(a,b) \in LW$ *there is an increasing map* $f \in DM_0$ *such that* $I(f) = (a_f\,, b_f) = (a,b)$.

This result is a generalization of a similar one obtained for expansive maps in [3].

3.3 The finite and the entropy zero cases.

Definition 4 *We will call a map $f \in DM_0$ essentially Morse-Smale if $a_f \in A^0_\infty$ and $b_f = \varphi(a_f)$.*

We will denote by $MS_0 \subset DM_0$ the set of essentially Morse-Smale maps . We call these maps Morse-Smale because its dynamics essentially reduces to a periodic orbit.

Lemma 11 *Given $f \in DM_0$ a "essentially Morse-Smale" map we have that $[f(0^+), f(0^-)] \cap \Gamma_f = \bigcup_{i=0}^{per(a_f)-1} I_i$, where $x \in I_0$ implies $I_f(x) = a_f$ and $I_f|_{I_j}$ is constant and equal to $a_j = \sigma^j(a_f)$ for $0 < j < per(a_f) - 1$.*

Definition 5 *We will call a map $f \in DM_0$ an entropy zero map if $b_f \leq \chi(a_f)$.*

One of the most interesting problems related with the bifurcation theory associated to a parameterized family of dynamical systems $\{f_\lambda\,;\, \lambda \in U \subset \mathbb{R}^k\}$, is to describe the set $\{\lambda \in U\,;\, f_\lambda$ is an entropy zero map$\}$ (see for instance [32], [14] and the references there in). Actually, see [28], this problem reduces to find the region where the previous definition applies. For our linear family of Lorenz maps we will solve this problem in section 4.1.

We observe that every essentially Morse-Smale map is an entropy zero map.

3.4 The topologically expansive cases

Definition 6 *A map $f \in DM_0$ will be called topologically expansive if $\bigcup_{i=0}^{\infty} f^{-i}(0)$ is a dense set in $[f(0^+), f(0^-)]$.*

Let denote by $TE_0 = \{f \in DM_0;\ f$ is a topologically expansive map$\}$ and $TE = I(TE_0) \subset LW$.

Note: It is clear that $EMS_0 \cap TE_0 = \emptyset$ and, as a consequence, $(LW \setminus TE) \neq \emptyset$.

In fact, we have the following result due to Hubbard and Sparrow ([3]).

Lemma 12 *([3]) $(a, b) \in TE$ if and only if $(a, b) \in LW$ and there are not finite sequences $d = 1d_1, \ldots, d_k$; $c = 0c_1 \ldots c_p$ such that $a = cb$ or $b = da$.*

4 Proof of Our Results

Let $Q = \{(\lambda, \mu) \in \mathbb{R}^2; \quad -1 \leq \mu \leq 1 \text{ and } -1 \leq \lambda \leq 1\}$.

Let $F : Q \to DM_0$ be the two-parameter family, $F_{\lambda,\mu} = F_{(\lambda,\mu)}$ given by

$$F_{\lambda,\mu}(x) = \begin{cases} (1+\lambda)x + 1, & x < 0 \\ (1+\mu)x - 1, & x > 0 \end{cases}$$

4.1 Primary Bifurcations

Assume $a \in A_0^\infty$.

Lemma 13 *There is a non empty region $L_a \subset Q$ such that $(\lambda, \mu) \in Q$ imply $a(\lambda, \mu) = a(F_{\lambda,\mu}) = a$.*

Proof: It is not difficult to see that the regions are given by the next figure.

