Bifurcation of the essential dynamics of Lorenz maps on the real line and the bifurcation scenario for Lorenz like flows: The contracting case. *

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Dedicated to Professor Jacob Palis on his sixtieth's birthday

1 Introduction

In a remarkable contribution, the meteorologist E.N. Lorenz [23] exhibited numerical evidence for the existence of a strange attractor in a quadratic system of ordinary differential equations in three variables. Some time later Afrajmovich, Bykov and Shilnikov([1], [2]) and Guckenheimer, Williams([15], [36], [16]) proposed the so called geometrical models for the behavior observed by Lorenz. An important feature of these models is the existence of a(partial) cross-section to the flow, as well as a smooth invariant foliation by curves. Using this, one can reduce the dynamics of the flow to that of an interval transformation with a discontinuity.These transformation, generically, divide into two disjoint classes: the expanding ones(those whose derivative, from both sides, at the discontinuity is infinity) and the contracting ones(those whose derivative, from both sides, at the discontinuity is zero). As observed in [16] and [2],there exist uncountably many conjugacy classes of such transformations. In fact the moduli space is essentially 2-dimensional and can be parameterized by the admissible kneading sequences(forward itineraries of the discontinuity).

In view of these results, it is natural to look for a bifurcation theory of these transformations and flows using symbolic dynamics. In this direction, de Melo and Martens showed([8]) the existence of parameterized families of contracting Lorenz flows that are topologically universal in the sense that given any geometric Lorenz flow, its dynamics is "essentially" the same as the dynamics of some element of the family. In the present work we describe the complete bifurcation diagram for an explicit natural choice of such an essentially universal family, namely the quadratic family

$$F_{\mu,\nu}(x) = \left\{ \begin{array}{l} -\mu + x^2 \ , \ x > 0 \\ \nu - x^2 \ , \ x < 0 \end{array} \right\},$$

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Our geometrical construction has several important consequences.One of them is that axiom A maps are dense in this family (this is also proved in [8]; see [14] for similar result for the logistic family). It also follows that the set of parameters whose corresponding Lorenz map has entropy zero is arc connected(see section 4.6) . Also, the combinatorial bifurcation diagramme allow us to reobtain, in a constructive way, the result of the Melo and Martens on the universality of the quadratic family. Moreover, we completely describe the exceptional combinatorics not realized in the model(a countable set). Also, as lemma 4 in section 3.1 shows,secondary bifurcations do not appear for injective Lorenz maps(as defined in section 2.1)

After this paper had been completed we became aware of the work of St. Pierre([33])about the bifurcation diagram asociated to periodic kneading sequences, which obtains, in an independent way, some of the results presented here. Also, Hubbard and Sparrow defined, in [18], a set of pair of sequences which model all the topological expanding dynamics exhibited by Lorenz maps.

On the other hand, starting with Arnold ([3]) there has been considerable interest in the bifurcation theory of the canonical family(see for instance [5] and the references there in). The most common approach, based on the notion of rotation number, is very useful when applied to homeomorphisms of the circle, but it seems delicate to apply it to non injective maps . In fact, a bifurcation diagram for the canonical family beyond the injectivity domain is still unknown. The best approximation seems to be the work by Boyland([5]).See also the work of Epstein, Keen and Tresser([10]).

Recently, in [21], we extended the Hubbard-Sparrow model for expanding maps to an universal model for the "essential dynamics" of Lorenz maps and we called it the *Lexicographical World* (in the sequel denoted by LW). This model includes the dynamics of all the interpolated maps used by Boyland ([5], page 359) in his construction.

It is clear that the structure of the Lexicographical world has to reflect in the bifurcation theory associated to any parameterized family of allowed maps. This is an extremely interesting problem focused for several authors in this and other contexts, see for instance [24],[7],[6],[11], [3], [5] .

This paper is organized as follows : In section 2 we state our results. In section 3 we describe the lexicographical world, in Section 4 we prove our results about the quadratic family $F_{\mu,\nu}$, in section 5 we extend our results to more general contracting families and, finally, in section 6 we will relate them to three dimensional geometric vector fields.

2 Statement of the results

2.1 The Set DM_0.

In the sequel DM_0 will denote the set of maps $f : (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$ such that:

(1) The restriction maps $f|_{(-\infty,0)} : (-\infty,0) \mapsto \mathbb{R}$ and $f|_{(0,\infty)} : (0,\infty) \to \mathbb{R}$ are continuous and non-decreasing maps.

(2)
$$f(0^+) = \lim_{x\downarrow 0} f(x) \in]-\infty, 0]$$
and
$$f(0^-) = \lim_{x\uparrow 0} f(x) \in [0, \infty[.$$

An element in DM_0 will be called *injective* if its restriction to the interval $]f(0^+)\,,\, f(0^-)[$ is an injective map.

We will say that $f \in DM_0$ is *increasing* if the restriction maps $f|_{(-\infty,0)} : (-\infty,0) \mapsto \mathbb{R}$ and $f|_{(0,\infty)} : (0,\infty) \to \mathbb{R}$ are increasing.

We call the elements in DM_0 *Lorenz maps*.

2.2 The Lexicographical Order

Let Σ_2 denote the set of sequences $\theta : \mathbb{N} \to \{0,1\}$ endowed with the topology given by the metric
$$d(\alpha\,,\beta) = \sum_{i=0}^{\infty} \frac{\overline{d}(\alpha_i\,,\beta_i)}{2^i},$$
where
$$\overline{d}(\alpha_i\,,\beta_i) = \begin{cases} 0\ ,\alpha_i = \beta_i \\ 1\ ,\alpha_i \neq \beta_i \end{cases}$$
Let $\sigma : \Sigma_2 \to \Sigma_2$ be the shift map $\sigma(\theta_0\,,\theta_1\,,\theta_2\,,\ldots) = (\theta_1\,,\theta_2\,,\ldots)$. Let Σ_0 and Σ_1 denote the sets $\{\theta \in \Sigma_2\ ;\ \theta_0 = 0\}$ and $\{\theta \in \Sigma_2\ ;\ \theta_0 = 1\}$ respectively. It is clear that $\Sigma_2 = \Sigma_0 \cup \Sigma_1$.

