Differentiable Structures of Central Cantor Sets

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Abstract

Central Cantor sets form a class of symmetric Cantor sets of the real line. Here we give a complete characterization of $C^{k+\alpha}$–regularity of these Cantor sets. We also give a classification of central Cantor sets up to global and local diffeomorphisms. Examples of central Cantor sets with special dynamical and measure–theoretical properties are also provided. Finally we calculate the fractal dimensions of an arbitrary central Cantor set.

Introduction. Central Cantor sets in the real line form a particular but interesting class of Cantor sets. They are constructed in a simple way, and are useful in providing examples in several mathematical contexts. For instance, in analysis they can be used to prove the sharpness of some recent results concerning the clasical Morse-Sard Theorem [M2]. Even more abundant are their applications in Dynamical Systems. In particular, they were used by R. Bowen [Bo] to construct a horseshoe of class C^1 with positive Lebesgue measure (see also our example IV.1), and by Samanni [Sa] and by Bamón–Plaza–Vera, [B-P-V] to give examples of regular Cantor sets whose self–arithmetic difference is a Cantor set with positive measure. Here we discuss a general class of such examples (see example IV.2). This is related to a conjecture of Palis (see [P]).

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Regular or dynamically defined Cantor sets constitutes an important class of subsets of the real line that naturally appear in dynamics (see [P-T], [P-T,1], [P-T,2], [P1], [P2], [M1], [N]). In the present work, we give a complete classifications, for any integer $k \geq 1$ and $\alpha \in [0,1]$, of central Cantor set that admit a structure of regular Cantor sets of class $C^{k+\alpha}$ (Theorem A), as well as a classification of central Cantor sets up to global and local diffeomorphisms, Theorem B. The present work, although self-contained, is a natural continuation of [B-P-V].

We are able to find the explicit and simple conditions for the characterization and classification of $C^{k+\alpha}$ regularity of central Cantot sets, given in Theorems A and B, because of the simple and symmetrical structure of this class of Cantor sets. This contrasts with the conditions for regularity of more general classes of Cantor sets in the line. See [S], [B-F], [P-R], [Pr-T].

I- Basic definitions and Statement of Results

Let $\underline{\lambda} = (\lambda_k)_{k\in\mathbb{N}^*}$ be a sequence of real numbers satisfying $0 < \lambda_k < \frac{1}{2}$, for all $k \in \mathbb{N}^*$, and let $I \subset \mathbb{R}$ be a closed interval. We define *the central Cantor set* $K_{\underline{\lambda},I}$ *associated to the sequence* $\underline{\lambda}$ *with support interval* I as follows: Let $P^0_{\underline{\lambda},I} = I$. For each $i \in \mathbb{N}^*$, $P^i_{\underline{\lambda},I}$ is the set consisting of 2^i closed disjoint intervals, obtained from $P^{i-1}_{\underline{\lambda},I}$ by removing from each of its intervals the central open interval of proportion $1-2\lambda_i$. We call these removed intervals the *gaps of the* i*–th step* of the construction of $K_{\underline{\lambda},I}$, and the intervals of $P^i_{\underline{\lambda},I}$ the *fundamental intervals* of the i–th step of the construction of $K_{\underline{\lambda},I}$. Finally define $K_{\underline{\lambda},I} = \cap_{i=0}^{\infty} P^i_{\underline{\lambda},I}$.

Let $a = |I|$ be the lenght of the interval I. Then the lenght of any n–th step fundamental interval is $a \cdot \pi_n$, where $\pi_n = \pi_n(\underline{\lambda}) = \Pi_{i=1}^n \lambda_i$, and the lenght of any n–th step gap is $a \cdot g_n$, where $g_n = g_n(\underline{\lambda}) = (1-2\lambda_n)\pi_{n-1}$.

Case we do not mention the interval I we will assume $I = [0,1]$, and we will use the notations $K_{\underline{\lambda}}$ and $P^i_{\underline{\lambda}}$ instead of $K_{\underline{\lambda},I}$ and $P^i_{\underline{\lambda},I}$.

Now let us define an important class of Cantor sets, which naturally appear in Dynamical Systems. We say that a Cantor set $K \subset \mathbb{R}$ is *dynamically defined or regular* if:

i) there exist compact disjoint intervals $\overline{K}_1, \overline{K}_2, \ldots, \overline{K}_r$ such that $K \subset \overline{K}_1 \cup \overline{K}_2 \cup \cdots \cup \overline{K}_r$ and the boundary of each $\overline{K}_i$ is contained in K;

ii) there is an expanding map $\psi : \overline{K}_1 \cup \overline{K}_2 \cup \cdots \cup \overline{K}_r \to \mathbb{R}$ such that:

a) for each $i = 1, \ldots, r$, $\psi(\overline{K}_i)$ is the convex hull of the union of some intervals $\overline{K}_j$'s;

b) for each $i = 1, \ldots, r$, and n large enough, $\psi^n(K \cap \overline{K}_i) = K$;

c) $K = \cap_{n=0}^{\infty} \psi^{-n} \left(\overline{K}_1 \cup \overline{K}_2 \cup \cdots \cup \overline{K}_r\right)$.

We call $\{\overline{K}_1, \overline{K}_2, \ldots, \overline{K}_r\}$ a *Markov partition* for K and $D = \overline{K}_1 \cup \overline{K}_2 \cup \cdots \cup \overline{K}_r$ a *Markov domain of* K.

We say that a Cantor set is a *regular of class* C^k *(resp.* $C^{k+\alpha}$*)* if ψ is a C^k (resp. $C^{k+\alpha}$) function. In many applications we restrict our attention to regular Cantor sets which are at least $C^{1+\varepsilon}$, for some $\varepsilon > 0$.

The main goal of this work is to characterize the central Cantor sets which are regular of a given class of differentiability. We also classify central Cantor sets up to global and local diffeomorphisms. We have the following results.

Theorem A. *Let K be a central Cantor set defined by the sequence $\underline{\lambda} = (\lambda_i)_{i \in \mathbb{N}^*}$. Then K admits a structure of regular Cantor set (of class at least C^1) if and only if there is $r \in \mathbb{N}^*$ such that the following conditions hold:*

a) $\displaystyle\lim_{n \to \infty} \frac{\lambda_{n+r}}{\lambda_n} = 1$;

b) $\displaystyle\lim_{n \to \infty} \frac{1 - 2\lambda_{n+r}}{1 - 2\lambda_n} = 1$;

c) the sequence $(\lambda_{n+1} \cdot \lambda_{n+2} \cdots \lambda_{n+r})_{n \in \mathbb{N}}$ converges to some $\lambda \neq 0$.

Let $r_0 = \min\{r \in \mathbb{N}^* :$ such that conditions a), b) and c) above hold$\}$, and let $k \in \mathbb{N}^*$. Then K admits a structure of regular Cantor set of class C^k if, and only if, the following equivalent conditions hold:

i) $\displaystyle\lim_{n \to \infty} \frac{|\lambda_{n+1} \cdot \lambda_{n+2} \cdots \lambda_{n+r_0} - \lambda|}{\pi_n^{k-1}} = 0$, *where* $\displaystyle\lambda = \lim_{n \to \infty} \lambda_{n+1} \cdots \lambda_{n+r_0}$;

ii) $\displaystyle\lim_{n \to \infty} \frac{|\lambda_{n+1} \cdot \lambda_{n+2} \cdots \lambda_{n+r_0} - \lambda|}{\lambda^{(k-1)n/r_0}} = 0$.

If $k > 1$ then these conditions are also equivalent to the following:

iii) *For each $j = 1, 2, \ldots, r_0$ the sequence $(\lambda_{r_0 n+j})_{n\in\mathbb{N}}$ converges to some $\mu_j \in (0, \frac{1}{2})$ such that $\lim\limits_{n\to\infty} \dfrac{|\lambda_{r_0 n+j} - \mu_j|}{\lambda^{(k-1)n}} = 0, \quad j = 1, 2, \ldots, r_0$. We have $\lambda = \mu_1 \cdot \mu_2 \cdots \mu_{r_0}$.*

Let $k \in \mathbb{N}^*$ and $\alpha \in (0, 1]$. Then K admits a structure of regular Cantor set of class $C^{k+\alpha}$ if and only if the following equivalent conditions hold:

i') $\limsup\limits_{n\to\infty} \dfrac{|\lambda_{n+1} \cdots \lambda_{n+r_0} - \lambda|}{\pi_n^{k-1+\alpha}} < \infty$;

ii') $\limsup\limits_{n\to\infty} \dfrac{|\lambda_{n+1} \cdots \lambda_{n+r_0} - \lambda|}{\lambda^{(k-1+\alpha)n/r_0}} < \infty$;

iii') $\limsup\limits_{n\to\infty} \dfrac{|\lambda_{r_0 n+j} - \mu_j|}{\lambda^{(k-1+\alpha)n}} < \infty, \qquad j = 1, 2, \ldots, r_0$.

Remarks. • Condition iii) above does not necessarily hold when $k = 1$. In fact, we will show that if $0 < a \le b \le \frac{1}{2}$ then there is a sequence $\underline{\lambda} = (\lambda_n)_{n\in\mathbb{N}}$ dense in $[a, b]$ such that $K_{\underline{\lambda}}$ is C^1. (see example IV.3).

• It follows from above that $\mu_j \neq 0, \quad j = 1, 2, \ldots, r_0$, since $\mu_1 \cdot \mu_2 \cdots \mu_{r_0} = \lambda$. Moreover, if K is regular of class $C^q, \;\; q = k$ or $q = k + \alpha$, with $q > 1$, then $\mu_j < \frac{1}{2}$, $j = 1, 2, \ldots, r_0$.

• The condition in order for K to admit a structure of regular Cantor set of class C^∞ is that the equivalent conditions i), ii) and iii) must be satisfied for each $k \in \mathbb{N}^*$.

