An extremal property of Chebyshev Polynomials

Carlos Gustavo T. de A. Moreira

Abstract

Let $B_n = \{p(x) \in \mathbb{R}[x] : |p(x)| \le 1, \forall x \in [-1,1]\}$, and let $f \colon \mathbb{R}_+^{n+1} \to \mathbb{R}$ be a symmetric convex function, increasing in each variable. For $p(x) = \sum_{j=0}^{n} a_j x^j$ we define $\overline{f}(p) = f(|a_0|, |a_1|, \ldots, |a_n|)$. We prove that $\overline{f}(p) \le \overline{f}(T_n)$ for every $p \in B_n$, where T_n is the $n-th$ Chebyshev Polynomial, and the equality holds if and only if $p = \pm T_n$. In particular, we have $\sum_{j=0}^{n} |a_j| \le \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2}$, when $\sum_{j=0}^{n} a_j x^j \in B_n$, and the equality holds if and only if $T_n(x) = \sum_{j=0}^{n} a_j x^j$.

1 Introduction

It is well known that $\cos(n\theta) = T_n(\cos\theta)$, for a real polynomial T_n, the $n-th$ Chebyshev polynomial. Chebyshev polynomials have many extremal properties. For instance, if $B_n = \{p(x) \in \mathbb{R}[x] : |p(x)| \le 1, \ \forall x \in [-1,1]\} \subset \{p(x) \in \mathbb{R}[x] : |p(\cos\left(\frac{j\pi}{n}\right)| \le 1, \ 0 \le j \le n\} = C_n$ then $T_n \in B_n$ and, if $p(x) = \sum_{j=0}^{n} a_j x^j \in C_n$, then $|a_n| \le 2^{n-1}$ and the equality holds if and only if $p = \pm T_n$ (see [R], chapter 2). Here we prove the following

Proposition 1: *Let $p(x) = \sum_{j=0}^{n} a_j x^j \in C_n$ and let $T_n(x) = \sum_{j=0}^{n} t_j^{(n)} x^j$. If n is even then $|a_0| \le |t_0^{(n)}| = 1$ and, for $0 \le i \le \frac{n}{2} - 1$, $|a_{2i+1}| + |a_{2i+2}| \le |t_{2i+2}^{(n)}|$. Moreover, when $p(x) \in B_n$, $|a_{2i+1}| + |a_{2i+2}| = |t_{2i+2}^{(n)}|$ for some i with $0 \le i \le \frac{n}{2} - 1$ if and only if $p = \pm T_n$. If n is odd then, for $0 \le i \le \frac{n-1}{2}$, $|a_{2i}| + |a_{2i+1}| \le |t_{2i+1}^{(n)}|$, and when $p(x) \in B_n$, the equality holds for some i with $0 \le i \le \frac{n-1}{2}$ if and only if $p = \pm T_n$.*

We use this proposition to prove

Theorem 1: *Let $f\colon \mathbb{R}_+^{n+1} \to \mathbb{R}$ be a symmetric convex function, increasing in each variable. For $p(x) = \sum_{j=0}^{n} a_j x^j$ we define $\overline{f}(p) = f(|a_0|, |a_1|, \ldots, |a_n|)$. We have $\overline{f}(p) \leq \overline{f}(T_n)$ for every $p \in C_n \supset B_n$. When $p \in B_n$, the equality holds if and only if $p = \pm T_n$.*

If $p(x) = \sum_{j=0}^{n} a_j x^j$, we define $\ell(p) = \sum_{j=0}^{n} |a_j|$. We conclude from theorem 1 above that if $p \in C_n$ then $\ell(p) \leq \frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{2} = \ell(T_n)$, and, when $p \in B_n$, the equality holds if and only if $p = \pm T_n$ (Corollary 1). This improves a result from [RZ], where this fact is proved for even or odd polynomials $p(x) \in B_n$. The equality $\ell(T_n) = \frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{2}$ follows from $T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x)$, which implies $\ell(T_{n+2}) = 2\ell(T_{n+1}) + \ell(T_n)$ (since the non-zero coefficients of $T_n(x)$, which is even for n even and odd for n odd, alternate signs); notice that $\ell(T_0) = \ell(T_1) = 1$.

