Sums of regular Cantor sets, dynamics and applications to number theory

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1. Introduction and main results

Let K_1, K_2 be regular ($C^{1+\varepsilon}$) Cantor sets. We say that K_1 has *stable intersection* with K_2 if for any pairs of regular Cantor sets $(\tilde{K}_1, \tilde{K}_2)$ near (K_1, K_2) (in the $C^{1+\varepsilon}$ topology) we have $\tilde{K}_1 \cap \tilde{K}_2 \neq \emptyset$.

We say that K_1 has d-stable intersection with K_2 $(d > 0)$ if for any $(\tilde{K}_1, \tilde{K}_2)$ near (K_1, K_2), we have $HD(\tilde{K}_1 \cap \tilde{K}_2) \geq d$.

Clearly, $\{t \in \mathbb{R} \mid K_1 \text{ has stable intersection with } K_2 + t\}$ is contained in the interior of $\{t \in \mathbb{R} \mid K_1 \cap (K_2 + t) \neq \emptyset\} = \{x - y,\ x \in K_1,\ y \in K_2\} = K_1 - K_2$.

The following theorem gives a positive answer to a conjecture by J. Palis (this is a joint work with J.C. Yoccoz).

Theorem: Let $\Omega = \{(K_1, K_2)$ regular Cantor sets, s.t. $HD(K_1) + HD(K_2) > 1\}$. There is an (open and) dense set $U \subset \Omega$ in the C^k topology $(1 < k \leq \infty)$ such that

$$(K_1, K_2) \in U \Rightarrow S(K_1, K_2) := \{t \in \mathbb{R} \mid K_1 \text{ has stable intersection with } K_2 + t\}$$

is (open and) dense in $K_1 - K_2$ and $HD((K_1 - K_2) \setminus S(K_1, K_2)) < 1$.

Moreover, given $(K_1, K_2) \in \Omega$ and $0 < d < HD(K_1) + HD(K_2) - 1$, we can approximate (K_1, K_2) by $(\tilde{K}_1, \tilde{K}_2)$ such that $S_d(K_1, K_2) = \{t \in \mathbb{R} \mid K_1$ has d-stable intersection with $K_2\}$ satisfies the above properties

2. Application to dynamics

Consider a first homoclinic bifurcation associated to a hyperbolic set Λ (a family of diffeomorphisms $\varphi_\mu \colon M^2 \to M^2$, $\mu \in (-1, 1)$ such that:

- φ_μ is persistently hyperbolic for $\mu < 0$;
- φ_0 presents a homoclinic tangency associated to the periodic saddle point p_0

Palis and Takens proved that if $HD(\Lambda) = HD(K^s) + HD(K^u) < 1$ then 0 is a point of full density of hyperbolicity in the parameter line.

The ideas of our previous theorem allow us to prove that with open and dense conditions, if $HD(\Lambda) > 1$, then there is an open set of parameters μ, with positive density at 0, corresponding to diffeomorphisms presenting stable homoclinic tangencies.

(This phenomena may persistentely coexist with positive density of hyperbolicity for positive parameters μ at $\mu = 0$).

3. Applications to number theory

Theorem: If K_1 and K_2 are regular Cantor sets, K_1 is of Class C^2 and not essentially affine then

$$HD(K_1 + K_2) = \min\{1, HD(K_1) + HD(K_2)\}.$$

This is a consequence of the main lemma in the proof of the previous theorem.

This is the main tool in the proof of the following results about the Markov and Lagrange spectra:

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. We define $k(\alpha) = \sup\{k > 0 \mid |\alpha - \frac{p}{q}| < \frac{1}{kq^2}$ have infinitely many solutions $\frac{p}{q} \in \mathbb{Q}\}$. The Lagrange spectrum L is the set $\{k(\alpha), \alpha \in \mathbb{R} \setminus \mathbb{Q}, k(\alpha) < \infty\}$.

The Markov spectrum can be defined as $M = \{ \inf_{(x,y) \in \mathbb{Z}^2 \setminus (0,0)} |f(x,y)|)^{-1}, f(x,y) = ax^2 + bxy + cy,\ a, b, c \in \mathbb{R},\ b^2 - 4ac = 1\}$.

It can be proved that L and M are closed sets and $L \subset M$.