In Σ_2 we consider the *lexicographical order*: $\theta < \alpha$ for any $\theta \in \Sigma_0$ and $\alpha \in \Sigma_1$ or $\theta < \alpha$ if there is $n \in \mathbb{N}$ such that $\theta_i = \alpha_i$ for $i = 0,1,2,\ldots,n-1$ and $\theta_n = 0$ and $\alpha_n = 1$.

For $a \leq b$ in Σ_2 let $[a\,,b]$ denote the interval $\{\theta \in \Sigma_2 |\, a \leq \theta \leq b\}$. $\Sigma_{a\,,b}$ will denote the set $\bigcap_{n=0}^{\infty} \sigma^{-n}([a\,,b])$.

2.3 The Set Σ_{a_f,b_f}.

For $f \in DM_0$ let $\Gamma_f = (\mathbb{R} \setminus \bigcup_{j=0}^{\infty} f^{-j}(0))$ denote the set of "continuity" of the map f.

For $x \in \Gamma_f$ we define $I_f(x) \in \Sigma_2$ by
$$I_f(x)(i) = 0 \quad \text{if } f^i(x) < 0 \text{ and } I_f(x)(i) = 1 \quad \text{if } f^i(x) > 0.$$

For $x = 0$ we define:
$$I_f(0^+) = \lim_{x\downarrow 0\,,x\in\Gamma_f} I_f(x)$$

and

$$I_f(0^-) = \lim_{x\uparrow 0\,,x\in\Gamma_f} I_f(x).$$

In the same way to any $x \in \bigcup_{j=0}^{\infty} f^{-j}(0)$ such that $f^i(x) \neq 0,\ 0 \leq i < n$; $f^n(x) = 0$ we associate the sequences :

$$I_f(x^+) = (I_f(x)(0), \ldots, I_f(x)(n-1),\ I_f(0^+))$$

and

$$I_f(x^-) = (I_f(x)(0), \ldots, I_f(x)(n-1),\ I_f(0^-)).$$

For $x \in \Gamma_f$ we define $I_f(x^+) = I_f(x^-) = I_f(x)$.

Let $I_f = \{I_f(x^+);\quad x \in [f(0^+), f(0^-)[\} \cup \{I_f(x^-);\quad x \in]f(0^+), f(0^-)]\}$.

Clearly $\sigma(I_f) \subset I_f$. Let us denote $a_f = I_f((f(0^+))^+)$ and $b_f = I_f((f(0^-))^-)$

Lemma 1 *([20])* $I_f = \bigcap_{n=0}^{\infty} \sigma^{-n}([a_f, b_f]) = \Sigma_{a_f,b_f}$.

We observe that associated to any $f \in DM_0$ we can define a continuous map

$$h : [f(0^+)\,, f(0^-)] \cap \Gamma_f \to \Sigma_{a_f\,,b_f} \subset \Sigma_2,$$

such that $h \circ f = \sigma \circ h$. The map h is given by $h(x) = I_f(x)$and collapses intervals into points. This map cannot be extended, continuously, to the set $\cup_{i=0}^{\infty} f^{-i}(0)$. There are two kinds of intervals that the map h can collapses: The wandering intervals and the intervals that are contained in the stable manifold of periodic sinks. An interval $I \subset [f(0^+)\,, f(0^-)]$ is called a *wandering interval* , for the map f, if for any $x \in I$ we have that x is a wandering point. We will call a point x a *nonwandering* point if for any neighborhood U_x of x and any positive integer N we can find $n \geq N$ such that $f^n(U_x) \cap U_x \neq \emptyset$. The set of nonwandering points of the map f is denoted by Ω_f . A point $x \notin \Omega_f$ is called a *wandering point*. Given any interval ,I, *the orbit* of this interval is the sequence of iterations $(f^n(I)\,, n \in \mathbb{N})$. Concerning the existence of wandering intervals we have the following:

Lemma 2 *([21]) Let $\{\varphi_\lambda\,, \lambda \in \mathbb{R}\} \subset DM_0$ be a one parameter family of C^2 increasing maps such that for each λ there are sequences $\lambda_n \to \lambda$ and $\mu_n \to \lambda$ with $\varphi_{\lambda_n}(x) > \varphi_\lambda(x)$ and $\varphi_{\mu_n}(x) < \varphi_\lambda(x)\,, \forall x$ then there is a residual set of parameters λ for which φ_λ has no wandering intervals.*

We observe that our two parameter family of contracting maps satisfy this property.

Definition 1 *Given $f, g \in DM_0$. We will say that f has* essentially *the same dynamics as g if $I_f = I_g$.*

We note that in this situation, up to the existence of some intervals where the itineraries of the points are the same, the dynamics of the maps f and g are topologically equivalent(see [9]).

2.4 The Lexicographical World

Let $Min_2 = \{a \in \Sigma_0 \ ; \ \sigma^k(a) \geq a \ , k \in \mathbb{N}\}$ and $Max_2 = \{b \in \Sigma_1 \ ; \ \sigma^k(b) \leq b \ , k \in \mathbb{N}\}$.

Definition 2 *The set* $LW = \{(a\,,b) \in Min_2 \times Max_2 \ ; \{a\,,b\} \subset \Sigma_{a\,,b}\}$ *will be called the* <u>lexicographical world</u>.