I would like to thank the participants of the list of discussion of problems of the Brazilian Mathematical Olympiad (see `www.mat.puc-rio.br/~nicolau/olimp/obm-l.html` and `www.obm.org.br/lista.htm`), particularly Eduardo Casagrande Stabel, who posed in the list the problem which motivated this note and Fábio Dias Moreira and Alex Corrêa Abreu, who gave valuable suggestions in the preparation of this manuscript. I also would like to thank Professors Ken McLaughlin, Dimitar Dimitrov, Vilmos Totik and Tamás Erdélyi, for the helpful conversations and suggestions about this work.

2 Proofs

Let us first prove Proposition 1. Let $p \in C_n$. Write $p(x) = g(x) + h(x)$, where $g(x) = \frac{p(x)+p(-x)}{2}$ is even and $h(x) = \frac{p(x)-p(-x)}{2}$ is odd. Clearly, g and h also belong to C_n. Since $|g(\cos(\frac{j\pi}{n})) + h(\cos(\frac{j\pi}{n}))| \leq 1$ and $|g(-\cos(\frac{j\pi}{n})) + h(-\cos(\frac{j\pi}{n}))| = |g(\cos(\frac{j\pi}{n})) - h(\cos(\frac{j\pi}{n}))| \leq 1$, we have $|g(\cos(\frac{j\pi}{n}))| + |h(\cos(\frac{j\pi}{n}))| \leq 1$, $0 \leq j \leq n$. We will compute $g(x)$ and $h(x)$ from their values in the points $w_j = \cos(\frac{j\pi}{n})$, $0 \leq j \leq n$. More precisely, we will write $g(x) = \tilde{g}(x^2)$, $h(x) = x\tilde{h}(x^2)$, and we will compute $\tilde{g}(x)$ from its values in w_j^2, $0 \leq j \leq \frac{n}{2}$ and $\tilde{h}(x)$ from its values in w_j^2, $0 \leq j < \frac{n}{2}$, by Lagrange interpolation (notice that $\deg(\tilde{g}) \leq \frac{n}{2}$ and $\deg(\tilde{h}) < \frac{n}{2}$). We will consider two cases:

i) n even, say $n = 2k$: In this cases, we have

$$\tilde{g}(x) = \sum_{j=0}^{k} \left(g(w_j) \cdot \prod_{\substack{r=0 \\ r\neq j}}^{k} \left(\frac{x - w_r^2}{w_j^2 - w_r^2} \right) \right) \quad \text{and} \quad \tilde{h}(x) = \sum_{j=0}^{k-1} \left(\frac{h(w_j)}{w_j} \cdot \prod_{\substack{r=0 \\ r\neq j}}^{k-1} \left(\frac{x - w_r^2}{w_j^2 - w_r^2} \right) \right),$$

so

$$g(x) = \sum_{j=0}^{k} \left(g(w_j) \cdot \prod_{\substack{r=0 \\ r\neq j}}^{k} \left(\frac{x^2 - w_r^2}{w_j^2 - w_r^2} \right) \right) \quad \text{and} \quad h(x) = x \sum_{j=0}^{k-1} \left(\frac{h(w_j)}{w_j} \cdot \prod_{\substack{r=0 \\ r\neq j}}^{k-1} \left(\frac{x^2 - w_r^2}{w_j^2 - w_r^2} \right) \right).$$