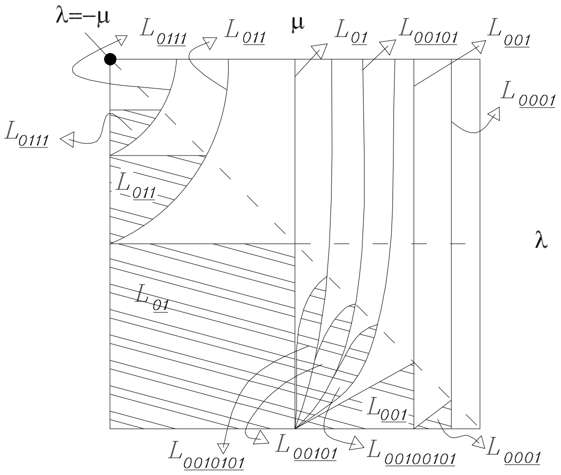

Figure 1:

Corollary 2 *a) For any $a \in (\overline{A_0^\infty} \setminus A_0^\infty)$ the set $L_a \subset Q$ is a curve.*

b) Let $G = \{(\lambda, \mu)\,;\, -1 \leq \mu \leq 1\,,\, -\mu < \lambda \leq 1\}$. Then

$$Q \setminus G = \{(\lambda\,, \mu)\,;\, F_{(\lambda,\mu)} \textit{is an entropy zero map}\}.$$

4.2 Secondary Bifurcations

In the sequel we are only interesting in the bifurcation scenario in $D = \overline{G}$.

Lemma 14 *There is* $(a,b) \in (TE \setminus I \circ F(D))$. *As a consequence: the linear family do not contains a universal model for topologically expanding maps.*

Proof: Let us consider $a_0 = \underline{01}$, $b_0 = \underline{10}$, $a_1 = \underline{(a_0)_-(b_0)_+} = \underline{0011}$, $b_1 = \underline{(b_0)_+(a_0)_-} = \underline{1100}$ and $a_{n+1} = \underline{(a_n)_-(b_n)_+}$, $b_{n+1} = \underline{(b_n)_+(a_n)_-}$; $n \geq 1$. The sequences $a = \lim_{n\to\infty} a_n$, $b = \lim_{n\to\infty} b_n$ satisfies the conditions of the Lemma in III.5.1. It is easy to see that associated to a_n and b_n there are curves $TE(a_n)$, $TE(b_n)$ that satisfies

(1) $(\lambda,\mu) \in TE(a_n)$ implies $a(F_{\lambda,\mu}) = a_n$ and $(\lambda,\mu) \in TE(b_n)$ implies $b(F_{\lambda,\mu}) = b_n$

In fact; for $a_0 = \underline{01}$ we have $TE(\underline{01}) = \{(0,\mu);\quad 0 \leq \mu \leq 1\}$ and for $b_0 = \underline{10}$ we have $TE(\underline{10}) = \{(\lambda, 0);\quad 0 \leq \lambda \leq 1\}$.

Let $A^0_\mu = [0^+, \mu]$ and its second image $F^2(A^0_\mu) = [-\lambda, (1+\lambda)(\mu(1+\mu)-1)+1]$. When $(1+\lambda)(\mu(1+\mu)-1)+1 = 0$ we get the sequence $b_1 = \underline{1100}$. That is

$$TE(\underline{1100}) = \left\{ \left(\frac{\mu(\mu+1)}{1-\mu(1+\mu)};\quad \mu \right);\quad 0 \leq \mu \leq -\frac{1+\sqrt{3}}{2} \right\}$$

In a symmetric way we get the curve $TE(\underline{0011}) = \{(\lambda, \mu(\lambda));\quad 0 \leq \lambda \leq -\frac{1+\sqrt{3}}{2}\}$.

Clearly, $TE(\underline{1100}) \subset \{(\lambda,\mu);\quad \mu < \lambda\}$ and $TE(\underline{0011}) \subset \{(\lambda,\mu);\quad \mu > \lambda\}$. Moreover $\lambda'(0) = \mu'(0) = 1$.