For $a \in Min_2$ its LW-fiber is the set $LW_0(a) = \{b \in Max_2 \ ; \ (a\,,b) \in LW\}$. For $b \in Max_2$ its LW-fiber is the set $LW_1(b) = \{a \in Min_2 \ ; \ (a\,,b) \in LW\}$.

Remark 1 *It is clear that given* $(a\,,b) \in LW$ *then* $\Sigma_{a,b} \neq \emptyset$.

Let us now consider $(a,b) \in LW$.

Lemma 3 *There is* $f \in DM_0$ *such that* $I_f = \Sigma_{a,b}$.

Proof: Let us consider the map $g : (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$ given by

$$g(x) = \begin{cases} 2x-1, & x > 0 \\ 2x+1, & x < 0 \end{cases}$$

In this case $I_g = \Sigma_2$. Let $x_a < 0$ and $x_b > 0$ be the points such that $I_g(x_a^+) = a$, $I_g(x_b^-) = b$.

Let $f : (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$ be the map defined by:

$$f(x) = \begin{cases} g(x), & x \leq \overline{x_b} \\ x_b, & \overline{x_b} \leq x < 0 \\ x_a, & 0 < x \leq \overline{x_a} \\ g(x), & \overline{x_a} \leq x \end{cases}$$

here $\overline{x_b} < 0 < \overline{x_a}$ are given by $g(\overline{x_b}) = x_b$ and $g(\overline{x_a}) = x_a$.

We will call this result the <u>*realization lemma*</u>. It is clear that the maps, in this construction, are non-decreasing but we want a realization lemma with increasing maps. In section 3 we prove a realization lemma with increasing maps.

Therefore, we have a surjective map $I : DM_0 \to LW$, $I(f) = (a_f, b_f)$ and $DM_0 = \bigcup_{(a,b)\in LW} I^{-1}(\{(a,b)\})$. Also, using C^0 proximity on compact sets, this map is continuous.

In this context the next definition is natural.

Definition 3 *Let* $\alpha : U \subset \mathbb{R}^k \to DM_0$ *be a map.*

1) *We will say that* α *is an* <u>a-surjective family</u> *if* $\forall a \in Min_2$ *there is a nonempty set ,* $A(a) \subset U$ *, such that* $a_f = a\,, \forall f \in \alpha(A(a))$ *.*

2) *We will say that* α *is an* <u>b-surjective family</u> *if* $\forall b \in Max_2$ *there is a nonempty set ,* $B(b) \subset U$ *, such that* $b_f = b\,, \forall f \in \alpha(B(b))$ *.*

3) We will say that α is an <u>LW-surjective family</u> *if $\forall\ (a,b) \in LW$ there exists a nonempty set $A(a,b) \subset U$ such that $I \circ \alpha(A(a,b)) = (a,b)$.*

It is clear that associated to any map α, as above, we have an $a(b\,, LW\,)$-decomposition of its domain. We will call this a(resp. $b\,, LW\,$)-decomposition the a(resp. $b\,, LW\,$) bifurcation theory defined by α.

Open Problem: Are there LW-surjective families?.

Certainly, this is a very hard problem and we believe that there is not a finite $k \in \mathbb{N}$ with this property. In the present paper we prove the following

Theorem 1 *The given two parameter family of quadratic contracting Lorenz maps is a-surjective, is b-surjective but it is not LW-surjective. Moreover, the respective $a\,, b\,, LW$ -bifurcation diagrammes are given.*

3 Symbolic Dynamics and the Lexicographical World

Here we introduce some results and notations that are necessary for our results. The results which we do not prove in this section are proved in [21]. In the sequel we will denote by $\underline{a} = \underline{a_0, a_1, ...a_n}$ the periodic sequence $(a_0, a_1, ..., a_n, a_0, a_1, ..., a_n, ...) \in \Sigma_2$. Assume that we have a countable set of sequences, A. We will say that two elements $a < b \in A$ are *consecutive in* A(or consecutive if the context is clear) if there not exist an element $c \in A$ such that $a < c < b$

3.1 Dynamical properties for sequences in LW

Let $\underline{a_1} \leq \underline{a_2}$ be two periodic sequences in Σ_0. The periodic sequence $m(a_1\,, a_2) = \underline{a_1\, a_2}$ will be called the <u>*average*</u> of the sequences $\underline{a_1}$ and $\underline{a_2}$.

<u>Example</u>: For $\underline{a_1} = \underline{01}\,, \underline{a_2} = \underline{011}$ we have $m(a_1\,, a_2) = \underline{01011}$.

Let $A_0 = \{\underline{0_n 1}\,, \underline{01_m}\,; n, m \in \mathbb{N} \setminus \{0\}\ \}$ and $A_{n+1} = A_n \cup \{\ m(a_1\,, a_2);\ \underline{a_1}\,, \underline{a_2} \in A_n$ are consecutive sequences $\}$. Set $A_\infty = \bigcup_{n=0}^{\infty} A_n$. The elements in A_∞ will be called <u>*primary sequences*</u>. As we will see in the next section, primary sequences are associated with primary bifurcations.

The elements in $\overline{A_\infty}$ are characterized by the following property:

Lemma 4 *$a \in \overline{A_\infty}$ if and only if*

$$(*) \qquad a \in Min_2 \text{ and } \sigma(a) \geq \sigma(b),$$

for $b = \sup\{\sigma^k(a)\,; k \in \mathbb{N}\}$.