ii) n odd, say $n = 2k + 1$: In this case, we have

$$\tilde{g}(x) = \sum_{j=0}^{k} \left(g(w_j) \cdot \prod_{\substack{r=0 \\ r\neq j}}^{k} \left(\frac{x - w_r^2}{w_j^2 - w_r^2} \right) \right) \quad \text{and} \quad \tilde{h}(x) = \sum_{j=0}^{k} \left(\frac{h(w_j)}{w_j} \cdot \prod_{\substack{r=0 \\ r\neq j}}^{k} \left(\frac{x - w_r^2}{w_j^2 - w_r^2} \right) \right),$$

so

$$g(x) = \sum_{j=0}^{k} \left(g(w_j) \cdot \prod_{\substack{r=0 \\ r\neq j}}^{k} \left(\frac{x^2 - w_r^2}{w_j^2 - w_r^2} \right) \right) \quad \text{and} \quad h(x) = x \sum_{j=0}^{k} \left(\frac{h(w_j)}{w_j} \cdot \prod_{\substack{r=0 \\ r\neq j}}^{k} \left(\frac{x^2 - w_r^2}{w_j^2 - w_r^2} \right) \right).$$

Notice that, since $w_i > w_{i+1}$ for $0 \leq i < i + 1 \leq \frac{n}{2}$, the signs of $\prod_{\substack{r=0 \\ r\neq j}}^{k} (w_j^2 - w_r^2)$ and of $\prod_{\substack{r=0 \\ r\neq j}}^{k-1} (w_j^2 - w_r^2)$ are $(-1)^j$. On the other hand, $\prod_{\substack{r=0 \\ r\neq j}}^{\ell} (x^2 - w_r^2)$ is an even polynomial of degree 2ℓ with alternating coefficients (the sign of the coefficient of $x^{2\ell-2u}$ is $(-1)^u$, for $0 \leq u \leq \ell$; all of these coefficients are non-zero, with the possible exception of the constant coefficient), where $\ell \in \{k - 1, k\}$. For $p(x) = \sum_{j=0}^{n} a_j x^j$, and $0 \leq i < n$, we define $\ell_i(p) = |a_i| + |a_{i+1}|$.

In the case i), if $p(x) = T_n(x)$, we have $h(x) \equiv 0$ and $g(w_j) = \cos(j\pi) = (-1)^j$, for $0 \leq j \leq k$, so

$$\ell_i(T_n) = \sum_{j=0}^{k} \ell_i \left(\prod_{\substack{r=0 \\ r\neq j}}^{k} \left(\frac{x^2 - w_r^2}{w_j^2 - w_r^2} \right) \right), \quad \forall\, i < n$$

by the above remarks (since in this case $g(w_j)\cdot\prod_{\substack{r=0\\r\neq j}}^{k}(\frac{1}{w_j^2-w_r^2})$ is positive for every j, with $0\le j\le k$). On the other hand, for an arbitrary polynomial $p\in C_n$, $|g(w_j)|+|h(w_j)|\le 1$, and, for $0\le j\le k-1$,

$$\prod_{\substack{r=0\\r\neq j}}^{k}\left(\frac{x^2-w_r^2}{w_j^2-w_r^2}\right)=\frac{x^2}{w_j^2}\cdot\prod_{\substack{r=0\\r\neq j}}^{k-1}\left(\frac{x^2-w_r^2}{w_j^2-w_r^2}\right)=\frac{x}{w_j}\left(\frac{x}{w_j}\cdot\prod_{\substack{r=0\\r\neq j}}^{k-1}\left(\frac{x^2-w_r^2}{w_j^2-w_r^2}\right)\right),$$

so

$$\ell_{2i+1}\left(\prod_{\substack{r=0\\r\neq j}}^{k}\left(\frac{x^2-w_r^2}{w_j^2-w_r^2}\right)\right)=\frac{1}{w_j}\ell_{2i+1}\left(\frac{x}{w_j}\cdot\prod_{\substack{r=0\\r\neq j}}^{k-1}\left(\frac{x^2-w_r^2}{w_j^2-w_r^2}\right)\right),\quad\forall\, i<k;$$