Now, for $(1+\lambda)(\mu(1+\mu)-1)+1 \leq \mu$ we have $F^2(A^0_\mu) \subset [-\lambda, \mu]$ and we can define $F_1 : [0^+, \mu] \to [-\lambda, \mu]$ by

$$F_1(x) = (1+\lambda)(1+\mu)x - \lambda.$$

Similarly, for $B^0_\lambda = [-\lambda, 0^-]$ we have $F^2(B^0_\lambda) = [(1+\mu)[1-\lambda(1+\lambda)]-1, \mu]$. For $-\lambda \leq (1+\mu)[1-\lambda(1+\lambda)]-1$ we have $F^2(B^0_\lambda) \subset [-\lambda, \mu]$ and we can define $F_1 : [-\lambda, 0^+] \to [-\lambda, \mu]$ by

$$F_1(x) = (1+\mu)(1+\lambda)x + \mu.$$

Using the map F_1 we can start again and we will obtain $TE(\underline{11010010})$ which is a curve $\{(\lambda(\mu);\mu);\quad 0 \leq \mu \leq \mu_1\}$ where $\lambda(\mu)$ satisfy the equation

$$(1+\lambda)^3(1+\mu)^3\mu - (1+\lambda)^2(1+\mu)^2\lambda - (1+\lambda)(1+\mu)\lambda + \mu = 0$$

and

$$8(1+\mu_1)^3\mu_1 - 4(1+\mu_1)^2 - 2(1+\mu_1) + \mu_1 = 0.$$

Inductively we get the result.

(2) $(0,0) \in \overline{TE(a_n)}$ and $(0,0) \in \overline{TE(b_n)}$

(3) $TE(a_n) \subset \{(\lambda, \mu) \in [0,1] \times [0,1]; \quad \lambda < \mu\}$
and
$TE(b_n) \subset \{(\lambda, \mu) \in [0,1] \times [0,1]; \quad \mu < \lambda\}$

(4) $TE(a_n) \to TE(a)$ and $TE(b_n) \to TE(b)$ with

$$TE(a) \subset \{(\lambda, \mu) \in [0,1] \times [0,1]; \quad \lambda < \mu\}$$

$$TE(b) \subset \{(\lambda, \mu) \in [0,1] \times [0,1]; \quad \mu < \lambda\}$$

(5) $(\lambda, \mu) \in TE(a)$ imply $a(F_{(\lambda,\mu)}) = a$ and$(\lambda, \mu) \in TE(b)$ imply $b(F_{(\lambda,\mu)}) = b$.

So, we conclude $(a, b) \in [TE \setminus I \circ F(D)]$ as we announced in the lemma.

3.5.3 Some remarks concerning the family $F_{\lambda,\mu}$.

a) Let $A_0, A_1, \ldots, A_\infty$ be defined as in I.5.(c). Let $a \in A_\infty$. There is a parameter value $(-\mu_a, \mu_a) \in D$ such that $a(-\mu_a, \mu_a) = a$; $b(-\mu_a, \mu_a) = b(a)$.

b) Associated to any $a \in A_\infty$ there are two curves $TE(a)$, $TE(b(a))$ such that

$$(\lambda, \mu) \in TE(a) \quad \text{imply} \quad a(F_{(\lambda,\mu)}) = a$$

and

$$(\lambda, \mu) \in TE(b(a)) \quad \text{imply} \quad b(F_{(\lambda,\mu)}) = b(a)$$

Moreover

$$TE(a) \cap TE(b(a)) = \{(-\mu_a, \mu_a)\}$$

For example, take $a = \underline{01_n}$, $b = \underline{1_n0}$. In this case we have

$$1 \xrightarrow{f} \mu \xrightarrow{f} (1+\mu)\mu - 1 \xrightarrow{f} (1+\mu)^2\mu - (1+\mu) - 1 \xrightarrow{f} \qquad \ldots$$

$$\xrightarrow{f} (1+\mu)^k\mu - (1+\mu)^{k-1} - \ldots - (1+\mu) - 1.$$

Therefore

$$f^{k+1}(1) = \xi_{k+1}(\mu) = (1+\mu)^k\mu - (1+\mu)^{k-1} - \ldots - 1$$

Observe:

1) $\xi_{k+1}(0) = 0 - \underbrace{1 - 1 \ldots - 1}_{k} = -k$

2) $\xi_{k+1}(1) = 2^k - (2^{k-1} + \ldots + 1) \; = \; 2^k - \frac{2^k - 1}{2-1} = 1.$

3) $\xi'_{k+1}(\mu) = k(1+\mu)^{k-1}\mu + (1+\mu)^k - (k-1)(1+\mu)^{k-2} - 1$ or, since

$\xi_{k+1}(\mu) = (1+\mu)^k\mu - \frac{(1+\mu)^k - 1}{1+\mu-1} = (1+\mu)^k\mu - \frac{(1+\mu)^k-1}{\mu}$, then

$$\xi'_{k+1}(\mu) = k(1+\mu)^{k-1}\mu + (1+\mu)^k - \frac{k(1+\mu)^{k-1}\mu - [(1+\mu)^k - 1]}{\mu^2}, \quad \mu \neq 0.$$

4) Let $\mu_k \pitchfork \xi_k(\mu_k) = 0$. Clearly $\mu_k \to 1$.

The equation $h_n = \mu_{n-1}$ correspond to $b = \underline{1_n o}$. These curve satisfies $h_n \to -1$ with $n \to \infty$ and so on.

(c) The curve $TE(a)$ transversally intersects the curves $\mu = \mu_0$ for $\mu_a \leq \mu_0 \leq 1$.

The curve $TE(b(a))$ transversally intersects the curves $\lambda = \lambda_0$ for $-\mu_a \leq \lambda_0 \leq 1$.

Moreover,for $a_n = 0_n 1 \ldots$, we have that $TE(a_n)$ converges to $\{(1,\mu);\ \ -1 \leq \mu \leq 1\}$ in the C^1-topology. For $b_n = 1_n 0 \ldots$, we have that $TE(b_n)$ converges to $\{(\lambda, 1);\ -1 \leq \lambda \leq 1\}$ in the C^1-topology.

(d) The curves $TE(10_n \ldots)$ converges to $\{(1,-1)\}$ and the curves $TE(01_n \ldots)$ converges to $\{(-1,1)\}$ with $n \to \infty$.

(e) Given $a \in A_\infty$, $b = b(a)$, we can find curves $TE(b_+\underline{a})$, $TE(b_+a_-\underline{b})$, $TE(a_-b_+\underline{a})$ and $TE(a_-\underline{b})$ such that:

(1) $(-\mu_a, \mu_a) \in \overline{TE(c)}$ for $c = b_+\underline{a},\ b_+a_-\underline{b},\ a_-b_+\underline{a},\ a_-\underline{b}$;

(2) $TE(a_-b_+\underline{a})$ and $TE(b_+\underline{a})$ are tangents to $TE(a)$ at $(-\mu_a, \mu_a)$;

(3) $TE(a - \underline{b})$ and $TE(b_+a_-\underline{b})$ are tangents to $TE(b)$ at $(-\mu_a, \mu_a)$

Figure (2) represents these essential facts concerning the geometry of these curves.

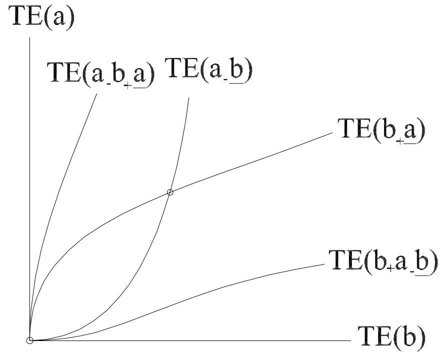

Figure 2:

(4) For $a \in A_\infty$ let

$$A_1^1(a) \;=\; \{c \in \Sigma_0; \quad c = \underline{a_- b_+ a^n}, \quad n \in \mathbb{N} \text{ or } c = \underline{a_(- b)^n b_+}, \quad n \in \mathbb{N}\}$$

.