To prove this lemma, let us define the renormalization map $R_{a,b} : \Sigma_2 \to \Sigma(a,b)$, by $R_{a,b}(a_0, a_1, \cdots) = (\tilde{a}_0, \tilde{a}_1, \cdots)$ where $\tilde{a}_i = a$ if $a_i = 0$ and $\tilde{a}_i = b$ if $a_i = 1$. Here $\Sigma(a,b) = \{\theta : \mathbb{N} \to \{a,b\}\}$. We have

Proposition 1 *Assume* $a \in \Sigma_0$ *satisfy* $(*)$ *then* $R_{0,01}(a)$ *satisfy* $(*)$ *in* $\Sigma_{0,01}$.

Proof: It is easy to see that

1.- $\alpha \leq \beta$ imply $R_{0,01}(\alpha) \leq R_{0,01}(\beta)$ and

2.- $\sigma_{0,01} \circ R_{0,01}(\alpha) = R_{0,01}(\sigma(\alpha))$, any $\alpha \in \Sigma_2$.

Now it is clear that if $\sigma^i(a) \geq a$ then $R_{0,01} \circ \sigma^i(a) \geq R_{0,01}(a)$ and $\sigma_{0,01}^i(R_{0,01}(a)) \geq R_{0,01}(a)$ any $i \in \mathbb{N}$.

If $\sigma(a) \geq \sigma(b)$ then $R_{0,01}(\sigma(a)) \geq R_{0,01}(\sigma(b))$ and $\sigma_{0,01}(R_{0,01}(a)) \geq \sigma_{0,01}(R_{0,01}(b))$.
Clearly; $R_{0,01}(b) = \sup\{\sigma_{0,01}^k(R_{0,01}(a)); k \in \mathbb{N}\}$. So we have the result announced in the proposition.

Proof of the Lemma Assume a satisfy (*) and $a = 00\cdots$.

Let $b = \sup\{\sigma^k(a); k \in \mathbb{N}\}$. In this situation we cannot have $b = (11\cdots)$.

In fact, otherwise $\sigma(b) = (1\cdots) > \sigma(a) = (0,\cdots)$, a contradiction.

Since $\sigma^k(b) \leq b$, any $k \in \mathbb{N}$, we have that b cannot have two consecutive $1's$ between its symbols . Therefore in this situation we have:

$$a = 0^{k_1}(01)^{k_2}0^{k_3}(01)^{k_4}\cdots = R_{0,01}(0^{k_1}1^{k_2}0^{k_3}1^{k_4}\cdots) = R_{0,01}(\tilde{a})$$

and $\tilde{a}$, clearly, satisfy (*).

Similarly , if $b = (11\cdots)$ and a satisfy (*) then $a = 01\cdots$ and $b = R_{10,1}(\tilde{b})$ some $\tilde{b}$ such that $\tilde{a} = \inf\{\sigma^k(\tilde{(b)}, k \in \mathbb{N}\}$ satisfy (*).

Since, the periodic sequences that satisfy (*) are obtained as

$$\begin{aligned} a &= \inf\left\{\sigma^k\left(R_{0,01}^{i_1} \circ R_{10,1}^{i_2} \circ \cdots \circ R_{10,1}^{i_j}(\underline{0})\right); k \in \mathbb{N}\right\} \quad \text{or} \\ a &= \inf\left\{\sigma^k\left(R_{0,01}^{i_1} \circ R_{10,1}^{i_2} \circ \cdots \circ R_{0,01}^{i_j}(1)\right); k \in \mathbb{N}\right\} \quad \text{or} \\ a &= \inf\left\{\sigma^k\left(R_{10,1}^{i_1} \circ R_{0,01}^{i_2} \circ \cdots \circ R_{10,1}^{i_j}(\underline{0})\right); k \in \mathbb{N}\right\} \quad \text{or} \\ a &= \inf\left\{\sigma^k\left(R_{10,1}^{i_1} \circ \cdots \circ R_{0,01}^{i_j}(\underline{1})\right); k \in \mathbb{N}\right\} \end{aligned}$$

we get the result $\square$.

Let us denote $A_\infty^0 = A_\infty$ and define, for any $a \in A_\infty^0$ the set:

$$A_1^1(a) = \{c \in \Sigma_0 \,;\, c = \underline{a_- b_+ a^n} \text{ or } c = \underline{a_- b^m b_+}\,, \text{for } n\,, m \in \mathbb{N}\}.$$

Here for $b = \underline{b_0 b_1 \ldots b_k 0}$ we have $b_+ = b_0 b_1 \ldots b_k 1$.

Observe that if $c_n = \underline{a_- b_+ a^n}$ then $c_n \to a_- b_+ \underline{a}$ and if $d_m = \underline{a_- b^m b_+}$ then $d_m \to a_- \underline{b}$.

Set

$$A_2^1(a) = \{m(a_1, a_2)\,;\ a_1, a_2 \in A_1^1(a) \text{are consecutive sequences}\} \cup A_1^1(a)$$

and in general

$$A_{n+1}^1(a) = \{m(a_1, a_2)\,;\ a_1, a_2 \in A_n^1(a) \text{are consecutive sequences}\} \cup A_n^1(a)\ , n \geq 2.$$

Let $A_\infty^1(a) = \bigcup_{n=1}^{n=\infty} A_n^1(a)$ and $A_\infty^1 = A_\infty^0 \cup \bigcup_{a \in A_\infty^0} A_\infty^1(a)$

At this stage we have to point out the following: Assume $a \in A_\infty^0$. Let $\tilde{a} = \sup\{\sigma^k(a); k \in \mathbb{N}$ and $\sigma^k(a) \in \Sigma_0\}$ and $\tilde{b} = \inf\{\sigma^k(a); k \in \mathbb{N}$ and $\sigma^k(a) \in \Sigma_1\}$. Consider $\Sigma_{\tilde{a},\tilde{b}} = \{\theta, \mathbb{N} \to \{\tilde{a}, \tilde{b}\}\}$ be the set of sequences of the two symbols $\tilde{a}$ and $\tilde{b}$. Replace $0 = \tilde{a}$ and $\tilde{b} = 1$ and define $A_0(a) = \{\underline{0_n 1}, \underline{01_m}; n, m \in \mathbb{N} \setminus \{0\}\}$ and $A_{n+1}(a) = A_n(a) \cup \{m(a_1, a_2); a_1, a_2 \in A_n(a)$ are consecutive sequences$\}$. Set $A_\infty(a) = \bigcup_{n=0}^{\infty} A_n(a)$ and $A_\infty^\star(a) = \{\inf\{\sigma^k(\alpha); k \in \mathbb{N}\}; \alpha = R_{\tilde{a},\tilde{b}}(\eta)\,;\ \eta \in A_\infty(a)\}$.