in particular,

$$\ell_{2i+1}\left(\prod_{\substack{r=0\\r\neq j}}^{k}\left(\frac{x^2-w_r^2}{w_j^2-w_r^2}\right)\right)\ge\ell_{2i+1}\left(\frac{x}{w_j}\cdot\prod_{\substack{r=0\\r\neq j}}^{k-1}\left(\frac{x^2-w_r^2}{w_j^2-w_r^2}\right)\right),$$

and the equality holds if and only if $j=0$. This implies that, if $p(x)=\sum_{j=0}^{n}a_jx^j$ and $T_n(x)=\sum_{j=0}^{n}t_j^{(n)}x^j$, then $|a_0|\le|t_0^{(n)}|=1$ and, for $0\le i<\frac{n}{2}$, $|a_{2i+1}|+|a_{2i+2}|\le|t_{2i+2}^{(n)}|$. If $|a_{2i+1}|+|a_{2i+2}|=|t_{2i+2}^{(n)}|$ for some i with $0\le i<\frac{n}{2}$, we should have $|g(w_j)|=1$ and $h(w_j)=0$ for $1\le j\le k$, and $g(w_{j+1})=-g(w_j)$ for $1\le j<j+1\le k$. We should also have $|g(w_0)|+|h(w_0)|=1$, and $g(w_0)g(w_1)\le 0$. This implies that $p(x)=\pm\left(T_n(x)+cx(x-1)\prod_{r=1}^{k-1}\left(\frac{x^2-w_r^2}{1-w_r^2}\right)\right)$ or $p(x)=\pm\left(T_n(x)+cx(x+1)\prod_{r=1}^{k-1}\left(\frac{x^2-w_r^2}{1-w_r^2}\right)\right)$, for some $c\in\mathbb{R}$. If $c\neq 0$, we would have $|p(w_1)|=|T_n(w_1)|=1$ and $p'(w_1)\neq 0$, which is not possible when $p\in B_n$.

In the case ii), the analysis is analogous: we use the fact that

$$\ell_{2i}\left(\frac{x}{w_j}\prod_{\substack{r=0\\r\neq j}}^{k}\left(\frac{x^2-w_r^2}{w_j^2-w_r^2}\right)\right)=\frac{1}{w_j}\cdot\ell_{2i}\left(\prod_{\substack{r=0\\r\neq j}}^{k}\left(\frac{x^2-w_r^2}{w_j^2-w_r^2}\right)\right)\ge\ell_{2i}\left(\prod_{\substack{r=0\\r\neq j}}^{k}\left(\frac{x^2-w_r^2}{w_j^2-w_r^2}\right)\right),$$

for $0\le i\le k$, $0\le j\le k$ (the equalities hold only if $j=0$), and so, if $p(x)=\sum_{j=0}^{n}a_jx^j$ and $T_n(x)=\sum_{j=0}^{n}t_j^{(n)}x^j$, then, for $0\le i\le k=\frac{n-1}{2}$, $|a_{2i}|+|a_{2i+1}|\le|t_{2i+1}^{(n)}|$ (the only case in which some of these equalities hold is when $p=\pm T_n$). This concludes the proof of Proposition 1. $\square$

Proof of Theorem 1:

We have $\overline{f}(p) = f(|a_0|, |a_1|, \ldots, |a_n|) \leq f(1, |a_1|, |t_2^{(n)}| - |a_1|, \ldots, |a_{n-1}|, |t_n^{(n)}| - |a_{n-1}|) \leq f(1, 0, |t_2^{(n)}|, 0, |t_4^{(n)}|, \ldots, 0, |t_n^{(n)}|) = \overline{f}(T_n)$. Here we used the fact that, if $g(x,y)$ is a convex symmetric function then $g(x,y) \leq g(x+y, 0)$, for $x, y > 0$, since

$$\begin{aligned} g(x,y) &= g\left(\frac{x}{x+y}(x+y,0) + \frac{y}{x+y}(0,x+y)\right) \\ &\leq \frac{x}{x+y} g(x+y,0) + \frac{y}{x+y} g(0,x+y) = g(x+y,0), \end{aligned}$$

since $g(x+y,0) = g(0,x+y)$. Notice that since f is strictly increasing in each variable, the equality only holds when $p = \pm T_n$.