• We will use Theorem A to provide examples, for each $q \ge 1$, $q = k$ or $q = k + \alpha$ and $c \in [0, 2)$, of central Cantor sets $K_{\underline{\lambda}}$ which are of class C^q but not of class $C^{q+\varepsilon}$, for all $\varepsilon > 0$, such that $K_{\underline{\lambda}} - K_{\underline{\lambda}} = \{\ x - y \ : \ x, y \in K_{\underline{\lambda}}\ \}$ is a Cantor set of c Lebesgue measure.

• To endow K with a regular structure we need a Markov partition with at least 2^{r_0} intervals. For example the central Cantor set $\{1/3, 1/4, 1/3, 1/4, 1/3, ...\}$ is C^r for every $r \in \mathbb{N} \cup \{\omega\}$ with a Markov partition of four intervals: $P^2_{\underline{\lambda}}$.

Theorem B. *Let $K_{\underline{\lambda},I}$ and $K_{\underline{\tilde{\lambda}},J}$ be two Central Cantor sets. Then there is a C^1 diffeomorphism $\varphi : I \to J$ such that $\varphi(K_{\underline{\lambda},I}) = K_{\underline{\tilde{\lambda}},J}$ if and only if the following conditions hold:*

i) $\lim_{n\to\infty} \dfrac{1-2\tilde{\lambda}_n}{1-2\lambda_n} = 1;$

ii) $\lim_{n\to\infty} \dfrac{\pi_n(\tilde{\underline{\lambda}})}{\pi_n(\underline{\lambda})} = a \neq 0 \, .$

In this case there is such a diffeomorphism satisfying $\varphi'(x) = a\,\frac{|J|}{|I|}$ *, for all* $x \in K_{\underline{\lambda},I}$ *, and* $\varphi\left(P^r_{\underline{\lambda},I}\right) = P^r_{\tilde{\underline{\lambda}},J}$ *, for all* $r \in \mathbb{N}$ *, i.e.,* φ *preserves the structure of central Cantor sets.*

Let $k \in \mathbb{N}^*$*. There is such a diffeomorphism of class* C^k *if and only if moreover*

iii) $\lim_{n\to\infty} \dfrac{\left|g_n(\tilde{\underline{\lambda}}) - a\, g_n(\underline{\lambda})\right|}{\left(g_n(\underline{\lambda})\right)^k} = 0 \, .$

If $\alpha \in (0,1]$ *then there is such a diffeomorphism of class* $C^{k+\alpha}$ *if and only if moreover*

iii') $\limsup_{n\to\infty} \dfrac{\left|g_n(\tilde{\underline{\lambda}}) - a\, g_n(\underline{\lambda})\right|}{\left(g_n(\underline{\lambda})\right)^{k+\alpha}} < \infty.$

Necessary and sufficient conditions for the existence of a local diffeomorphism (resp. a local diffeomorphism of class C^q, $q = k$ or $q = k+\alpha$, $k \in \mathbb{N}^*$, $\alpha \in (0,1]$), $\varphi : U \to V$ such that $\varphi(U \cap K_{\underline{\lambda},I}) = V \cap K_{\tilde{\underline{\lambda}},J}$, where U and V are open intervals and $U \cap K_{\underline{\lambda},I} \neq \emptyset$ are that the corresponding conditions above should be verified for some $\tilde{\underline{\lambda}}^{(p)}$, $p \in \mathbb{Z}$, instead of $\tilde{\underline{\lambda}}$ where $\tilde{\underline{\lambda}}^{(p)} = \tilde{\lambda}_{n+p}$, if $n+p \geq 1$, and $\tilde{\lambda}^{(p)}_n = 1/3$, otherwise.

As a consequence of Theorem B, we conclude that if $K_{\underline{\lambda}}$ has a structure of C^q-regular Cantor set, $q = k$ or $q = k+\alpha$, $k \in \mathbb{N}^*$, $\alpha \in (0,1]$, and if $q > 1$ then there is a C^q diffeomorphism $\varphi : [0,1] \to [0,1]$ such that $\varphi(K_{\underline{\lambda}}) = K_{\tilde{\underline{\lambda}}}$, where $\tilde{\underline{\lambda}}$ is the periodic sequence defined by $\tilde{\lambda}_n = \mu_{\overline{n}}$, where $\overline{n} \in \{1, 2, \ldots, r_0\}$ satisfy $n \equiv \overline{n} \pmod{r_0}$, $\mu_{\overline{n}}$ being as in Theorem A. Note that $K_{\tilde{\underline{\lambda}}}$ is an affine Cantor set in the sense of [P-T].

Finally we compute the fractal dimensions of a central Cantor set: for each interval I and sequence $\underline{\lambda} = (\lambda_n)_{n\in\mathbb{N}}$,

- $\tau(K_{\underline{\lambda},I}) = \tau(K_{\underline{\lambda}}) = \inf_{n\in\mathbb{N}} \dfrac{\lambda_n}{1-2\lambda_n}$;

- $\tau_{loc}(K_{\underline{\lambda},I}) = \tau_{loc}(K_{\underline{\lambda}}) = \liminf\limits_{n\to\infty} \dfrac{\lambda_n}{1-2\lambda_n}$;

- $d(K_{\underline{\lambda},I}) = d(K_{\underline{\lambda}}) = \limsup\limits_{n\to\infty} \dfrac{-n\,\log 2}{\log \pi_n(\underline{\lambda})}$;

- $HD(K_{\underline{\lambda},I}) = HD(K_{\underline{\lambda}}) = \liminf\limits_{n\to\infty} \dfrac{-n\,\log 2}{\log \pi_n(\underline{\lambda})}$.

II - Structure of diffeomorphisms of central Cantor sets

In this section we shall prove results that characterize combinatorially diffeomorphisms and local diffeomorphisms of central Cantor sets.

Proposition 1 *Let $K_1 = K_{\underline{\lambda},J_1}$ and $K_2 = K_{\underline{\tilde{\lambda}},J_2}$ be two central Cantor sets, and let U, V be two intervals such that $\partial U \cap K_1 = \emptyset$ and $\partial V \cap K_2 = \emptyset$. Let $\varphi : U \to V$ be a C^1 diffeomorphism such that $\varphi(U\cap K_1) = V \cap K_2$. Then for any $\varepsilon > 0$ there exist $m, n_0 \in \mathbb{N}$ such that*

- *$U \cap K_1 = (\cup_{r=1}^{m} I_r) \cap K_1$, where I_r is a fundamental interval of the n_0–th step of the construction of K_1, $r = 1, 2, \ldots, m$;*

- *$|I_1| = |I_2| = \cdots = |I_m| < \varepsilon$;*

- *$\varphi(I_r)$ is a fundamental interval of the construction of K_2, $r = 1, 2, \ldots, m$.*

Remark. We can substitute the hypotheseses $\partial U \cap K_1 = \emptyset$ and $\partial V \cap K_2 = \emptyset$ by $U \cap K_1$ and $V \cap K_2$ are open and closed in K_1 and K_2, respectively.

Proof. First we give some definitions. Let K be a Cantor set. A *gap* of K is a connected component of its complement, perhaps unbounded. We say that an interval J between two gaps U_1, U_2 of K is a *bridge* if $|U_1| \geq \frac{6}{5}\,|U_2|$ or $|U_2| \geq \frac{6}{5}\,|U_1|$, and there are no gaps U contained in J with $|U| \geq \frac{5}{6}\min\left(|U_1|, |U_2|\right)$. Note that a bridge of a central Cantor set must necessarily be a fundamental interval of some step of its construction. In fact, a bridge must lie between two gaps of different steps, U_1 , U_2 where U_1, say, correspond to a more advanced step than U_2, and if the interval between U_1 and U_2 is not a fundamental interval then it must contain a gap of the same step of U_1, and hence with the same lenght of U_1, thus it cannot be a bridge.

We say that the n–th step of the construction of a central Cantor set $K_{\underline{\lambda},I}$ is a *good step* if $g_n(\underline{\lambda}) \geq \frac{3}{2}\, g_m(\underline{\lambda})$, for all $m > n$. In this case, if $g_k(\underline{\lambda}) \geq \frac{6}{5}\, g_n(\underline{\lambda})$ then any fundamental interval bounded by a gap of the k–th step and a gap of the n–th step of the construction of $K_{\underline{\lambda},I}$ must be a bridge. Also, note that there are arbitrarily advanced good steps for any $K_{\underline{\lambda},I}$. In fact, if it is not the case, then there exist sufficiently large n_0 such that for $n \geq n_0$ there is $m > n$ with $g_m(\underline{\lambda}) > \frac{2}{3}\, g_n(\underline{\lambda})$, but this imply that for a positive constant (fixed) C and infinitely many $n \in \mathbb{N}$, $g_n(\underline{\lambda}) > C \cdot \left(\frac{2}{3}\right)^n$, which contradicts the fact $g_n(\underline{\lambda}) < \pi_{n-1}(\underline{\lambda}) < \left(\frac{1}{2}\right)^{n-1}$.

Now we can decompose $U = \widetilde{U}_1 \cup \widetilde{I}_1 \cup \widetilde{U}_2 \cup \cdots \cup \widetilde{I}_\ell \cup \widetilde{U}_{\ell+1}$, for some $\ell \in \mathbb{N}$ in such a way that for each i, $\widetilde{U}_i$ is a gap of K_1, in a more precise way $\widetilde{U}_1$ and $\widetilde{U}_{\ell+1}$ are intersections of gaps of K_1 with U, and $\widetilde{I}_j$ are fundamental intervals of K_1 such that

$$\sup_{x,y \in S\widetilde{I}_j} \frac{|\varphi'(y)|}{|\varphi'(x)|} < \frac{10}{9}\,.$$

Here, if $\widetilde{I}_j = [a, a+\delta]$, $S\widetilde{I}_j$ denote the interval $[\, a - 2\delta\, ,\, a + 3\delta\,]$.