Lemma 2 $\bar{A_\infty^\star}(a) = \bar{A}_\infty^1(a)$.

As before, let $\sigma_{\tilde{a},\tilde{b}} : \Sigma_{\tilde{a},\tilde{b}} \to \Sigma_{\tilde{a},\tilde{b}}$ be the shift map. We have

Lemma 3 $\alpha \in \bar{A_\infty^\star}(a)$ if and only if

$$(\star)\ \ \alpha \in Min_2(\tilde{a}, \tilde{b}) \text{ and } \sigma_{\tilde{a},\tilde{b}}(\alpha) \geq \sigma_{\tilde{a},\tilde{b}}(\beta) \text{ for } \beta = \sup\{\sigma_{\tilde{a},\tilde{b}}^k(\alpha); k \in \mathbb{N}\}$$

Inductively, for any $a \in A_\infty^n(a)$,let

$$A_1^{n+1}(a) = \{c \in \Sigma_0\,;\ c = \underline{a_- b_+ a^j} \text{or } c = \underline{a_- b^k b_+} \text{ for } j, k \in \mathbb{N}\}$$

Now , we define

$$A_2^{n+1}(a) = \{m(a_1, a_2)\,;\ a_1, a_2 \in A_1^{n+1}(a) \text{ are consecutive sequences}\} \bigcup A_1^{n+1}(a)$$

and

$$A_{m+1}^{n+1}(a) = \{m(a_1, a_2)\,;\ a_1, a_2 \in A_m^{n+1}(a) \text{ are consecutive sequences}\} \bigcup A_m^{n+1}(a)\ , m \geq 2.$$

As before, define $A_\infty^{n+1}(a) = \bigcup_{m=1}^{\infty} A_m^{n+1}(a)$; $A_\infty^{n+1} = \bigcup_{a \in A_\infty^n} A_\infty^{n+1}(a) \cup A_\infty^n$ and finally, $A_\infty^\infty = \bigcup_{j=0}^{\infty} A_\infty^j$.

Note A similar construction; as we did in Lemma 2 and Lemma 3, for $a \in A_\infty^\circ$, we can do for any $a \in A_\infty^\infty$.

The elements in $(A_\infty^\infty \setminus A_\infty^0)$ will be called secondary sequences. As we will see in the next section secondary sequences are associated with secondary bifurcations.

Let denote by B^{∞}_{∞} the set $\{\sup\{\sigma^k(a)\,,k\,\in\,\mathbb{N}\}\,,a\,\in\,A^{\infty}_{\infty}\}$. We will denote by $b(a)$ the sequence $\sup\{\sigma^k(a)\,,k\,\in\,\mathbb{N}\}$ for $a\in\Sigma_0$ and by $a(b)$, the sequence $\inf\{\sigma^k(b)\,,\,k\in\mathbb{N}\}$ for $b\in\Sigma_1$. Clearly, $b(a)\in Max_2$ and $a(b)\in Min_2$.

It is clear that $\Sigma_{a,\underline{1}}\neq\phi$ for any $a\in\Sigma_0$. Hence we can define maps $\varphi\,,\psi\,,\chi:\Sigma_0\to\Sigma_1$ by :

$$\varphi(a)\;=\;\inf\{b\in\Sigma_1|\,\Sigma_{a\,,b}\neq\phi\,\},$$
$$\psi(a)\;=\;\inf\{b\in\Sigma_1\,;\,\Sigma_{a\,,b}\text{ contains}\,\infty\text{-elements}\}$$

and

$$\chi(a)\;=\;\inf\{b\in\Sigma_1\,;\,\Sigma_{a\,,b}\text{ is uncountable}\}.$$

Clearly, $a_1\leq a_2$ imply $\varphi(a_1)\leq\varphi(a_2)$ $,\psi(a_1)\leq\psi(a_2)$ and $\chi(a_1)\leq\chi(a_2)$ and for all $c\in\Sigma_1$ such that $c<\varphi(a)$ we have $\Sigma_{a\,,c}=\Phi$.

<u>Examples</u> : For any $0\underline{01}\leq a\leq\underline{01}$ we have $\varphi(a)=\underline{10}$ $,\psi(\underline{01})=\chi(\underline{01})=1\underline{10}$. Also $\varphi(\underline{0})=\psi(\underline{0})=\chi(\underline{0})=1\underline{0}\,;\varphi(0\underline{1})=\psi(0\underline{1})=\underline{1}\,;\varphi(\underline{0_n1})=\underline{10_n}$ $,\psi(\underline{0_n1})=\chi(\underline{0_n1})=1\underline{10_n}$ and $\varphi(\underline{01_m})=\underline{1_m0}$ $,\psi(\underline{01_m})=\chi(\underline{01_m})=\underline{11_m0}$.