The argument in the case ii) is analogous. $\square$

Corollary 1: If $p \in B_n$ then $\ell(p) \leq \ell(T_n) = \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2}$, and the equality holds if and only if $p = \pm T_n$.

Corollary 2: If $p(x) = \sum\limits_{j=0}^{n} a_j x^j \in B_n$ then $\ell^{(\alpha)}(p) := \sum\limits_{j=0}^{n} |a_j|^\alpha \leq \sum\limits_{j=0}^{n} |t_j^{(n)}|^\alpha = \ell^{(\alpha)}(T_n)$, for every $\alpha \geq 1$. The equality holds if and only if $p = \pm T_n$.

Remark: The conclusion of Corollary 2 is false if $0 < \alpha < 1$. Indeed, if we take $f_n(x) = \frac{x+1}{2} + (-1)^n \cdot \frac{x-1}{2}$, then for $\varepsilon \in (0,1)$, $(1-\varepsilon)T_n + \varepsilon f_n \in B_n$, and $\ell^{(\alpha)}((1-\varepsilon)T_n + \varepsilon f_n) = = (1-\varepsilon)^\alpha \ell^{(\alpha)}(T_n) + \varepsilon^\alpha > \ell^{(\alpha)}(T_n)$, if $\varepsilon > 0$ is small.

3 General comments and Problems

Professor Tamás Erdélyi remarked that if $0 \leq a < b$, $B_n(a,b) = \{p(x) \in \mathbb{R}[x] \mid |p(x)| \leq 1,\ \forall\, x \in [a,b]\}$ and $T_n^{(a,b)}(x) = T\left(\frac{2x-(a+b)}{b-a}\right) = \sum\limits_{j=0}^{n} t_j^{(n,a,b)} x^j$ is the normalized Chebyshev polynomial then, for any $p(x) = \sum\limits_{j=0}^{n} a_j x^j \in B_n(a,b)$ we have $|a_j| \leq |t_j^{(n,a,b)}|$ for $0 \leq j \leq n$. Prof. Erdélyi's argument is as follows: If $0 < a < b$ and $0 \leq j \leq n$ then $\{x^r \mid 0 \leq r \leq n,\ r \neq j\}$ is a Chebyshev system with respect to $[a,b]$ (this is easy to check using induction is n, by considering the derivative of a polynomial in the space generated by such a set). Now, by Theorem 2.10 of [R], the best approximation on $[a,b]$ to $\lambda t_j^{(n,a,b)}x^j$ by polynomials of the form $\sum_{i=0}^{j-1} a_i x^i + \sum_{\ell=j+1}^{n} a_\ell x^\ell$ is $\lambda\left(t_j^{(n,a,b)}x^j - T_n^{(a,b)}(x)\right)$, for any $\lambda \neq 0$. In particular, if $|\lambda| > 1$ then the maximum norm of this best approximation is larger than 1, and this clearly implies the result. The case $a = 0$ follows by continuity. We remark that this result also follows from the technique of the previous section, by using Lagrange interpolation at the points $w_k = w_k^{(a,b)} = \frac{(b-a)\cos\left(\frac{k\pi}{n}\right)+a+b}{2}$, $0 \leq k \leq n$ to write $p(x)$, and observing that the norm of each coefficient is maximized when $|p(w_k)| = 1$ for $0 \leq k \leq n$ and $p(x_{w+1}) = -p(w_k)$ for $0 \leq k \leq n-1$ (here it is not necessary to decompose $p(x)$ in its even and odd part, as we did before). Notice that the same arguments can be used to show the (strongest) results obtained b y changing $B_n(a,b)$ by $C_n(a,b) = \{p(x) \in \mathbb{R}[x] \mid |p(w_k^{(a,b)})| \leq 1,\ 0 \leq k \leq n\}$ in the hypothesis.