Let $n_0 \in \mathbb{N}$ such that the n_0–th step of the construction of K_1 is a good step, the fundamental intervals of this n_0–th step have lenght less than ε and each of the gaps of this step have lenght less than $\frac{1}{2}\min_{1\leq i\leq \ell+1}\left(\left|\widetilde{U}_1\right|\right)$. We will prove that the fundamental intervals of this n_0–th step contained in U, say $I_1, I_2, \ldots, I_r$ satisfies our statements.

We must prove that $\varphi(I_j)$ is a fundamental interval of K_2 , $1 \leq j \leq r$. Let us consider the decomposition of one of these intervals $\widetilde{I}_r$, corresponding to the n_0–th step of the construction of K_1, say $\widetilde{I}_r = I_m \cup U_1 \cup I_{m+1} \cup \cdots \cup U_{2^s-1} \cup I_{m+2^s-1}$, where U_i are gaps, and the decomposition is given in increasing order of the intervals. Then, by the above construction, I_m lies between $\widetilde{U}_r$ and U_1, where $\left|\widetilde{U}_r\right| > 2\, |U_1|$, and any gap contained in I_m have length less than or equal to $\frac{2}{3}\,|U_1|$, so, since $\sup_{x,y\in S\widetilde{I}_r} \frac{|\varphi'(y)|}{|\varphi'(x)|} < \frac{10}{9}$ (distortion estimated of φ in $S\widetilde{I}_r$), $\varphi(I_m)$ lies between $\varphi(\widetilde{U}_r)$ and $\varphi(U_1)$, where $\left|\varphi(\widetilde{U}_r)\right| > \frac{9}{5}\,|\varphi(U_1)|$ and any gap contained in $\varphi(I_m)$ have lenght less than or equal to $\frac{20}{27}\,|\varphi(U_1)| < \frac{5}{6}\,|\varphi(U_1)|$, hence $\varphi(I_m)$ is a bridge and therefore a fundamental interval.

Now let us consider the decomposition of $\widetilde{I}_r$ corresponding to a good step of the construction of K_1 such that the gaps of this step have length less than $\frac{1}{2}\,\min_{1\leq i\leq 2^s}\ (|U_i|)$, say $\widetilde{I}_r = I'_1 \cup U'_1 \cup I'_2 \cup \cdots \cup U'_{2^p-1} \cup I'_{2^p}$. Note that each interval I_ℓ is decomposed in a union of gaps and 2^{p-s} of the intervals I'_j, $m \leq \ell \leq m + 2^s - 1$. More precisely, I_{m+r} is the convex hull of $\cup_{i=1}^{2^{p-s}} I'_{2^{p-s}r+i}$. We know that $\varphi(I_m)$ is a fundamental interval of K_2 and, by the same reason, $\varphi(I'_1)$ is also a fundamental interval of K_2, say of the n_1–th step.

Now we will prove that φ send the gaps $U'_1, U'_3, \ldots, U'_{2^p-1}$ of the same step of the fundamental intervals I'_j in the gaps of the n_1–step contained in $\varphi(\widetilde{I}_r)$. First of all, $\varphi(U'_1)$ is the first gap after $\varphi(I'_1)$, which is the first gap of the n_1–th step contained in $\varphi(\widetilde{I}_r)$. Suppose by induction, that $\varphi(U'_{2k-1})$ is the k–th gap of the n_1–th step contained in $\varphi(\widetilde{I}_r)$.

We have $\left|U'_{2k+1}\right| = \left|U'_{2k-1}\right|$, and the left end of U'_{2k+1} is at least $2 \cdot |I'_{2k}|$ after the right end of U'_{2k-1}. Then, because of the distortion estimate of φ, $\left|\varphi(U'_{2k+1})\right| > \frac{5}{6} \left|\varphi(U'_{2k-1})\right|$, and the left end of $\varphi(U'_{2k+1})$ is at least $\frac{5}{3} |\varphi(I'_{2k})| \geq \frac{3}{2} |\varphi(I'_1)|$ after the right end of $\varphi(U'_{2k-1})$. Thus, either $\varphi(U'_{2k+1}) = V_{k+1}$, the $(k+1)$–th gap of the n_1–th step contained in $\varphi(\widetilde{I}_r)$ or either $\varphi(U'_{2k+1})$ is after V_{k+1}. The last case can not occur, because $|V_{k+1}| = |V_k| = \left|\varphi(U'_{2k-1})\right|$, and the left end of V_{k+1} is at least $2\ |\varphi(I'_1)|$ after the right end of $\varphi(U'_{2k-1})$, and so, by the same argument applied with φ^{-1} instead of φ, $\varphi^{-1}(V_{k+1})$ would be a gap of K_1 between U'_{2k-1} and U'_{2k+1}, which lies at least $\frac{3}{2}\ |I'_1| = \frac{3}{2}\ |I'_{2k}|$ after the right end of U'_{2k-1}, satisfying $|\varphi^{-1}(V_{k+1})| \geq \frac{3}{4} \left|U'_{2k-1}\right|$, which is absurd, then our claim is proved.

Since, $U'_{2^{p-s}r} = U_r$, $1 \leq r \leq 2^s - 1$, we have, by the construction of I'_j that $U'_{2^{p-s}r}$ is a gap that lies between $U'_{2^{p-s}r-1}$ and $U'_{2^{p-s}r+1}$, and $|U'_{2^{p-s}r}| \geq 2\ |U_{2^{p-s}r-1}|$, so $\varphi(U'_{2^{p-s}r})$ is a gap that lies between $V_{2^{p-s-1}r}$ and $V_{2^{p-s-1}r+1}$ and $|\varphi(U'_{2^{p-s}r})| \geq \frac{3}{2} |V_{2^{p-s-1}r}|$, so $\varphi(U'_{2^{p-s}r})$ must be the unique gap of older step between $V_{2^{p-s-1}r}$ and $V_{2^{p-s-1}r+1}$. This proves that $\varphi(I_{m+r})$ is the convex hull of the union of 2^{p-s} consecutive fundamental intervals of the n_1–th step of the construction of K_2, $0 \leq r \leq 2^s - 1$, and since $\varphi(I_m)$ is itself a fundamental interval of the construction of K_2, all the $\varphi(I_{m+r})$, $0 \leq r \leq 2^s - 1$ must also be fundamental intervals of the same step of the construction of K_2 .

Corollary 1 *Let $K_{\underline{\lambda},I}$ and $K_{\underline{\tilde{\lambda}},J}$ be two central Cantor sets, and let $\varphi : I \to J$ be a C^1 diffeomorphism such that $\sup_{x,y\in I} \frac{|\varphi'(y)|}{|\varphi'(x)|} \leq 4$ and $\varphi(K_{\underline{\lambda},I}) = K_{\underline{\tilde{\lambda}},J}$. Then φ map the decomposition of the m–th step of $K_{\underline{\lambda},I}$ into the decomposition of the m–th step of $K_{\underline{\tilde{\lambda}},J}$, i.e., $\varphi(P^m_{\underline{\lambda},I}) = P^m_{\underline{\tilde{\lambda}},J}$, for all $m \in \mathbb{N}$.*

Proof. Let $m \in \mathbb{N}$, and consider the decomposition $I = I_1 \cup U_1 \cup \cdots \cup U_{2^m-1} \cup I_{2^m}$ of the m–th step of the construction of $K_{\underline{\lambda},I}$. By Proposition 1, there exist $n_0 \in \mathbb{N}$ such that if $I = I'_1 \cup U'_1 \cup \ldots \cup U'_{2^{n_0}-1} \cup I'_{2^{n_0}}$ is the n_0–th step of the construction of $K_{\underline{\lambda},I}$ then

$|I_1'| = |I_2'| = \cdots = |I_{2^{n_0}}'| < |I_1|$, and $\varphi(I_r')$ is a fundamental interval of the construction of K_2, $r = 1, 2, \ldots, 2^{n_0}$, but then all $\varphi(I_r')$ must be fundamental intervals of the n_0–th step of the construction of K_2, otherwise there will be fundamental intervals $\varphi(I_{r_1}')$ of a step before the n_0–th and $\varphi(I_{r_2}')$ of a step after the n_0–th, and we will have $\frac{|\varphi(I_{r_1}')|}{|\varphi(I_{r_2}')|} > 4$, which is a contradiction, since $|I_{r_1}'| = |I_{r_2}'|$ and $\sup_{x,y\in I} \frac{|\varphi'(y)|}{|\varphi'(x)|} \le 4$. This proves our result for the n_0–th step of the construction of K_1 and a fortiori for the m-th step, which is less advanced than the n_0-th.

Corollary 2 *Under conditions of Proposition 1, φ map decompositions of I_r in fundamental intervals and gaps in the corresponding ones of $\varphi(I_r)$, $1 \le r \le m$, provided the I_r are conveniently chosen. In particular, φ maps sufficiently small fundamental intervals of K_1 in fundamental intervals of K_2.*

Proof. This is an immediate consequence of Corollary 1, since if K is a central Cantor set and I is a fundamental interval of K then $I \cap K$ is itself a central Cantor set, and we have $\sup_{x,y\in I_r} \frac{|\varphi'(y)|}{|\varphi'(x)|} < \frac{10}{9} < 4$, $1 \le r \le m$, for I_r obtained as in the proof of Proposition 1.

Example II.1 Let $\lambda \in (0, \frac{1}{2})$, and $K_\lambda = K_{\underline{\lambda}}$, where $\lambda_n = \lambda$, for all $n \in \mathbb{N}$. Then there are C^∞- diffeomorphisms $\varphi : [0,1] \to [0,1]$ such that $\varphi(K_\lambda) = K_\lambda$, whose distortion $D\varphi = \sup_{x,y\in[0,1]} \frac{|\varphi'(y)|}{|\varphi'(x)|}$ are arbitrarily close to $1/\lambda^2$. In fact, define

$$\varphi(x) = \begin{cases} \dfrac{x}{\lambda}, & x \in [0, \lambda^2] \\[2ex] x + 1 - 2\lambda + \lambda^2 , & x \in [\lambda(1-\lambda), \lambda] \\[2ex] \lambda x + 1 - \lambda , & x \in [1-\lambda, 1] . \end{cases}$$

Note that $[0, \lambda]$ is a fundamental interval of K_λ but $\varphi([0, \lambda]) = [0, 1 - \lambda + \lambda^2]$ is not. Letting λ tend to $\frac{1}{2}$, we see that the constant 4 in Corollary 1 is the best possible.