Markov proved that $L \cap (-\infty, 3) = M \cap (-\infty, 3) = \{k_1 = \sqrt{5} < k_2 = 2\sqrt{2} < k_3 = \frac{\sqrt{221}}{5} < \dots\}$ where (k_n) is a sequence of algebraic numbers converging to 3.

M. Hall proved that L contains a whole half-line (for instance $[6, \infty)$), as a consequence of the fact that $K + K$ is an interval, where K is the (regular) Cantor set of the numbers in $[0, 1]$ whose continued fractions consist of 1,2,3 and 4.

Theorem:

- If $d(t) := HD(L \cap (-\infty, t))$, then $d(t)$ is also equal to $HD(M \cap (-\infty, t))$, and is a continuous surjective function from $\mathbb{R}$ to $[0, 1]$.

  Moreover,

  i) $d(t) = \min\{1, 2 \cdot HD(k^{-1}(-\infty, t))\}$

  ii) $\max\{t \in \mathbb{R} \mid d(t) = 0\} = 3$

  iii) $d(\sqrt{12}) = 1$.

- The set of accumulation points of L is perfect $(L' = L'')$
- $0 < HD(M \setminus L) < 1$.

If the continued fraction of α is $[a_0, a_1, a_2, \dots]$ then we have $k(\alpha) = \limsup_{n \to \infty} (\alpha_n + \beta_n)$, where

$$\alpha_n := [a_n, a_{n+1}, a_{n+2}, \dots]$$

and

$$\beta_n := [0, a_{n-1}, a_{n-2}, \dots, a_1].$$

Let $\Sigma = \mathbb{N}^{\mathbb{Z}}$ be the set of all bi-infinite sequences of positive integers. If $\underline{\theta} = (a_n)_{n \in \mathbb{Z}} \in \Sigma$, define $f(\underline{\theta}) = \alpha_0 + \beta_0 = [a_0, a_1, a_2, \dots] + [0; a_{-1}, a_{-2}, \dots]$. We have

$L = \{\limsup_{n\to\infty} f(\sigma^n \underline{\theta}), \underline{\theta} \in \Sigma\}$ and $M = \{\sup_{n\in\mathbb{Z}} f(\sigma^n \underline{\theta}), \underline{\theta} \in \Sigma\}$, where $\sigma: \Sigma \to \Sigma$ is the shift defined by $\sigma((a_n)_{n\in\mathbb{Z}}) = (a_{n+1})_{n\in\mathbb{Z}}$.

From this is clear that $L \subset M$, and the relation between sums of Cantor sets of continued functions and the geometric structures of L and M.

If $\varphi: M^2 \to M^2$ is a diffeomorphism, Λ is a hyperbolic set for φ and $f: M^2 \to \mathbb{R}$ is a differentiable function we may define sets that are analogous to the classical Markov and Lagrange spectra, the dynamical spectra

$$L(\Lambda, f) = \{\limsup_{n\to\infty} f(\varphi^n(x)), x \in \Lambda\}$$

$$M(\Lambda, f) = \{\sup_{n\in\mathbb{Z}} f(\varphi^n(x)), x \in \Lambda\}.$$

We have results analogous of those of the previous theorem for $L(\Lambda, f)$ and $M(\Lambda, f)$ that hold generically in an open set of (φ, f) that contains an open and dense set of $\{(\varphi, f)$ such that φ is conservative$\}$, exept for the last one $(0 < HD(M \setminus L) < 1)$, that may be false. In this open set it is also generically true that if $d_{\Lambda,f}(t) = HD(L(\Lambda, f) \cap (-\infty, t))$ then $\inf\{t \in \mathbb{R} \mid d_{\Lambda,f}(t) = 1\}$ belongs to the closure of the interior of $L(\Lambda, f)$.

Sketch of the proof of the main theorem:

Roughly speaking, a regular Cantor set k in the real line is defined by an iterated function system of contractions (say $f_1, f_2, \ldots, f_n$) whose images are disjoint that satisfies some combinatorial conditions (related to the existence of a Markov partition). Some compositions $f_i \circ f_j$ may be not defined, so let $B = \{(i,j),\ 1 \le i,\ j \le n$ such that $f_i \circ f_j$ is defined$\}$. Let $\Sigma^- = \{\underline{\theta} = (\theta_k)_{k\le 0},\ \theta_k \in (1, 2, \ldots, n\}$ and $(\theta_k, \theta_{k+1}) \in B,\ \forall k\}$.