3.2 The Morse-Smale and the Entropy Zero cases

Definition 4 *a.- We will call a map $f\in DM_0$* <u>Morse-Smale</u> *if $a_f\in A^0_{\infty}$ and $b_f=\varphi(a_f)$.*

b.- We will call a map $f\in DM_0$ <u>essentially Morse-Smale</u> *if $a_f\in A^{\infty}_{\infty}$ and $b_f=\varphi(a_f)$.*

We will denote by $MS_0\subset DM_0$ the set of Morse-Smale and essentially Morse-Smale maps . We call these maps Morse-Smale because its dynamics essentially reduces to a periodic orbit.

Lemma 5 *([21]) Given $f\in DM_0$ be a "Morse-Smale" map we have that $[f(0^+),f(0^-)]=\bigcup_{i=0}^{per(a_f)-1}\overline{I_i}$, where $x\in I_0$ implies $I_f(x)=a_f$ and $I_f|_{I_j}$ is constant and equal to $a_j=\sigma^j(a_f)$ for $0<j\leq per(a_f)-1$.*

Lemma 6 *Let $a\in A^0_{\infty}$. There is an injective $f\in DM_0$ such that $a_f=a$ and $b_f=\varphi(a)$.Moreover, this map satisfy $f(f(0^+))=f(f(1^-))$.*

Proof: Let $a=a_0<a_1<\ldots<(a_{\rm last}=a_p)<0\underline{1}<1\underline{0}<(b_{\rm first}=b_{p+1})<b_{p+2}<\ldots<b_k=b(a)$, denote the consecutive elements in $\{\sigma^j(a);\quad j\in\mathbb{N}\}$. Let $\ell<0<r$ and $I=[\ell,r]$. Consider closed intervals $L_0,L_1,\ldots,L_k$ such that $\bigcup_{i=0}^k L_i=[\ell,r]$; $\text{left}(L_0)=\ell$, $\text{right}(L_0)=\text{left}(L_1),\ldots,\text{right}(L_p)=0=\text{left}(L_{p+1}),\ldots,\text{right}(L_{k-1})=\text{left}(L_k)$; $\text{right}(L_k)=r$. Here $\text{left}[x,y]=x$ and $\text{right}[x,y]=y$.

Now, define an injective map f that sends $L_{p+1}\to L_0$, $L_{p+2}\to L_1,\ldots,L_k\to L_{k-p-1}$ and $L_0\to L_{k-p}$, $L_1\to L_{k-p+1},\ldots,L_p\to L_k$. This map satisfy the lemma. $\Box$

The next result follows immediately from this lemma:

Corollary 1 *Let* $a \in \overline{A^0_\infty} \setminus A^0_\infty$ *. There is an injective map* $f \in DM_0$ *such that* $a_f = a$ *and* $b_f = \varphi(a)$ $\Box$.

We note that these maps can be considered as bijective maps on the circle.

Lemma 7 *Let* $a \in A^\infty_\infty$ *. There is* $f \in DM_0$ *an increasing map, such that* $a_f = a$ *and* $\varphi(a) = b_f$ *.*

Proof: Let

$$a = a_0 < a_1 < \ldots < a_p < 0\underline{1} < 1\underline{0} < a_{p+1} < \ldots < a_k.$$

denote the consecutive elements in $\{\sigma^j(a)\,,\, j \in \mathbb{N}\}$. Associated with $a_s\,, 0 \leq s \leq k$,consider closed intervals $L_s\,, s = 0, \ldots k$, such that

$$L_0 \leq L_1 \leq \ldots \leq L_p \leq L_{p+1} \leq \ldots \leq L_k,$$

$L_i \cap L_{i+1} =$ one point or $L_i \cap L_{i+1} = \emptyset$, otherwise.

Let $\eta : \{0\,,1\,,\ldots\,,k\} \to \{0\,,1\,,\ldots\,,k\}$ be the permutation defined by $\sigma(a_j) = a_{\eta(j)}$. Clearly, $\eta(p) = k$ and $\eta(p+1) = 0$. Now, define an increasing map $f \in DM_0$ which satisfies $f(L_i) = L_{\eta(i)}$ and is increasing between L_i and L_{i+1} when $L_i \cap L_{i+1} = \emptyset$ $\Box$

In a similar way we obtain

Lemma 8 *Associated to any* $a \in Min_2$ *there is an increasing map* $f \in DM_0$ *such that* $a_f = a$ *and* $b_f = \varphi(a_f)$ *.*

In general we get

Lemma 9 *For* $a \in A^\infty_\infty$ *and* $d \in B^\infty_\infty \cap LW_0(a)$ *. There is an increasing map* $f \in DM_0$ *such that* $a_f = a\,, b_f = d$ *.*

As a consequence of these lemmas we have

Proposition 2 *Given* $(a,b) \in LW$ *there is an increasing map* $f \in DM_0$ *such that* $I(f) = (a_f\,,b_f) = (a,b)$ *.*

This result is a generalization of a similar result obtained for expansive maps in [18].

Definition 5 *We will call a map* $f \in DM_0$ *an* <u>entropy zero</u> *map if* $b_f \leq \chi(a_f)$ *.*

One of the most interesting problems related with the bifurcation theory associated to a parameterized family of dynamical systems $\{f_\lambda\,;\,\lambda \in U \subset \mathbb{R}^k\,\}$, is to describe the set $\{\lambda \in U\,;\, f_\lambda$ is an entropy zero map$\}$ (see for instance [26], [4] and the references there in). For our quadratic family of Lorenz maps we will prove some results, in this direction, in section 4.6.

We observe that any Morse-Smale or essentially Morse-Smale map is an entropy zero map.

4 Proof of the Results.

4.1 The Contracting Family.

Let us consider the two parameter family $F_{(\lambda,\mu)} : (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$, $\mu \geq 0$, $\nu \geq 0$, of maps in DM_0

$$F_{\mu,\nu}(x) = \begin{cases} -\mu + x^2 \ , \ x > 0 \\ \nu - x^2 \ , \ x < 0 \end{cases}$$

In this section we will provide the bifurcation theory associated to this family.