Proposition 2 *Let $K_1 = K_{\underline{\lambda},I}$ and $K_2 = K_{\underline{\tilde\lambda},J}$ be two central Cantor, and let $\varphi : U \to V$ be a C^1 diffeomorphism, where U and V are intervals such that* $\text{int}(U) \cap K_1 \neq \emptyset$ *and $\varphi(U \cap K_1) = V \cap K_2$. Then there exist $p \in \mathbb{Z}$ such that* $\lim_{n\to\infty} \frac{1-2\tilde\lambda_{n+p}}{1-2\lambda_n} = 1$ *and* $\lim_{n\to\infty} \frac{\pi_{n+p}(\underline{\tilde\lambda})}{\pi_n(\underline{\lambda})} = a$, *for some $a \neq 0$. We also have that $\varphi'|_{K_1\cap U}$ is locally constant. Let*

$k \in \mathbb{N}^*$. If φ is C^k, then there exist $p \in \mathbb{Z}$ as before such that

$$\lim_{n\to\infty} \frac{\left|g_{n+p}(\tilde{\underline{\lambda}}) - a\, g_n(\underline{\lambda})\right|}{(g_n(\underline{\lambda}))^k} = 0$$

and

$$\lim_{n\to\infty} \frac{\left|\pi_{n+p}(\tilde{\underline{\lambda}}) - a\, \pi_n(\underline{\lambda})\right|}{(\pi_n(\underline{\lambda}))^k} = 0\,.$$

If $\alpha \in (0,1]$ and $\varphi \in C^{k+\alpha}$ then there exist $p \in \mathbb{Z}$ as before such that

$$\limsup_{n\to\infty} \frac{\left|g_{n+p}(\tilde{\underline{\lambda}}) - a\, g_n(\underline{\lambda})\right|}{(g_n(\underline{\lambda}))^{k+\alpha}} < \infty$$

and

$$\limsup_{n\to\infty} \frac{\left|\pi_{n+p}(\tilde{\underline{\lambda}}) - a\, \pi_n(\underline{\lambda})\right|}{(\pi_n(\underline{\lambda}))^{k+\alpha}} < \infty\ .$$

Proof. We can, if necessary, reduce the interval U to satisfy hypothesis of Proposition 1. We will restrict our attention to $\varphi|_{I_j}$, where I_j is one of the fundamental intervals obtained in Proposition 1, Corollary 2. I_j is a fundamental interval of the n_0–th step of the construction of K_1, for some $n_0 \in \mathbb{N}$, and $\varphi(I_j)$ is a fundamental interval of the (n_0+p)–th step of the construction of K_2, for some $p \in \mathbb{Z}$, such that, for $r \in \mathbb{N}$, φ map fundamental intervals and gaps of the (n_0+r)–th step of the construction of K_1 contained in I_j in fundamental intervals and gaps of the (n_0+p+r)–th step of the construction of K_1. In particular, if $x \in K_1 \cap I_j$, and $I_n(x)$ is the fundamental interval of the n–th step containing x, then $\varphi'(x) = \lim_{n\to\infty} \frac{|\varphi(I_n(x))|}{|I_n(x)|} = \frac{|J|}{|I|} \cdot \lim_{n\to\infty} \frac{\pi_{n+p}(\tilde{\underline{\lambda}})}{\pi_n(\underline{\lambda})}$, so $\lim_{n\to\infty} \frac{\pi_{n+p}(\tilde{\underline{\lambda}})}{\pi_n(\underline{\lambda})}$ exists and it is equal to some $a \neq 0$ such that $\varphi'(x) = a\,\frac{|J|}{|I|}$, for all $x \in K_1 \cap I_j$. If $U_n(x)$ is the gap of the n–th step of the construction of K_1 contained in $I_{n-1}(x)$ then $\lim_{n\to\infty} \frac{1-2\tilde{\lambda}_{n+p}}{1-2\lambda_n} = \lim_{n\to\infty} \frac{|\varphi(U_n(x))|/|\varphi(I_{n-1}(x))|}{|U_n(x)|/|I_{n-1}(x))|} = \frac{\lim_{n\to\infty} |\varphi(U_n(x))|/|U_n(x)|}{\lim_{n\to\infty} |\varphi(I_{n-1}(x))|/|I_{n-1}(x)|} = \frac{\varphi'(x)}{\varphi'(x)} = 1$. Let $k \in \mathbb{N}^*$ and suppose φ is C^k. If $k=1$ the results are proved. If $k>1$, since $\varphi'|_{K_1\cap I_j}$ is constant, $\varphi^{(r)}|_{K_1\cap I_j}$ identically 0, for $r = 2,3,\ldots,k$. Then $\left|\frac{\pi_{n+p}(\tilde{\underline{\lambda}})}{\pi_n(\underline{\lambda})} - a\right| = \frac{|I|}{|J|} \cdot \left|\frac{|\varphi(I_n(x))|}{|I_n(x)|} - \varphi'(x)\right| = \frac{|I|}{|J|}\,|\varphi'(c(n,x)) - \varphi'(x)|$, for some $c(n,x) \in I_n(x)$. Since, $\varphi^{(r)}(x) = 0$, $2 \le r \le k$, we have $\lim_{y\to x} \frac{|\varphi'(y)-\varphi'(x)|}{|y-x|^{k-1}} = 0$, and since $|c(n,x) - x| \le |I_n(x)| = |I| \cdot \pi_n(\underline{\lambda})$, we have $\lim_{n\to\infty} \frac{|\varphi'(c(n,x))-\varphi'(x)|}{(\pi_n(\underline{\lambda}))^{k-1}} = 0$, and then $\lim_{n\to\infty} \frac{\left|\pi_{n+p}(\tilde{\underline{\lambda}})-a\,\pi_n(\underline{\lambda})\right|}{(\pi_n(\underline{\lambda}))^k} = 0$. Let (U_n) be a sequence of gaps of the n–th step of the construction of K_1 contained in I_j, and x_n be the left end of U_n. Then $\left|\frac{g_{n+p}(\tilde{\underline{\lambda}})}{g_n(\underline{\lambda})} - a\right| = \frac{|I|}{|J|}\left|\frac{|\varphi(U_n)|}{|U_n|} - \varphi'(x_n)\right| = \frac{|I|}{|J|}\ |\varphi'(c_n) - \varphi'(x_n)|$, for some $c_n \in U_n$, and our result about the $g_n' s$ follows from $\lim_{n\to\infty} \frac{|\varphi'(c_n)-\varphi'(x_n)|}{|c_n-x_n|^{k-1}} = 0$, which follow from $\varphi'(c_n) = \varphi'(x_n) + \frac{\varphi^{(k)}(\xi_n)}{(k-1)!}\ (c_n - x_n)^{k-1}$, for some $\xi_n \in (x_n, c_n) \subset U_n$. We have $\lim_{n\to\infty}\ \varphi^{(k)}(\xi_n) = 0$, since φ is C^k. The claims in case φ is $C^{k+\alpha}$ follows exactly as before from $|\varphi'(y) - \varphi'(x)| \leq M\,|x-y|^{k+\alpha-1}$, for all $y \in I_j$, and all $x \in K_1 \cap I_j$, where M is a constant. This follow from $\varphi'(y) - \varphi'(x) = \frac{\varphi^{(k)}(c)}{(k-1)!}\ (y-x)^{k-1}$, where c is between x and y, and $\left|\varphi^{(k)}(c)\right| = \left|\varphi^{(k)}(c) - \varphi^{(k)}(x)\right| \leq M\,|c-x|^{\alpha} \leq M\,|y-x|^{\alpha}$, for $k \geq 2$, since φ is $C^{k+\alpha}$, and, for $\varphi \in C^{1+\alpha}$, $|\varphi'(y) - \varphi'(x)| \leq M\,|y-x|^{\alpha}$.

Remark. The conclusions of Proposition 2 hold for any p, such that there is a fundamental interval I_j of the n_0–th step of the construction of K_1 as in Proposition 1, Corollary 2 such that $\varphi(I_j)$ is a fundamental interval of the (n_0+p)–th step of the construction of K_2. In this case we have $\varphi'(x) = a\,\frac{|J|}{|I|}$, for all $x \in K_1 \cap I_j$, where $a = \lim_{n\to\infty}\ \frac{\pi_{n+p}(\tilde{\underline{\lambda}})}{\pi_n(\underline{\lambda})}$.

Corollary *If $K_{\underline{\lambda}}$ admits a structure of regular Cantor (at least of class C^1), then there is $r \in \mathbb{N}^*$ such that* $\lim_{n\to\infty}\ \frac{\lambda_{n+r}}{\lambda_n} = 1$, *and* $\lim_{n\to\infty}\ \frac{1-2\lambda_{n+r}}{1-2\lambda_n} = 1$, *and the sequence* $(\lambda_{n+1}\cdot\lambda_{n+2}\cdots\lambda_{n+r})_{n\in\mathbb{N}}$ *converges to some* $\lambda \neq 0$. *If $K_{\underline{\lambda}}$ admits a structure of regular Cantor C^k (resp. $C^{k+\alpha}$) then there exist r as before such that* $\lim_{n\to\infty}\ \frac{|\lambda_{n+1}\cdot\lambda_{n+2}\cdots\lambda_{n+r}-\lambda|}{\pi_n^{k-1}} = 0$ *(resp.* $\limsup_{n\to\infty}\ \frac{|\lambda_{n+1}\cdot\lambda_{n+2}\cdots\lambda_{n+r}-\lambda|}{\pi_n^{k+\alpha-1}} < \infty$.