For $\underline{\theta} = (\ldots, \theta_{-k}, \ldots, \theta_{-1}, \theta_0) \in \Sigma^-$ and $m \in \mathbb{N}$ we define $\underline{\theta}^m = (\theta_{-m}, \theta_{1-m}, \ldots, \theta_{-1}, \theta_0)$ and $f_{\underline{\theta}^m} = f_{\theta_{-m}} \circ f_{\theta_{1-m}} \circ \cdots \circ f_{\theta_0}$. Let $B(\underline{\theta}^m)$ be the affine map such that $B(\underline{\theta}^m) \circ f_{\underline{\theta}^m}$ is orientation preserving and have the interval $[0,1]$ as image. It is a well-know result that for each $\underline{\theta} \in \Sigma^-$ there exists $k_{\underline{\theta}} = \lim_{m\to\infty} (B_{\underline{\theta}^m} \circ f_{\underline{\theta}^m})$ and the $k_{\underline{\theta}}$ depend Hölder-continuously in $\underline{\theta}$ with the metric $d(\underline{\theta}, \underline{\tilde{\theta}}) = |I(\underline{\theta} \cap \underline{\tilde{\theta}})|$, where $\underline{\theta} \wedge \underline{\tilde{\theta}} = (\theta_{-k}, \ldots, \theta_{-1}, \theta_0)$, $\theta_i = \tilde{\theta}_i$ for $i \geq -k$ and $\theta_{-k-1} \neq \tilde{\theta}_{-k-1}$, and $I(\underline{\theta} \wedge \underline{\tilde{\theta}})$ is the image of $f_{\underline{\theta}^k}$ (the dependence on $\underline{\theta}$ is even Lipschtz if the maps f_i are of class C^2).

The image of K by the maps $k_{\underline{\theta}}$ are the *limit geometries* of K. They are also regular Cantor sets, and intervals of the construction of a limit geometry intersected to the limit geometry are small copies of another limit geometry. This allows us to define renormalization operators as follows:

Let K, K' be two regular Cantor sets, $K_{\underline{\theta}} = k_{\underline{\theta}}(K)$ and $K'_{\underline{\theta}'} = k'_{\underline{\theta}'}(K')$ be two limit geometries of K and K', $\underline{\alpha}$ and $\underline{\beta}$ be two finite sequences, $\lambda \in \mathbb{R}^*$ and $t \in \mathbb{R}$. We define the renormalization operator $R_{\underline{\alpha},\underline{\beta}}$ by $R_{\underline{\alpha},\underline{\beta}}: \Sigma^- \times \Sigma^- \times \mathbb{R}^* \times \mathbb{R}$, $R_{\underline{\alpha},\underline{\beta}}(\underline{\theta}, \underline{\theta}', \lambda, t) = (\underline{\theta\alpha}, \underline{\theta'\beta}, \lambda(d-c)/(b-a), (\lambda c + t - a)/(b-a))$ where $[a,b]$ is the image of $I_{\underline{\alpha}}$ by $K_{\underline{\theta}}$ and $[c,d]$ is the image of $I'_{\underline{\beta}}$ by $k'_{\underline{\theta}'}$.

A point $(\underline{\theta}, \underline{\theta}', \lambda, t) \in \Sigma^{'-} \times \Sigma^- \times \mathbb{R}^* \times \mathbb{R}$ can be viewed as a relative position of two limit geometries ($K_{\underline{\theta}}$ and $\lambda K'_{\underline{\theta}'} + t$). $R_{\underline{\alpha},\underline{\beta}}(\underline{\theta}, \underline{\theta}', \lambda, t)$ is the relative position of two intervals of the construction of $K_{\underline{\theta}}$ and $\lambda K'_{\underline{\theta}'} + t$ (that are $k_{\underline{\theta}}(I_{\underline{\alpha}})$ and $\lambda k'_{\underline{\theta}'}(I'_{\underline{\beta}}) + t$).

The strategy of the proof of the main theorem is based on the following proposition (that is not difficult to prove):

Proof: If there is a compact set $\mathcal{L} \subset \Sigma^- \times \Sigma^- \times \mathbb{R}^* \times \mathbb{R}$ with non-empty interior such that for each $(\underline{\theta}, \underline{\theta}', \lambda, t) \in \mathcal{L}$ there is a renormalization operator $R_{\underline{\alpha},\underline{\beta}}$ such that $R_{\underline{\alpha},\underline{\beta}}(\underline{\theta}, \underline{\theta}', \lambda, t) \in \operatorname{int} \mathcal{L}$ then $K_{\underline{\theta}}$ has stable intersection with $\lambda K'_{\underline{\theta}'} + t$ for any $(\underline{\theta}, \underline{\theta}', \lambda, t) \in \mathcal{L}$.