(i) Associated with any $c = \underline{a_- b_+ a^n}$ we can find a direction $v_n \in \mathbb{R}^2$, $v_n \to v_a$, $v_a =$ the direction of the curve $TE(a)$ at $(-\mu_a, \mu_a)$;

$*$ Associated with any $c = \underline{a_- b^n b_+}$ we can find a direction $v_n \in \mathbb{R}^2$, $v_n \to v_b$, $v_b =$ the direction of the curve $TE(b)$ at $(-\mu_a, \mu_a)$

(ii) Associated with any $c = \underline{a_- b_+ a^n}$ $(c = \underline{a_- b^m b_+})$ we can find two curves $TE(c)$ and $TE(\underline{b_+ a^n a})$ (resp. $TE(c)$, $TE(\underline{b_+ a_- b^m})$) tangents to the direction v_n (resp. v_m) and located is different sides of the direction v_n (resp. v_m) that satisfies

$$(\lambda, \mu) \in TE(c) \quad \text{imply} \quad a(F_{(\lambda,\mu)}) = c$$

$$(\lambda, \mu) \in TE(\underline{b_+ a^n a}) \quad \text{imply} \quad b(F_{(\lambda,\mu)}) = \underline{b_+ a^n a}$$

$$\Big[\text{resp.} \quad (\lambda, \mu) \in TE(c) \quad \text{imply} \quad a(F_{(\lambda,\mu)}) = c \qquad \text{and}$$

$$(\lambda, \mu) \in TE(\underline{b_+ a_- b^m}) \quad \text{imply} \quad b(F_{(\lambda,\mu)}) = \underline{b_+ a_- b^m}\Big]$$

Figure (3) represents these essential facts.

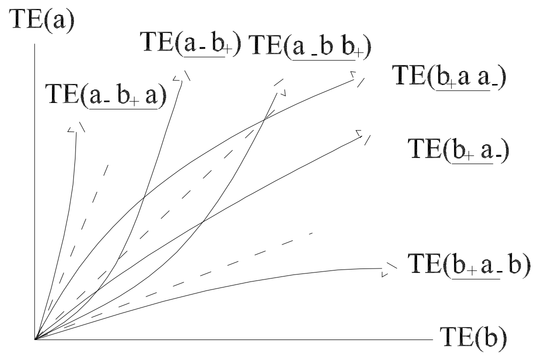

Figure 3:

(iii) Given $c \in A_1^1(a)$, $, b = b(c)$, we can find curves $TE(b_+ \underline{c})$, $TE(b_+ c_- \underline{b})$, $TE(c_- b_+ \underline{c})$ and $TE(c_- \underline{b})$ such that:

(a) $(\lambda, \mu) \in TE(b_+ \underline{c})$ imply $b(F_{(\lambda,\mu)}) = b_+ \underline{c}$;

- $(\lambda, \mu) \in TE(b_+ c_- \underline{b})$ imply $b(F_{(\lambda,\mu)}) = b_+ c_- \underline{b}$;
- $(\lambda, \mu) \in TE(c_- b_+ \underline{c})$ imply $a(F_{(\lambda,\mu)}) = c_- b_+ \underline{c}$;
- $(\lambda, \mu) \in TE(c_- \underline{b})$ imply $a(F_{(\lambda,\mu)}) \;=\; c_- \underline{b}$;

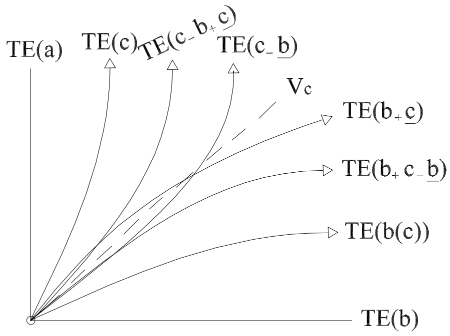

Figure 4:

(b) These four curves are tangents to the corresponding direction associated to c;

(c) $(-\mu_a, \mu_a) \in \overline{TE(d)}$ for $d = b_+\underline{c}$, $b_+c_-\underline{b}$, $c_-b_+\underline{c}$ and $c_-\underline{b}$.