Proof. If $K_{\underline{\lambda}}$ is a regular Cantor defined by ψ as defined in section I., this is a direct consequence of Proposition 2 (and the above remark) applied to $K_1 = K_2 = K_{\underline{\lambda}}$, and V being some connected component of the domain of ψ, $U = \psi(V)$ and $\varphi = (\psi|_V)^{-1}$.

III - Construction of diffeomorphisms between central Cantor sets

Proposition 3 *Let $K_1 = K_{\underline{\lambda},I}$ and $K_2 = K_{\tilde{\underline{\lambda}},J}$ be two central Cantor sets such that* $\lim_{n\to\infty} \frac{1-2\tilde{\lambda}_n}{1-2\lambda_n} = 1$ *and* $\lim_{n\to\infty}\ \frac{\pi_n(\tilde{\underline{\lambda}})}{\pi_n(\underline{\lambda})} = \alpha \neq 0$. *Then there is a C^1 diffeomorphism $\varphi : I \to J$ such that $\varphi(K_{\underline{\lambda},I}) = K_{\tilde{\underline{\lambda}},J}$, and $\varphi(P^r_{\underline{\lambda},I}) = P^r_{\tilde{\underline{\lambda}},J}$ for all $r \in \mathbb{N}$, and thus* $\varphi'(x) = \alpha\,\frac{|J|}{|I|}$, *for all* $x \in K_{\underline{\lambda},I}$.

Let $k \in \mathbb{N}^*$. If $\lim_{n\to\infty}\ \frac{\left|g_n(\tilde{\underline{\lambda}})-\alpha\, g_n(\underline{\lambda})\right|}{(g_n(\underline{\lambda}))^k} = 0$ *then we can obtain φ of class C^k.*

Let $\beta \in (0,1]$. If $\limsup_{n\to\infty}\ \frac{\left|g_n(\tilde{\underline{\lambda}})-\alpha\, g_n(\underline{\lambda})\right|}{(g_n(\underline{\lambda}))^{k+\beta}} < \infty$, *then we obtain φ of class $C^{k+\beta}$.*

Proof. We will define φ in the gaps of $K_{\underline{\lambda},I}$, the definition of φ in $K_{\underline{\lambda},I}$ being done by continuity. Let $n \in \mathbb{N}$, $I = I_1^{(n)} \cup U_1^{(n)} \cup \cdots \cup U_{2^n-1}^{(n)} \cup I_{2^n}^{(n)}$, and $J = J_1^{(n)} \cup V_1^{(n)} \cup \cdots \cup V_{2^n-1}^{(n)} \cup J_{2^n}^{(n)}$, be the corresponding decompositions of the n–th step of the constructions of K_1 and K_2, respectively. We will have $\varphi(I_r^{(n)}) = J_r^{(n)} and\ \varphi(U_r^{(n)}) = V_r^{(n)}$, for each r, n. In particular, if $x \in K_1$, $\varphi'(x) = \lim_{n\to\infty} \dfrac{\left|J_r^{(n)}\right|}{\left|I_r^{(n)}\right|} = \lim_{n\to\infty} \frac{|J|\cdot\pi_n(\tilde{\underline{\lambda}})}{|I|\cdot\pi_n(\underline{\lambda})}$, since $\left|I_r^{(n)}\right| = |I|\cdot\pi_n(\underline{\lambda})$ and $\left|J_r^{(n)}\right| = |J|\cdot\pi_n(\tilde{\underline{\lambda}})$, for all r, so $\varphi'(x) = \alpha\,\frac{|J|}{|I|}$, for all $x \in K_{\underline{\lambda},I}$.

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $\psi(x) \geq 0$, for all $x \in \mathbb{R}$, and $\psi(x) = 0$, $x \notin [0,1]$, and $\int_0^1 \psi(x)\,dx = 1$. Say $\psi(x) = \frac{\tilde{\psi}(x)}{\int_0^1 \tilde{\psi}(t)\,dt}$, where

$$\tilde{\psi}(x) = \begin{cases} e^{\frac{1}{x(x-1)}}, & x \in [0,1] \\ 0, & x \notin [0,1] \,, \end{cases}$$

Let $U = (a,b)$ be a gap of the n–th step of $K_{\underline{\lambda},I}$ and let $V = (c,d)$ be the corresponding gap of the n–th step of $K_{\tilde{\underline{\lambda}},J}$. We have $|U| = |I|\cdot g_n(\underline{\lambda})$ and $|V| = |J|\cdot g_n(\tilde{\underline{\lambda}})$. We define, for n large enough $\varphi|_U$ as follows: $\varphi(a) = c$ and for $x \in [a,b]$, $\varphi'(x) = \frac{|J|}{|I|}\left(\alpha + \left(\frac{g_n(\tilde{\underline{\lambda}})}{g_n(\underline{\lambda})} - \alpha\right)\psi\left(\frac{x-a}{b-a}\right)\right)$. Since $\lim_{n\to\infty} \frac{g_n(\tilde{\underline{\lambda}})}{g_n(\underline{\lambda})} = \alpha$, this expression for $\varphi'(x)$ is strictly positive for $n \geq n_0$, where n_0 is a sufficiently large positive integer. For the gaps (a,b) of the steps before the n_0–th, we define φ in $[a,b]$ as follows: where (c,d) is the gap of K_2 corresponding to (a,b), set $\varphi(a) = c$, and for $x \in [a,b]$, set $\varphi'(x) = \alpha\,\frac{|J|}{|I|}\cdot\exp\left(\rho\,\psi\left(\frac{x-a}{b-a}\right)\right)$, where $\rho \in \mathbb{R}$ is choosen so that $\int_a^b \varphi'(x)\,dx = d - c$.

The function φ just constructed satisfies $\varphi(K_{\underline{\lambda},I}) = K_{\tilde{\underline{\lambda}},J}$, and is a C^1–diffeomorphism, since $\varphi'|_{I-K_{\underline{\lambda},I}}$ is strictly positive and tends to $\alpha\,\frac{|J|}{|I|}$ in the gaps of the n–th step, uniformly when $n \to \infty$ and $\lim_{n\to\infty} \dfrac{\left|\varphi(I_r^{(n)})\right|}{\left|I_r^{(n)}\right|} = \alpha\,\frac{|J|}{|I|}$, uniformly, so $\varphi'|_{K_{\underline{\lambda},I}} \equiv \alpha\,\frac{|J|}{|I|}$, and φ' is continuous in I. Suppose we have $\lim_{n\to\infty} \frac{\left|g_n(\tilde{\underline{\lambda}}) - \alpha\,g_n(\underline{\lambda})\right|}{(g_n(\underline{\lambda}))^k} = 0$. Then φ is of class C^k, the derivative of order k of φ in a gap (a,b) of the n–th step are given, for $k \geq 2$, by $\varphi^{(k)}(x) = \frac{|J|}{|I|^k}\,\frac{\left(g_n(\tilde{\underline{\lambda}}) - \alpha\,g_n(\underline{\lambda})\right)}{(g_n(\underline{\lambda}))^k}\cdot\psi^{(k)}\left(\frac{x-a}{b-a}\right)$, which tends to 0 uniformly when $n \to \infty$, and $\varphi^{(k)}|_{K_{\underline{\lambda},I}} \equiv 0$. So $\varphi^{(k)}$ a continuous function in I.

Now suppose $\beta \in (0,1]$ and $\limsup_{n\to\infty} \frac{\left|g_n(\tilde{\underline{\lambda}}) - \alpha\,g_n(\underline{\lambda})\right|}{(g_n(\underline{\lambda}))^{k+\beta}} < \infty$. Let $x, y \in I$. Since $\varphi^{(k)}$ is constant in $K_{\underline{\lambda},I}$, for all $k \in \mathbb{N}^*$, we have $\left|\varphi^{(k)}(x) - \varphi^{(k)}(y)\right| \leq 2\max\left(\left|\varphi^{(k)}(x) - \varphi^{(k)}(\tilde{a})\right|\right.$, $\left.\left|\varphi^{(k)}(y) - \varphi^{(k)}(\tilde{b})\right|\right)$, where $\tilde{a} = x$ (resp. $\tilde{b} = y$) if $x \in K_{\underline{\lambda},I}$ (resp. $y \in K_{\tilde{\underline{\lambda}},J}$) and $\tilde{a}$ (resp. $\tilde{b}$)

is an end of the gap containing x (resp. y). Thus we can suppose that x and y are in the closure of the same gap (a,b), and so $\left|\varphi^{(k)}(x)-\varphi^{(k)}(y)\right| = \frac{|J|}{|I|^k}\frac{\left|g_n(\tilde{\underline{\lambda}})-\alpha g_n(\underline{\lambda})\right|}{(g_n(\underline{\lambda}))^k}\cdot\left|\psi^{k+1}(\xi)\right|\cdot\frac{|x-y|}{b-a}$, for some $\xi\in[0,1]$, where (a,b) is a gap of the n-th step of $K_{\underline{\lambda},I}$, and so

$$\frac{\left|\varphi^{(k)}(x)-\varphi^{(k)}(y)\right|}{|x-y|^\beta} = \frac{|J|}{|I|^{k+\beta}}\cdot\frac{\left|g_n(\tilde{\underline{\lambda}})-\alpha g_n(\underline{\lambda})\right|}{(g_n(\underline{\lambda}))^{k+\beta}}\cdot\left|\varphi^{(k+1)}(\xi)\right|\left(\frac{|x-y|}{b-a}\right)^{1-\beta},$$

which is bounded (since $\varphi^{(k+1)}$ is bounded and $|x-y|\le b-a$) by

$$\frac{|J|}{|I|^{k+\beta}}\cdot\sup_{n\in\mathbb{N}}\frac{\left|g_n(\tilde{\underline{\lambda}})-\alpha\, g_n(\underline{\lambda})\right|}{(g_n(\underline{\lambda}))^{k+\beta}}\cdot\sup_{x\in[0,1]}\left|\psi^{(k+1)}(x)\right|.$$

So φ is of class $C^{k+\beta}$.