Such a set $\mathcal{L}$ is called a compact recurrent set.

In the previous proposition, if we require that for each $(\underline{\theta}, \underline{\theta}', \lambda, t) \in \mathcal{L}$ there are renormalization operators $R_{\underline{\alpha}_i,\underline{\beta}_i}$, $1 \leq i \leq j = j(\underline{\theta}, \underline{\theta}', \lambda, t)$ such that $\sum_{i=1}^{j} \lambda_i^d > 1$, where $\lambda_i = |k_{\underline{\theta}}(I_{\underline{\alpha}_i})|$ then $K_{\underline{\theta}}$ has d-stable interseciton with $\lambda K'_{\underline{\theta}'} + t$ for any $(\underline{\theta}, \underline{\theta}', \lambda, t) \in \mathcal{L}$.

Such a set $\mathcal{L}$ is called a compact d-recurrent set.

We prove that we can always perturb a pair of regular Cantor sets (K, K') such that $HD(K) + HD(K') > 1$ in order to have a compact set as in the previous proposition.

A theorem by Marstrand shows that under our hypothesis, for any $(\underline{\theta}, \underline{\theta}') \in \Sigma^- \times \Sigma^{'-}$ there is a set with positive measure of $(\lambda, t) \in \mathbb{R}^* \times \mathbb{R}$ for which $K_{\underline{\theta}} \cap (\lambda K'_{\underline{\theta}'} + t)$ has positive Hausdorff dimension. For $(\underline{\theta}, \underline{\theta}', \lambda, t)$ in this set there are many candidates for renormalization operators $R_{\underline{\alpha},\underline{\beta}}$ (when some point of $K_{\underline{\theta}} \cap (\lambda K'_{\underline{\theta}'} + t)$ belongs to $k_{\underline{\theta}}(I_{\underline{\alpha}}) \cap k'_{\underline{\theta}'}(I'_{\underline{\beta}})$) that can return to the same set, and small neighbourhoods of this set give reasonable candidates for the set $\mathcal{L}$ in the proposition, and some of these sets $\mathcal{L}$ will work after most of small random perturbations of the functions that define K.

The main lemma that we use in the proof is the Scale recurrence lemma, that shows that, given a set $L \subset \Sigma^- \times \Sigma^{'-} \times \mathbb{R}^*$ such that for some large number $A > 0$ the measure of $\{\lambda \in [-A, A] \mid (\underline{\theta}, \underline{\theta}', \lambda) \notin L\}$ is small for any $(\underline{\theta}, \underline{\theta}')$ there exists a set F contained in a small neighbourhood of L such that the measure of $\{\lambda \in [-A, A] \mid (\underline{\theta}, \underline{\theta}', \lambda) \notin F\}$ is small for any $(\underline{\theta}, \underline{\theta}')$ that is compact recurrent set at the level of $\Sigma^- \times \Sigma^{'-} \times \mathbb{R}^*$ (we forget the coordinate t), and indeed a d-recurrent set with d very close to $HD(K) + HD(K')$ at the level of $\Sigma^- \times \Sigma^{'-} \times \mathbb{R}^*$.

The Scale recurrence lemma holds with the hypothesis of the set K being essentially non-affine. This means that there are $\underline{\theta}, \tilde{\underline{\theta}} \in \Sigma^-$ and $x \in K_{\tilde{\underline{\theta}}}$ such that $(k_{\underline{\theta}} \circ k_{\tilde{\underline{\theta}}}^{-1})''(x) \neq 0$. It is applied to the set given by Marstrand's theorem, and it is necessary since the renormalization operators are very expansive in respect to the coordinate t, but essentially neutral in the coordinate λ, so we use the lemma to obtain a compact recurrent set at the level of $(\underline{\theta}, \underline{\theta}', \lambda)$ and then use perturbations to obtain a recurrent compact set in $\Sigma^- \times \Sigma^{'-} \times \mathbb{R}^* \times \mathbb{R}$.

The Scale recurrence lemma can be used to show the formula of the dimension of $K + K'$ that we used in the discussion about the Markov and Lagrange Spectra.

In the references below there are some papers related to the results discussed here:

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