Figure (4) represents these essential facts.

noindent **Remark:** $\forall\ a \in A_\infty$ we have that $\inf A_1^1(a) = a_-b_+\underline{a}$ and $\sup A_1^1(a) = a_-\underline{b}$.

(5) Given $a \in A_\infty$ let $A_2^1(a) = A_1^1(a) \cup \{m(a_1, a_2)\,;\, a_1, a_2 \in A_1^1(a)$ are consecutive sequences$\}$.
beginitemize

(i) Associated with $c = m(a_1, a_2)$ we can find a direction, $v_c \in \mathbb{R}^2$, located between v_{a_1} and v_{a_2};

(ii) Associated with $c = m(a_1, a_2)$ we can find two curves, $TE(c)$ and $TE(b(c))$, tangents to the direction v_c and located in different sides of the direction v_c that satisfies:

$$(\lambda, \mu) \in TE(c) \text{ imply } a(F_{(\lambda,\mu)}) = c;$$

$$(\lambda, \mu) \in TE(b(c)) \text{ imply } b(F_{(\lambda,\mu)}) = b(c);$$

(iii) Given $c \in A_2^1(a)$ we can continuous as in (4.iii).

In the same way, as we did with A_1^1andA_2^1 we can continuous with the set $A_{n+1}^1(a) = A_n^1(a) \cup \{m(a_1, a_2)\,; a_1, a_2 \in A_n^1$ are consecutive sequences$\}$, $n \geq 2$.
Let $A_\infty^1(a) = \bigcup_{n=1}^\infty A_n^1(a)$ and $A_\infty^1 = \bigcup_{a \in A_\infty^0} A_\infty^1(a) \cup A_\infty^0$.
Inductively, we make the same construction for elements in A_∞^{n+1}, where: $A_\infty^{n+1} = \bigcup_{a \in A_\infty^n} A_\infty^n(a) \cup A_\infty^n$.

Let $A_\infty^\infty = \bigcup_{n=0}^\infty A_\infty^n$. The same construction, as we did in (4.iii), holds for any $c \in A_\infty^\infty$.

(6) Let us now consider $a \in A_\infty^\infty$ and $b \in B_\infty^\infty$ such that $TE(a) \cap TE(b) \neq \emptyset$ we have:

(i) The intersection between these curves is transversal and contains a unique point $P(a, b)$;

(ii) Define $A(a, b) = \{c \in \Sigma_0; \quad c = \underline{a_- b_+ a^n}; \quad n \in \mathbb{N} \text{or } c = \underline{a_- b^m b_+}; \quad m \in \mathbb{N}\}$. Associated with any $c \in A(a, b)$ we can make the same construction that as we did in (4) and (5) before.

(7) Let $a \in A_\infty^\infty$ and $d \in Max_2$ be a sequence which is eventually periodic such that $TE(d) \cap TE(a) \neq \emptyset$. We have:

(i) The intersection between these two curves is transversal and contains a unique point $P(a, d)$;

(ii) We can find a curve $TE(a_- d)$, tangent to $TE(d)$ and such that $P(a, d) \in TE(a_- d)$ which satisfy:

$$(\lambda, \mu) \in TE(a_- d) \quad \text{imply} \quad a(F_{(\lambda,\mu)}) = a_- d.$$

Figure (5) represents these facts.