Corollary *Let $K_1=K_{\underline{\lambda},I}$ and $K_2=K_{\tilde{\underline{\lambda}},J}$ be two central Cantor sets, and $p\in\mathbb{Z}$ such that $\lim_{n\to\infty}\frac{1-2\tilde{\lambda}_{n+p}}{1-2\lambda_n}=1$ and $\lim_{n\to\infty}\frac{\pi_{n+p}(\tilde{\underline{\lambda}})}{\pi_n(\underline{\lambda})}=\alpha\neq 0$. Then, if $m\in\mathbb{N}^*$, $m+p\in\mathbb{N}^*$, $\tilde{I}$ is a fundamental interval of the m–th step of K_1 and $\tilde{J}$ is a fundamental interval of the $(m+p)$–th step of K_2, then there exist a C^1 diffeomorphism $\varphi:\tilde{I}\to\tilde{J}$ such that $\varphi(K_1\cap\tilde{I})=K_2\cap\tilde{J}$, $\varphi(P^r_{\underline{\lambda},I}\cap\tilde{I})=P^{r+p}_{\tilde{\underline{\lambda}},J}\cap\tilde{J}$, for $r\ge m$, and thus $\varphi'(x)=\alpha\frac{|J|}{|I|}$, for all $x\in K_1\cap\tilde{I}$. Let $k\in\mathbb{N}$ and $\beta\in(0,1]$. Then, if $\lim_{n\to\infty}\frac{\left|g_{n+p}(\tilde{\underline{\lambda}})-\alpha g_n(\underline{\lambda})\right|}{(g_n(\underline{\lambda}))^k}=0$ (resp. $\limsup_{n\to\infty}\frac{\left|g_{n+p}(\tilde{\underline{\lambda}})-\alpha g_n(\underline{\lambda})\right|}{(g_n(\underline{\lambda}))^{k+\beta}}<\infty$) then we can obtain φ of class C^k (resp. of class $C^{k+\beta}$).*

Remark. If $\alpha=0$ the result still valid, except for φ be not a diffeomorphism, but is a homeomorphism with $\varphi'(x)=0\Leftrightarrow x\in K_1$. This is used in [M2] to give examples related to critical situations of Morse-Sard's theorem.

IV - Proof of Theorems

Proof of Theorem A. Let us first prove the stated equivalences. If $k=1$ then the conditions i) and ii) are the same. Let $P_n=\lambda_{n+1}\cdot\lambda_{n+2}\cdots\lambda_{n+r_0}$. If $q>1$, ($q=k$ or $q=k+\alpha$) then each of the conditions i), ii), i'), ii') implies that P_n converges exponentially to λ, i.e., there are constants $M>0$ and $C\in(0,1)$ such that $|P_n-\lambda|\le M\cdot C^n$, for all $n\in\mathbb{N}$. So $\Pi_{n=1}^{\infty}(P_{r_0 n}/\lambda)$ converges exponentially, so $\pi_n^{q-1}=O\left(\lambda^{(q-1)n/r_0}\right)$ and $\lambda^{(q-1)n/r_0}=O\left(\pi_n^{q-1}\right)$. (We are using the notations $a_n=O(b_n)\Leftrightarrow\limsup_{n\to\infty}\frac{a_n}{b_n}<\infty$, and $a_n=o(b_n)\Leftrightarrow\lim_{n\to\infty}\frac{a_n}{b_n}=0$.)

So i) $\Leftrightarrow$ ii) and i') $\Leftrightarrow$ ii'). If $k > 1$ (resp. $k + \alpha > 1$)

$\frac{\lambda_{r_0(n+1)+j}}{\lambda_{r_0 n+j}} = \frac{P_{r_0 n+j}}{P_{r_0 n+j-1}} = \frac{\lambda + o(\lambda^{(k-1)n})}{\lambda + o(\lambda^{(k-1)n})}$, (resp. $\frac{\lambda + O(\lambda^{(k+\alpha-1)n})}{\lambda + O(\lambda^{(k+\alpha-1)n})} = 1 + o(\lambda^{(k-1)n})$)
(resp. $1 + O(\lambda^{(k+\alpha-1)n})$).

So $\lambda_{r_0(n+1)+j} = \lambda_{r_0 n+j} + o(\lambda^{(k-1)n})$ (resp. $\lambda_{r_0 n+j} + O(\lambda^{(k+\alpha-1)n})$), and thus $(\lambda_{r_0 n+j})$ converges exponentially to some μ_j, and $|\lambda_{r_0 n+j} - \mu_j| =$
$\sum_{m=n}^{\infty} \left|\lambda_{r_0 m+j} - \lambda_{r_0(m+1)+j}\right| = \sum_{m=n}^{\infty} o\left(\lambda^{(k-1)m}\right)$ (resp. $\sum_{m=n}^{\infty} O\left(\lambda^{(k+\alpha-1)m}\right) =$
$o\left(\lambda^{(k-1)n}\right)$) (resp. $O\left(\lambda^{(k+\alpha-1)n}\right)$). If some of the μ_j is equal to $1/2$ then $(\lambda_{r_0 n+j})$ converges exponentially to $1/2$, contradicting $\lim_{n\to\infty} \frac{1-2\lambda_{n+r_0}}{1-2\lambda_n} = 1$. Moreover, $\mu_1 \cdot \mu_2 \cdot \mu_3 \cdots \mu_{r_0} = \lim_{n\to\infty} P_{r_0 n} = \lambda \neq 0$, so $\mu_j \neq 0$, for all j. Thus we proved $\mu_j \in (0, 1/2)$, for all j. Reciprocally, if $\lambda_{r_0 n+j} = \mu_j + o(\lambda^{(k-1)n})$ (resp. $\lambda_{r_0 n+j} = \mu_j + O(\lambda^{(k+\alpha-1)n})$ then $P_n = \lambda + o(\lambda^{(k-1)n/r_0})$ (resp. $P_n = \lambda + \lambda^{(k+\alpha-1)n/r_0}$.) Then $i') \Leftrightarrow ii') \Leftrightarrow iii')$ and, if $k > 1$, $i) \Leftrightarrow ii) \Leftrightarrow iii)$. Proposition 2, Corollary shows that if K is C^k (resp. K is $C^{k+\alpha}$) then conditions a), b), c) and condition i) (resp. i')) hold for some r instead of r_0. We will show that this holds also for $r = r_0$. In the case C^1 there is nothing to prove. If $k > 1$ (resp. $k + \alpha > 1$), $\lim_{n\to\infty} \frac{\lambda_{n+r_0}}{\lambda_n} = 1$ implies $\mu_i = \mu_j$ if $i \equiv j \pmod{r_0}$, so $\lim_{n\to\infty} \frac{|\lambda_{rn+j} - \mu_j|}{\lambda^{(k-1)n}} = 0$, $1 \leq j \leq r$, if and only if, $\lim_{n\to\infty} \frac{\left|\lambda_n - \mu_{\overline{n}_{(r)}}\right|}{\pi_n^{k-1}} = 0$ (resp. $\limsup_{n\to\infty} \frac{|\lambda_{rn+j} - \mu_j|}{\lambda^{(k-1)n}} < \infty$, $1 \leq j \leq r$, if and only if $\limsup_{n\to\infty} \frac{\left|\lambda_n - \mu_{\overline{n}_{(r)}}\right|}{\pi_n^{k+\alpha-1}} < \infty$), where $\overline{n}_{(p)} \in \{1, 2, \ldots, p\}$ satisfies $n \equiv \overline{n}_{(p)} \pmod{p}$, but, since $\mu_{\overline{n}_{(r)}} = \mu_{\overline{n}_{(r_0)}}$ (because $r_0 = \min\{p \in G \ / \ \ p > 0\}$, where $G = \{p \in \mathbb{Z} \ / \ \ \lim_{n\to\infty} \frac{\lambda_{n+p}}{\lambda_n} = \lim_{n\to\infty} \frac{1-2\lambda_{n+p}}{1-2\lambda_n} = 1 \ \}$ is a subgroup of $\mathbb{Z}$, so $G = r_0\,\mathbb{Z}$ and $r_0 | r$, thus $\overline{n}_{(r)} \equiv \overline{n}_{(r_0)} (\operatorname{mod} r_0)$), we see that the validity of i) (resp. of i')) for r and for r_0 are equivalent. This shows that necessity ot the conditions.