4.2 Primary Bifurcations.

We call the bifurcations in this section *primary bifurcations* because they appears as bifurcation curves for injective maps in DM_0 and can be studied by using rotation numbers(see for instance [5], [3], [13] and [17]).

4.2.1 Fixed points:

The fixed points of the map $F_{(\mu,\nu)}$ are given by

(a) $y_{\pm}(\nu) = \frac{-1 \pm \sqrt{1+4\nu}}{2}$ defined for $\nu \geq -1/4$ and

(b) $x_{\pm}(\mu) = \frac{1 \pm \sqrt{1+4\mu}}{3}$ defined for $\mu \geq -1/4$.

We have $y_{-}(\nu) \leq y_{+}(\nu) \leq 0 \leq x_{-}(\mu) \leq x_{+}(\mu)$.

4.2.2 Preimages of 0

(a) $\nu - x^2 = 0$ imply $x = -\sqrt{\nu} = y_1(\nu)$, $\nu \geq 0$

$\nu - x^2 = -\sqrt{\nu}$ imply $x = -\sqrt{\nu + \sqrt{\nu}} = y_2(\nu)$

$\nu - x^2 = y_n(\nu)$ imply $x = -\sqrt{\nu - y_n(\nu)} = y_{n+1}(\nu)$, $n \geq 2$;

(b) $-\mu + x^2 = 0$ imply $x = \sqrt{\mu} = x_1(\mu)$, $\mu \geq 0$

$-\mu + x^2 = \sqrt{\mu}$ imply $x = \sqrt{\mu + \sqrt{\mu}} = x_2(\mu)$

$-\mu + x^2 = y_n(\mu)$ imply $x = \sqrt{\mu + x_n(\mu)} = x_{n+1}$, $n \geq 2$.

4.2.3 Curves of bifurcation (first generation)

(a) $F(0^{-})$ = fixed point of the right hand side

$$\nu = \frac{1 \pm \sqrt{1+4\mu}}{2}$$

(b) $F(0^+) =$ fixed point of the left hand side

$$-\mu = \frac{-1 \pm \sqrt{1+4\nu}}{2}$$

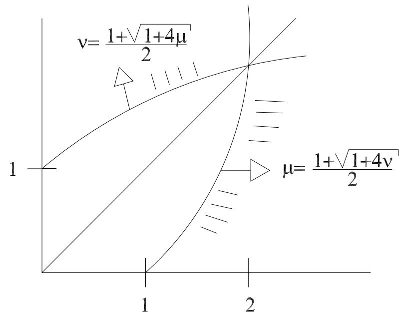

Figure 1:

We note(see figure 2 and 3) that:

$$(\mu,\nu) \in \mathbb{R}^2 \text{ such that } \nu \geq \frac{1+\sqrt{1+4\mu}}{2}, \quad \mu \geq 0 \text{ imply } b(F_{(\mu,\nu)}) = \underline{1}$$

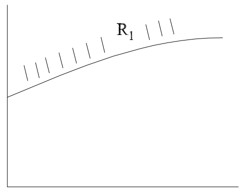

Figure 2:

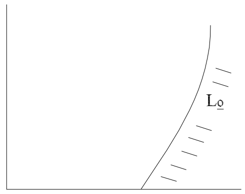

Figure 3:

$$(\mu,\nu) \in \mathbb{R}^2 \text{ such that } \mu \geq \frac{1+\sqrt{1+4\nu}}{2}, \quad \nu \geq 0 \text{ imply } a(F_{(\mu,\nu)}) = \underline{0}$$

Given $a \in Min_2$ and $b \in Max_2$ let denote by

$$R_b = \{(\mu, \nu) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+; \quad b(F_{(\mu,\nu)}) = b\}$$

$$L_a = \{(\mu, \nu) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+; \quad a(F_{(\mu,\nu)}) = a\}$$

(c) $F(0^+) = n$-th preimage of zero $= y_n(\nu)$.

These equation define the curves $\mu_n(\nu) = -y_n(\nu)$ that satisfies:

$$(\mu, \nu) \in \{(\mu_n(\nu); \nu); \quad \nu \geq 0\} \quad \text{imply} \quad a(F_{(\mu,\nu)}) = \underline{0_n 1}.$$

These curves converges to the curve $(\mu(\nu)\,,\,\nu)$ (see figure 4) where $\mu(\nu)$ is given by

$$\mu(\nu) \;=\; \frac{1+\sqrt{1+4\nu}}{2}\,, \nu \geq 0\,.$$

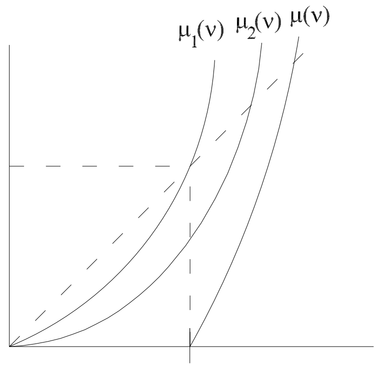

Figure 4:

In fact, we note that $\mu_1(\nu) = \sqrt{\nu}$, $\mu_2(\nu) = -y_2(\nu) = \sqrt{\nu + \sqrt{\nu}}, \ldots, \mu_{n+1}(\nu) = -y_{n+1}(\nu) = \sqrt{\nu + \mu_n(\nu)}$.

Hence, taking the limit when $n \to \infty$, we get $\mu(\nu) = \sqrt{\nu + \mu(\nu)}$, so $\mu^2 = \nu + \mu$ and, consequently, $\mu(\nu) = \dfrac{1+\sqrt{1+4\nu}}{2}$ as announced.