Prof. Erdélyi also observed that this result for $[a,b] = [0,1]$ gives another proof to our estimates in $[-1,1]$: let $f(x) \in B_n = B_n(-1,1)$ (the argument for $f(x) \in C_n = C_n(-1,1)$ is analogous). Write $f(x) = \tilde{g}(x^2) + x\tilde{h}(x^2)$ as in Section 2.

Let us first consider the case when the degree of $f(x)$ is even, say $2n : f(x) = \sum_{k=0}^{2n} a_k x^k$. We consider now the polynomials $p_1(x) = \tilde{g}(x^2) + x^2\tilde{h}(x^2)$ and $p_2(x) = \tilde{g}(x^2) - x^2\tilde{h}(x^2)$. Notice that by the argument in the beggining of Section 2, $|p_i(x)| \leq |\tilde{g}(x^2)| + |x^2\tilde{h}(x^2)| \leq |\tilde{g}(x^2)| + |x\tilde{h}(x^2)| \leq 1$, $\forall\, x \in [-1,1]$, $i = 1,2$. We can write $p_i(x) = \tilde{p}_i(x^2)$, $i = 1,2$, where

$$\tilde{p}_1(x) = a_0 + \sum_{k=1}^{n}(a_{2k-1} + a_{2k})x^k, \quad \tilde{p}_2(x) = a_0 + \sum_{k=1}^{n}(a_{2k} - a_{2k-1})x^k.$$

Now, since $\tilde{p}_i \in B_n(0,1)$, $i = 1,2$, we have $|a_{2k} + a_{2k-1}| \leq |t_k^{(n,0,1)}|$ and $|a_{2k} - a_{2k-1}| \leq |t_k^{(n,0,1)}|$, and so $|a_{2k}| + |a_{2k-1}| \leq |t_k^{(n,0,1)}|$ for $0 \leq k \leq n$. On the other hand, $T_n^{(0,1)}(x) = T_n(2x-1)$, so $T_n^{(0,1)}(x^2) = T_n(2x^2-1) = T_n \circ T_2(x) = T_{2n}(x)$, and so $|t_k^{(n,0,1)}| = |t_{2k}^{(2n)}|$, thus we get $|a_0| \leq 1$, $|a_{2i+1}| + |a_{2i+2}| \leq |t_{2i+2}^{(2n)}|$ for $0 \leq i < \frac{n}{2}$, and we can conclude as in Section 2.

For the case where the degree of $f(x)$ is odd, say $2n+1$: $f(x) = \sum_{k=0}^{2n+1} a_k x^k$, we consider the polynomials $p_1(x) = x\tilde{g}(x^2) + x\tilde{h}(x^2)$ and $p_2(x) = x\tilde{g}(x^2) - x\tilde{h}(x^2)$. We have $|p_i(x)| \leq |x\tilde{g}(x^2)| + |x\tilde{h}(x^2)| \leq |\tilde{g}(x^2)| + |x\tilde{h}(x^2)| \leq 1$, $\forall\, x \in [-1,1]$, $i = 1,2$. Notice now that $\{x^{2r+1},\ 0 \leq r \leq n,\ r \neq j\}$ is a Chebyshev system with respect to $\left[\cos\left(\frac{n\pi}{2n+1}\right), 1\right]$ for any j with $0 \leq j \leq n$, and so the best approximation on $\left[\cos\left(\frac{n\pi}{2n+1}\right), 1\right]$ to $t_{2j+1}^{(2n+1)} \cdot x^{2j+1}$ is $\left(t_{2j+1}^{(2n+1)} \cdot x^{2j+1} - T_{2j+1}(x)\right)$.