To prove the sufficiency of the conditions, note that $\lim_{n\to\infty} \frac{\left|g_{n+r_0}(\underline{\lambda}) - \lambda g_n(\underline{\lambda})\right|}{(g_n(\underline{\lambda}))^k} = 0$ (resp. $\limsup_{n\to\infty} \frac{\left|g_{n+n_0}(\underline{\lambda}) - \lambda g_n(\underline{\lambda})\right|}{(g_n(\underline{\lambda}))^{k+\alpha}} < \infty$), since $\frac{g_{n+r_0}(\underline{\lambda})}{g_n(\underline{\lambda})} = \frac{1-2\lambda_{n+r_0}}{1-2\lambda_n} \cdot \frac{\pi_{n+r_0-1}(\underline{\lambda})}{\pi_{n-1}(\underline{\lambda})} = (1 + o(\pi_n^{k-1}))(1 + o(\pi_n^{k-1})$ (resp. $(1 + O(\pi_n^{k+\alpha-1}))(\lambda + O(\pi_n^{k+\alpha-1}))) = \lambda + O(\pi_n^{k-1})$ (resp. $\lambda + \mathcal{O}(\pi_n^{k+\alpha-1})$), and $\pi_n(\underline{\lambda}) = \mathcal{O}(g_n(\underline{\lambda}))$, provided $k > 1$ (resp. $k + \alpha > 1$), since $\limsup_{n\to\infty} \lambda_n < 1/2$. The case of class C^1 is trivial. Now we can use Proposition 3, Corollary, in the following way: If $I = I_1 \cup U_1 \cup I_2 \cup \cdots \cup U_{2^{r_0}-1} \cup I_{2^{r_0}}$ is the decomposition of the r_0–th step of the construction of K then there are diffeomorphisms $\varphi_j : I \to I_j$ of class C^k (resp. $C^{k+\alpha}$) such that $\varphi_j(K) = K \cap I_j$ and φ_j map decompositions of the construction of K into corresponding decompositions of the construction of $K \cap I_j$ (so $\varphi_j'|_K \equiv \lim_{n\to\infty} \lambda_{n+1} \cdot \lambda_{n+2} \cdots \lambda_{n+r_0} = \lambda$). Let $\psi : P_{\underline{\lambda},I}^{r_0} \to I$ be defined by $\psi|_{I_j} = \varphi_j^{-1}$. Then $\psi'|_K \equiv \lambda^{-1} > 1$. Then there is

$k_0 \in \mathbb{N}$ and $\sigma > 1$ such that $\psi'(x) > \sigma$, for all $x \in P^{k_0 r_0}_{\underline{\lambda},I}$. By the construction we have $\psi(P^{(k+1)r_0}_{\underline{\lambda},I}) = P^{kr_0}_{\underline{\lambda},I}$. So, if $\delta = \inf\{(\psi^{k_0})'(x) \mid x \in P^{k_0 r_0}_{\underline{\lambda},I}\}$ and $N \in \mathbb{N}$ satisfy $\delta \cdot \sigma^N > 1$, then $\psi^{N+k_0} : P^{(N+k_0)r_0}_{\underline{\lambda},I} \to I$ is an expansive function of class C^k (resp. $C^{k+\alpha}$) that makes K be a regular Cantor of class C^k (resp. $C^{k+\alpha}$), because $(\psi^{N+k_0})'(x) = (\psi^{k_0})'(\psi^N(x)) \cdot \Pi_{k=0}^{N-1} \psi'(\psi^k(x)) \geq \delta \cdot \sigma^N$, since $\psi^k(x) \in P^{k_0 r_0}_{\underline{\lambda},I}$, for all $i \leq N$.

Example IV.1 . The Lebesgue measure of a central Cantor $K_{\underline{\lambda}}$ is $\lim_{n\to\infty} 2^n \pi_n(\underline{\lambda})$. Given $a \in (0,1)$ there is a C^1 Cantor set $K_{\underline{\lambda}}$ with Lebesgue measure equal to a. For this we take

$$\lambda_n = \frac{1}{2} \cdot \frac{1 - \frac{1-a}{n}}{1 - \frac{1-a}{n+1}} .$$

If K is $C^{1+\varepsilon}$ Cantor set then its Lebesgue measure is 0.

Example IV.2 . If $\lambda_n < 1/3$ for all $n \in \mathbb{N}$ then $K_{\underline{\lambda}} - K_{\underline{\lambda}} = \{\, x - y \,/\ \ x, y \in K_{\underline{\lambda}} \,\}$ is a Cantor set with measure $2 \cdot \Pi_{n=1}^{\infty}\ (3\lambda_n)$ (see [Sa] or [B-P-V]). Given $q \geq 1$ ($q = k$ or $q = k + \alpha$) and $a \in (0,2)$ we can give example of a central Cantor $K_{\underline{\lambda}}$ of class C^q but not of class $C^{q+\varepsilon}$, for all $\varepsilon > 0$ such that $m(K_{\underline{\lambda}} - K_{\underline{\lambda}}) = a$, where m denote the Lebesgue measure in $\mathbb{R}$. For this we take

$$\lambda_n = \frac{1}{3} \frac{1 - \frac{3^q(2-a)}{2\cdot 3^{\sqrt{n}+(q-1)n}}}{1 - \frac{3^q(2-a)}{2\cdot 3^{\sqrt{n+1}+(q-1)(n+1)}}}$$

if $q < \infty$, and

$$\lambda_n = \frac{1}{3} \cdot \frac{1 - \frac{3(2-a)}{2\cdot 3^{n^2}}}{1 - \frac{3(2-a)}{2\cdot 3^{(n+1)^2}}} ,$$

if $q = \infty$.

We remark that if $\lambda_n < 1/3$, for all n, $\lim_{n\to\infty} \lambda_n = 1/3$ and $K_{\underline{\lambda}}$ is C^q , $\quad q > 1$, then $K_{\underline{\lambda}} - K_{\underline{\lambda}}$ is a Cantor with positive Lebesgue measure, because $\underline{\lambda} = (\lambda_n)_{n\in\mathbb{N}}$ converges exponentially to $1/3$ and so $\Pi_{n=1}^{\infty}(3\lambda_n) > 0$.

Example IV.3 . Let $0 \leq a < b \leq 1/2$. The K_λ is of class C^1, where

$$\lambda_{2n-1} = \frac{a+b}{2} + \left(\frac{b-a}{2}\right)\left(1 - \frac{1}{\sqrt{n}}\right)\sin(\log n)$$
$$\lambda_{2n} = \frac{ab}{\lambda_{2n-1}}$$

is dense in $[a, b]$.

Proof of Theorem B. The necessity and sufficiency of the conditions for local diffeomorphisms follow from Proposition 2 and Proposition 3, Corollary, respectively. The suficiency of the conditions for diffeomorphisms (i) and (ii)) follow from Proposition 3. We must only prove that these conditions are also necessary. By Proposition 1, Corollary 2, we can decompose $I = I_1 \cup U_1 \cup I_2 \cup \cdots \cup U_{2^{n_0}-1} \cup I_{2^{n_0}}$, where I_j are the fundamental intervals of the n_0–th step of the construction of $K_{\underline{\lambda}}, I$, such that $\varphi(I_j)$ is a fundamental interval of the construction of $K_{\underline{\tilde{\lambda}},J}$, say of the $n(j)$–th step, for $j = 1, 2, \ldots, 2^{n_0}$. Moreover, φ map decompositions of I_j in fundamental intervals and gaps in the corresponding decompositions of $\varphi(I_j)$. A diffeomorphism between two central Cantor sets with this last property is called a *good diffeomorphism.* Let $A = \{p \in \mathbb{Z}$ / for sufficiently large $n \in \mathbb{N}$, given fundamental intervals $\tilde{I}$ of the n–th step of $K_{\underline{\lambda},I}$ and $\tilde{J}$ of the $(n+p)$–th step of $K_{\underline{\tilde{\lambda}},J}$ there is a good diffeomorphism $\psi : \tilde{I} \to \tilde{J}$ at least as differentiable as φ such that $\psi(\tilde{I} \cap K_{\underline{\lambda},I}) = \tilde{J} \cap K_{\underline{\tilde{\lambda}},J}\}$, and $B = \{p \in \mathbb{Z}$ / for sufficiently large $n \in \mathbb{N}$, given fundamental intervals $\tilde{I}$ of the n–th step of $K_{\underline{\tilde{\lambda}},J}$ and $\tilde{J}$ of the $(n+p)$–th step of $K_{\underline{\tilde{\lambda}},J}$ there is a good diffeomorphism $\psi : \tilde{I} \to \tilde{J}$ at least as diferentiable as φ such that $\psi(\tilde{I} \cap K_{\underline{\tilde{\lambda}},J}) = \tilde{J} \cap K_{\underline{\tilde{\lambda}},J}\}$. Then:

- $n(j) - n_0 \in A$, $\ j = 1, 2, \ldots, 2^{n_0}$ (take restrictions of $\varphi|_{I_j}$)

- $p, q \in A \Rightarrow p - q \in B$; and $\ p, q \in B \Rightarrow p - q \in B$ (be diffeomorphic by a good diffeomorphism is an equivalence relation). In particular, $n(i) - n(j) \in B$, for all i, j.

This implies that $B = d \cdot \mathbb{Z}$ for some $d \in \mathbb{Z}$, and $A = p + d\mathbb{Z}$, for all $p \in A$. If $0 \notin A$ then $d > 1$ and $X \equiv p \pmod{d}$, for all $X \in A$, where $d \nmid p$, so we can take $p \in \{1, 2, \ldots, d-1\}$ such that $n(j) \equiv n_0 + p \pmod{d}$, for all j. Let $N = \max(n(j) + p)$. Then J contains 2^N fundamental intervals of the N–th step of $K_{\underline{\tilde{\lambda}},J}$ and $\varphi(I_j)$ contains $2^{N-n(j)}$ of these intervals, so $\sum_{j=1}^{2^{n_0}} 2^{N-n(j)} = 2^N$ but $n(j) \equiv n_0 + p \pmod{d}$, so $2^{N-n(j)} \equiv 2^{N-n_0-p} \pmod{2^d - 1}$, and thus $2^N = \sum_{j=1}^{2^{n_0}} 2^{N-n(j)} \equiv 2^{n_0} 2^{N-n_0-p} = 2^{N-p} \pmod{2^d - 1}$, so $(2^d - 1) | 2^{N-p}(2^p - 1)$, contradiction. So $0 \in A$, and the result follows from Proposition 2.

Example IV.4. The Central Cantor sets defined by the sequences $(\frac{1}{3}, \frac{1}{4}, \frac{1}{3}, \frac{1}{4}, \ldots)$ and $(\frac{1}{4}, \frac{1}{3}, \frac{1}{4}, \frac{1}{3}, \ldots)$ are not diffeomorphics but are locally diffeomorphics.

V - Calculation of numerical invariants of central Cantor sets

i) Thickness: Given a Cantor set K and a gap U of K, let I_U be the interval bounded by U and the closest gap V that is larger than U. We define $\tau(U) = \dfrac{|I_U|}{|U|}$ and $\tau(K) = \inf\{\tau(U) \ / \ U \text{ is a bounded gap of } K\}$, the *thickness* of K.