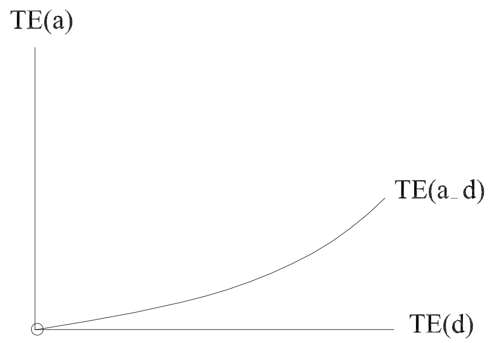

Figure 5:

(8) Let $b \in B_\infty^\infty$ and $c \in Min_2$ be an eventually periodic sequence such that $TE(c) \cap TE(b) \neq \emptyset$. We have that:

(i) The intersection between these two curves is transversal and contains a unique point $P(c, b)$;

(ii) We can find a curve, $TE(b_+c)$, tangent to $TE(c)$ such that $P(c,b) \in \overline{TE(b_+c)}$ which satisfy

$$(\lambda,\mu) \in TE(b_+c) \quad \text{imply} \quad b(F_{(\lambda,\mu)}) = b_+c.$$

Figure (6) represents these facts

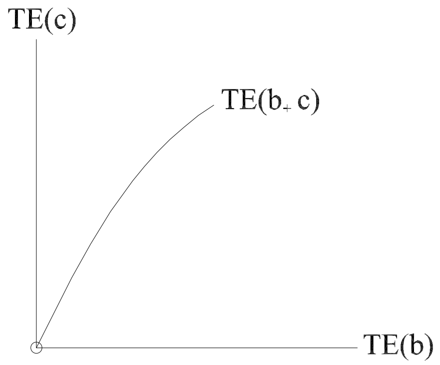

Figure 6:

(9) Let $a \in A^{\infty}_{\infty}$ and $d \in Max_2$ be a sequence which is not periodic nor eventually periodic such that $TE(a) \cap TE(d) \neq \emptyset$. We have that:

(i) The intersection between these two curves is transversal and contains a unique point $P(a,d)$;

(ii) We can find a curve, $TE(a_-d)$, tangent to $TE(d)$ and such that $P(a,d) \in \overline{TE(a_-d)}$ which satisfy:

$$(\lambda,\mu) \in TE(a_-d) \quad \text{imply} \quad a(F_{(\lambda,\mu)}) = a_-d.$$

(10) Let $b \in B^{\infty}_{\infty}$ and $c \in Min_2$ be a sequence which is not periodic nor eventually periodic such that $TE(b) \cap TE(c) \neq \emptyset$. We have that:

(i) The intersection between these two curves is transversal and contains a unique point $P(c,b)$;

(ii) We can find a curve, $TE(b_+c)$, tangent to $TE(c)$ and such that $P(c,b) \in \overline{TE(b_+c)}$ which satisfy:

$$(\lambda,\mu) \in TE(b_+c) \quad \text{imply} \quad b(F_{(\lambda,\mu)}) = b_+c.$$

(11) $F_{(\lambda,\mu)}$ is conjugated to $F_{(\tilde{\lambda},\tilde{\mu})}$ if and only if

$$a(F_{(\lambda,\mu)}) = a(F_{(\tilde{\lambda},\tilde{\mu})})$$

and

$$b(F_{(\lambda,\mu)}) = b(F_{(\tilde{\lambda},\tilde{\mu})}).$$

In fact, any $(\lambda, \mu) \in D$ represents a unique equivalence class of elements in $F(D)$.

Remark 3 *Theorem 1 is a consequence of the results in section 4.*

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R. Labarca
Departamento de Matematica y CC
Universidad de Santiago de Chile
Casilla 307 Correo 2
Santiago - Chile
and
C. Moreira
IMPA
Estrada Dona Castorina 110
CEP 22460-320
Jardim Botanico
Rio de Janeiro
Brazil