Since $p_1(x) = \sum_{j=0}^{n}(a_{2j} + a_{2j+1})x^{2j+1}$ and $p_2(x) = \sum_{j=0}^{n}(a_{2j} - a_{2j+1})x^{2j+1}$, this implies, as before, $|a_{2j}| + |a_{2j+1}| \le |t_{2j+1}^{(2n+1)}|$ for $0 \le j \le n$, and we conclude again as in Section 2.

Now we will consider the corresponding problems for general intervals $[a, b]$. For instance, we want to find polynomials $f(x) = \sum_{j=0}^{n} a_j x^j$ belonging to $B_n(a, b) = \{p(x) = \sum_{j=0}^{n} b_j x^j \in \mathbb{R}[x] \mid |p(x)| \le 1, \ \forall\, x \in [a, b]\}$ which maximize $\ell(f) = \sum_{j=0}^{n} |a_j|$.

Let us first notice that if $[a, b] = [-c, c]$, with $0 < c \le 1$, and $f \in C_n(a, b) \supset B_n(a, b)$ then $\tilde{f}(x) = f(cx)$ belongs to $C_n(1, 1) = C_n$. If $f(x) = \sum_{j=0}^{n} a_j x^j$ then $\tilde{f}(x) = \sum_{j=0}^{n} a_j \cdot c^j \cdot x^j$.

If n is even, Proposition 1 gives $|a_{2i+1} \cdot c^{2i+1}| + |a_{2i+2} \cdot c^{2i+2}| \le |t_{2i+2}^{(n)}|$ for $0 \le i < \frac{n}{2}$, and so $|a_{2i+1}| + |a_{2i+2}| \le \frac{|a_{2i+1}|}{c} + |a_{2i+2}| \le \frac{|t_{2i+2}^{(n)}|}{c^{2i+2}} = |t_{2i+2}^{(n,-c,c)}|$, since $T_n^{(-c,c)}(x) = T_n(\frac{x}{c})$. If n is odd, we have $|a_{2i} \cdot c^{2i}| + |a_{2i+1} \cdot c^{2i+1}| \le |t_{2i+1}^{(n)}|$ for $0 \le i < \frac{n}{2}$, and so $|a_{2i}| + |a_{2i+1}| \le \frac{|a_{2i}|}{c} + |a_{2i+1}| \le \frac{|t_{2i+1}^{(n)}|}{c^{2i+1}} = |t_{2i+1}^{(n,-c,c)}|$. This implies that Theorem 1 holds if we replace $B_n and C_n$ by $B_n(-c, c) and C_n(-c, c)$ and T_n by $T_n^{(-c,c)}$ when $0 < c \le 1$, and, in particular, $\ell(f) \le \ell(T_n^{(-c,c)})$, for every $f \in C_n(-c, c) \supset B_n(-c, c)$.

However, the situation is quite different for general intervals $[a, b]$, and even for symmetric intervals $[-c, c]$, with $c > 1$. For instance, we have the following

Proposition 2: For any $c > 1$ and $n \ge 1$, there is $f(x) \in B_n(-c, c)$ such that $\ell(f) > \ell(T_n^{(-c,c)})$.

Proof: We will show that if $\delta > 0$ is small enough then $\ell(T_n^{(-c,c)} = \ell(T_n(\frac{x}{c})) < \ell(T_n(\frac{x+\delta}{c+\delta}))$. Notice that $x \in [-c, c] \Rightarrow \frac{x+\delta}{c+\delta} \in [\frac{\delta-c}{c+\delta}, 1] \subset [-1, 1]$, and so $T_n(\frac{x+\delta}{c+\delta}) \in B_n(-c, c)$.