If $K = K_{\underline{\lambda}}$ is a central Cantor set, then the interval I_U in the above definition must be a fundamental interval of either U or V so, if I_U is a fundamental interval of the n–th step, we have $\tau(U) = \dfrac{|I_U|}{|U|} = \dfrac{|I_U|}{\min(|U|\,,\,|V|)} \geq \dfrac{\pi_n(\underline{\lambda})}{g_n(\underline{\lambda})} = \dfrac{\lambda_n}{1-2\lambda_n}$. On the other hand, if U is the first gap of the n–th, $\tau(U) = \dfrac{\pi_n(\underline{\lambda})}{g_n(\underline{\lambda})} = \dfrac{\lambda_n}{1-2\lambda_n}$. Therefore, $\tau(K) = \inf\limits_{n\in\mathbb{N}} \dfrac{\lambda_n}{1-2\lambda_n}$.

Given $p \in K$ we define $\tau_{loc}(k,p) = \limsup\limits_{\varepsilon_1,\varepsilon_2\to 0} \tau\left(\overline{K \cap (p-\varepsilon_2\,,\,p+\varepsilon_2)}\right)$. By the above arguments we can conclude that $\tau_{loc}\left(K_{\underline{\lambda}}\,,\,p\right) = \liminf\limits_{n\to\infty} \dfrac{\lambda_n}{1-2\lambda_n}$, for all $p \in K_{\underline{\lambda}}$.

ii) Limit Capacity : Let K be a Cantor set and $\varepsilon > 0$. We define $N(K,\varepsilon)$ as the minimum number of intervals of length ε needed to cover K, and the *limit capacity* $d(K)$ of K is defined by $d(K) = \limsup\limits_{\varepsilon\to 0} -\dfrac{\log N(K,\varepsilon)}{\log\varepsilon}$.

If $K = K_{\underline{\lambda}}$ is a central Cantor set then we can cover $K_{\underline{\lambda}}$ by 2^{n+1} intervals of lenght ε for each $\varepsilon > 0$ satisfying $\pi_{n+1}(K_{\underline{\lambda}}) \leq \varepsilon < \pi_n(\underline{\lambda})$, and so $d\left(K_{\underline{\lambda}}\right) \leq \limsup\limits_{n\to\infty} -(n+1)\dfrac{\log 2}{\log \pi_n(\underline{\lambda})} = \limsup\limits_{n\to\infty} -n\dfrac{\log 2}{\log \pi_n(\underline{\lambda})}$.

On the other hand, if $a_1 < a_2 < \cdots < a_{2^n}$ are the left ends of the fundamental intervals of the n–th step of $K_{\underline{\lambda}}$, we have $a_{i+1} - a_i > \pi_n(\underline{\lambda})$, for each i, so, if $\pi_{n+1}(\underline{\lambda}) \leq \varepsilon < \pi_n(\underline{\lambda})$ then and interval of lenght ε contains at most one of the a_i's and so we must at least 2^n intervals of lenght ε to cover $K_{\underline{\lambda}}$, since $a_i \in K_{\underline{\lambda}}$, for all i. Thus $N(K_{\underline{\lambda}},\varepsilon) \geq 2^n$ and, $-\dfrac{\log N(K_{\underline{\lambda}},\varepsilon)}{\log\varepsilon} \geq -\dfrac{n\,\log 2}{\log \pi_n(\underline{\lambda})}$ and so $d\left(K_{\underline{\lambda}}\right) = \limsup\limits_{n\to\infty} -\dfrac{n\,\log 2}{\log \pi_n(\underline{\lambda})}$.

iii) Hausdorff dimension. Given a Cantor set $K \subset \mathbb{R}$ a finite cover of K by open intervals $\mathcal{U} = \{U_i\}_{i\in I}$ we define the diameter of $\operatorname{diam}\mathcal{U} = \max\{|U_i|\ ,\ i \in I\}$. Given $\alpha \in \mathbb{R}$, we define $H_\alpha(\mathcal{U}) = \sum_{i\in I} |U_i|^\alpha$. The Hausdorff α–measure of K is $m_\alpha(K) = \lim_{\varepsilon\to 0}(\inf\{H_\alpha(\mathcal{U}) \,/\, \mathcal{U} \text{ is a covering of } K \text{ with } \operatorname{diam}(\mathcal{U}) < \varepsilon\})$. The Hausdorff dimension of K is defined as being the unique real number $HD(K)$ such that

$$\begin{cases} m_\alpha(K) = +\infty\,, & \alpha < HD(K) \\ m_\alpha(K) = 0\,, & \alpha > HD(K)\,. \end{cases}$$

If $K = K_{\underline{\lambda}}$, K can be covered by 2^n intervals of lenght $\pi_n(\underline{\lambda})$, so, if $\alpha > \liminf_{n\to\infty} -\dfrac{n\,\log 2}{\log \pi_n(\underline{\lambda})}$, there are infinitely many $n \in \mathbb{N}$ such that $2^n\,(\pi_n(\underline{\lambda}))^\alpha < 1$, and thus $HD(K_{\underline{\lambda}}) \le \alpha$.

Let $\mathcal{U}$ be a covering of $K_{\underline{\lambda}}$ by m interval of small lenght and let $\alpha \in \mathbb{R}$. Let $n \in \mathbb{N}$ such that $m < 2^{n-1}$. Then the set C of fundamental intervals of the n–th step which are strictly contained in some interval of $\mathcal{U}$ has at least 2^{n-1} elements. Now, if $U_i \in \mathcal{U}$, let $n(U_i) = \#\{I \subset C \ / \ I \subset U_i\}$, and choose a U_{i_0} that minimizes $\dfrac{|U_i|^\alpha}{n(U_i)}$. Since $\sum n(U_i) \ge 2^{n-1}$ and $\sum |U_i|^\alpha = H_\alpha(\mathcal{U})$, we have $\dfrac{|U_{i_0}|}{n(U_{i_0})} \le \dfrac{H_\alpha(\mathcal{U})}{2^{n-1}}$. If $2^k \le n(U_{i_0}) < 2^{k+1}$, then U_{i_0} constains 2^k consecutive fundamental intervals of the n–th step, and so it contains a whole fundalmental interval of the $(n-k+1)$–th step. Thus $|U_{i_0}| \ge \pi_{n-k+1}(\underline{\lambda})$ and $\dfrac{H_\alpha(\mathcal{U})}{2^{n-1}} \ge \dfrac{|U_{i_0}|^\alpha}{n(U_{i_0})} > \dfrac{(\pi_{n-k+1}(\underline{\lambda}))^\alpha}{2^{n+1}}$ thus $H_\alpha(\mathcal{U}) > 2^{n-k-2}\,(\pi_{n-k+1}(\underline{\lambda}))^\alpha$. This proves that for each covering $\mathcal{U}$ of $K_{\underline{\lambda}}$ there is $N \in \mathbb{N}$ such that $(*)$ $\pi_N(\underline{\lambda}) \le \operatorname{diam}(\mathcal{U})$ and $2^N \cdot (\pi_N(\underline{\lambda}))^\alpha < 8 \cdot H_\alpha(\mathcal{U})$, in particular, if $\alpha < \liminf_{n\to\infty} -\dfrac{n\,\log 2}{\log \pi_n(\underline{\lambda})}$, we have $\lim_{n\to\infty} 2^n\,(\pi_n(\underline{\lambda}))^\alpha = +\infty$ implies $H_\alpha(\mathcal{U}) = +\infty$, by $(*)$. So we have $HD(K_{\underline{\lambda}}) \ge \alpha$, which proves that $HD(K_{\underline{\lambda}}) = \liminf\limits_{n\to\infty} -\dfrac{n\,\log 2}{\log \pi_n(\underline{\lambda})}$.

Using the same arguments we can conclude that for each $\alpha \in \mathbb{R}$, $\alpha > 0$, $\frac{1}{8}\liminf_{n\to\infty} 2^n \cdot (\pi_n(\underline{\lambda}))^\alpha \le m_\alpha(K_{\underline{\lambda}}) \le \liminf_{n\to\infty} 2^n \cdot (\pi_n(\underline{\lambda}))^\alpha$.

In resume, the above results show that for each interval I and sequence $\underline{\lambda} = (\lambda_n)_{n\in\mathbb{N}}$,

- $\tau(K_{\underline{\lambda},I}) = \tau(K_{\underline{\lambda}}) = \inf\limits_{n\in\mathbb{N}} \dfrac{\lambda_n}{1-2\lambda_n}$;
- $\tau_{loc}(K_{\underline{\lambda},I}) = \tau_{loc}(K_{\underline{\lambda}}) = \liminf\limits_{n\to\infty} \dfrac{\lambda_n}{1-2\lambda_n}$;
- $d(K_{\underline{\lambda},I}) = d(K_{\underline{\lambda}}) = \limsup\limits_{n\to\infty} \dfrac{-n\,\log 2}{\log \pi_n(\underline{\lambda})}$;

- $HD(K_{\underline{\lambda},I}) = HD(K_{\underline{\lambda}}) = \liminf_{n\to\infty} \dfrac{-n\,\log 2}{\log \pi_n(\underline{\lambda})}$.

Moreover, for each $\alpha > 0$, $m_\alpha(K_{\underline{\lambda},I}) = |I|^\alpha \cdot m_\alpha(K_{\underline{\lambda}})$ and $m_\alpha(K_{\underline{\lambda}})$ satisfies $\frac{1}{8} \liminf_{n\to\infty} 2^n \cdot (\pi_n(\underline{\lambda}))^\alpha \le m_\alpha(K_{\underline{\lambda}}) \le \liminf_{n\to\infty} 2^n \cdot (\pi_n(\underline{\lambda}))^\alpha$.

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