In order to do this, notice that $T_n(\frac{x+\delta}{c+\delta}) = \sum_{j=0}^{n} t_n^{(n)}(\frac{x+\delta}{c+\delta})^j = \sum_{j=0}^{n} \frac{t_j^{(n)}}{c^n} \cdot \frac{(x^j + j\delta x^{j-1} + O(\delta^2))}{(1 + \frac{j\delta}{c} + O(\delta^2))} = \sum_{j=0}^{n} \frac{t_n^{(n)}}{c^n} \cdot (x^j + j\delta x^{j-1} + O(\delta^2))(1 - \frac{j\delta}{c} + O(\delta^2)) = \sum_{j=0}^{n} \frac{t_j^{(n)}}{c^j}(x^j + j\delta(\frac{c-1}{c})c^{j-1}) + O(\delta^2)$. Therefore, since T_n is an even or odd polynomial, we have $\ell(T_n(\frac{x+\delta}{c+\delta})) = \sum_{j=0}^{n} \frac{|t_j^{(n)}|}{c^j} + \delta(\frac{c-1}{c}) \sum_{j=0}^{n} \frac{j|t_j^{(n)}|}{c^j} + O(\delta^2) > \sum_{j=0}^{n} \frac{|t_j^{(n)}|}{c^j} = \ell(T_n^{(-c,c)})$ for δ small enough. $\square$

Examples: If $c > 1$ and $n = 1$ then, $\ell(f)$ for $f \in B_n(-c, c)$ is maximum when $f(x) \equiv 1$

or $f(x) \equiv -1$. Indeed, if $f(x) = ax + b \in B_n(-c, c)$ then $|ax| + |b| \leq 1$, $\forall\, x \in [-c, c]$, so $|a| + |b| \leq |ac| + |b| \leq 1$.

If $c > 1$ and $n = 2$ then $\ell(f)$ for $f \in B_n(-c, c)$ is maximum when

$$f(x) = \pm\frac{2(c+1)^2}{(c^2+2c-1)^2}(\pm x - \frac{c-1}{c+1})^2 - 1.$$

Indeed, by the Exercise 2.6.11 of [R], we may restrict our attention to polynomials $f(x) \in B_n(-c, c)$ such that the total multiplicity with which the values 1 and -1 are assumed by $f(x)$, in $[-c, c]$ is at least 3. By considering $\pm f(-x)$, we may suppose that $f(-c) = 1$ and $f(\alpha) = -1$, $f'(\alpha) = 0$, for some $\alpha \in [0, c]$, so $f(x) = \frac{2}{(\alpha+c)^2} \cdot (x - \alpha)^2 - 1$. We have $\ell(f) = \frac{2}{(\alpha+c)^2} \cdot (1 + 2\alpha) + |\frac{2\alpha^2}{(\alpha+c)^2} - 1| = \frac{2}{(\alpha+c)^2} \cdot (1 + 2\alpha) + 1 - \frac{2\alpha^2}{(\alpha+c)^2}$, which is maximum for $\alpha = \frac{c-1}{c+1}$. In this case, $\ell(f) = 1 + \frac{4}{c^2+2c-1}$. For instance, when $c = 2$, $f(x) = (\frac{18}{49})(x - \frac{1}{3})^2 - 1 = (\frac{18}{49})x^2 - (\frac{12}{49})x - \frac{47}{49}$ and $\ell(f) = \frac{11}{7} > \frac{3}{2} = \ell(\frac{x^2}{2} - 1) = \ell(T_2^{(-2,2)})$.

In general, it is an interesting (and apparently difficult) problem to characterize the polynomials $f(x) \in B_n(a, b)$ which maximize $\ell(f)$ for general intervals $[a, b]$ (and even for symmetric intervals $[-c, c](x)$.

4 References

[R] Theodore J. Rivlin, Chebyshev Polynomials, Wiley Interscience, 1990.

[RZ] M. Reimer and K. Zeller, Abschätzung der Teilsummen reeler Polynome, Math. Zeitschr. **99** (1967), 101–104.

Instituto de Matemática Pura e Aplicada - IMPA

Estrada Dona Castorina, 110, Jardim Botânico

22460-320, Rio de Janeiro - Brasil

e-mail: gugu@impa.br