We also observe that the curves $(\mu_n(\nu),, \nu)$ transversally intersect the curve $(\mu\,, \dfrac{1+\sqrt{1+4\mu}}{2})$, any $n \in \mathbb{N}$.

(d) $F(0^-) = n$-th preimage of zero $= x_n(\mu)$.

These equation define the curves $\nu_n(\mu) = x_n(\mu)$ that satisfies:

$$(\mu, \nu) \in \{(\mu, \nu_n(\mu)); \quad \mu \geq 0\} \quad \text{imply} \quad b(F_{(\mu,\nu)}) = \underline{1_n 0}.$$

These curves converges to the curve $(\mu\,, \nu(\mu))$ (see figure 5) where $\nu(\mu)$ given by

$$\nu(\mu) \;=\; \frac{1+\sqrt{1+4\mu}}{2}\,, \mu \geq 0\,.$$

Let us now prove the following

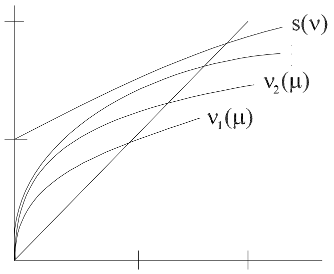

Figure 5:

Proposition 3 *The curve* $C_{\underline{10}} = \{(\mu, \nu_1(\mu)),\ \mu \in]0, \infty[\,\}$ *transversally intersect the curve* $C_{\underline{0_n 1}} = \{\mu_n(\nu), \nu),\ \nu \geq 0\}$ $,\forall n \in \mathbb{N}$.

Proof: For $C_{\underline{10}}$ we have $C_{\underline{10}} = \{(\mu, \sqrt{\mu}); \mu > 0\}$. Hence $C_{\underline{10}}$ is described by $\mu(\nu) = \nu^2$, $\nu > 0$.

Now $C_{\underline{01}}$ is described as $\mu_1(\nu) = \sqrt{\nu}$. We have $C_{\underline{01}} \cap C_{\underline{10}} = \{(1,1)\}$. and $\mu_1'(\nu) = \frac{1}{2\sqrt{\nu}}$; $(\nu^2)' = 2\nu$. So $\mu_1(\nu)$ is transversal to $\mu(\nu) = \nu^2$ at $\nu = 1$. Note $\mu_1'(\nu) \leq \frac{1}{2}$ and $\mu'(\nu) \geq 2$, all $\nu \geq 1 = \nu_1$.

Inductively, assume that $C_{\underline{10}}$ transversally intersect $C_{\underline{0n1}}$ at a point (ν_n^2, ν_n) with $\mu_n'(\nu) \leq 1/2$ and $\mu'(\nu) \geq 2$, all $\nu \geq \nu_n \geq \nu_{n-1}$.

For the curve $C_{\underline{0_{n+1}1}}$ we have $\mu_{n+1}(\nu) = \sqrt{\nu + \mu_n(\nu)}$ and $\mu_{n+1}'(\nu) = \dfrac{1 + \mu_n'(\nu)}{2\mu_{n+1}(\nu)}$

Let $(\nu_{n+1}^2, \nu_{n+1}) \in C_{\underline{0_{n+1}1}} \cap C_{\underline{10}}$. We have $\mu'(\nu) = 2\nu \geq 2\nu_{n+1} > 2\nu_n \geq 2\nu_{n-1} \geq \cdots \geq 2$ and $\mu_{n+1}'(\nu) = \dfrac{1 + \mu_n'(\nu)}{2\mu_{n+1}(\nu)} \leq \dfrac{3/2}{4} < 1/2$, any $\nu \geq \nu_{n+1} > \nu_n$. So, we get the result. $\square$

Let us now prove

Proposition 4 *The curve* $C_{\underline{1m0}} = \{(\mu, \nu_m(\mu));\ \mu \in]0, \infty[\,\}$ *transversally intersect the curve* $C_{\underline{0_n 1}}$ $,\forall n \in \mathbb{N}$.

Proof: For $m = 2$ we have $\nu_2(\mu) = \sqrt{\mu + \sqrt{\mu}}$ then $\nu_2'(\mu) = \dfrac{1 + \frac{1}{2\sqrt{\mu}}}{2\nu_2(\mu)} = \dfrac{1 + \nu_1'(\mu)}{2\nu_2(\mu)}$.

Since $\dfrac{1 + \frac{1}{2\sqrt{\mu}}}{2\nu_2(\mu)} < 1$ we get $\nu_2'(\mu) < 1$. Let denote by $\tilde{\mu_2}(\nu)$ its inverse map. We have $\tilde{\mu_2}'(\nu) > 1$. Since $\mu_n'(\nu) < 1$ we obtain the result in this case.

Inductively, assume $\nu_m'(\mu) < 1$ and that $\{(\mu\,, \nu_m(\mu))\}$ transversally intersect the curve $C_{\underline{0_n 1}}$.

Since $\nu'_{m+1}(\mu) = \dfrac{1+\nu'_m(\mu)}{2\nu_{m+1}(\mu)} < \dfrac{1+1}{2\nu_{m+1}(\mu)} < 1$, we get the result. $\square$

(e) Let us now assume that $F(0^+) < y_1(\nu)$.

In this situation there is $\overline{x_1}(\mu,\nu)$ such that $-\mu + \overline{x_1}^2 = y_1(\nu)$; that is $\overline{x_1}(\mu,\nu) = \sqrt{\mu - \sqrt{\nu}}$.

If we ask for the condition $F(0^-) = \overline{x_1}(\mu,\nu)$ we get the curve $\overline{\mu_1}(\nu) = \nu^2 + \sqrt{\nu}$(see figure 6). This curve is tangent to $\mu_1(\nu)$ at $\nu = 0$ and transversally intersect $\mu_n(\nu)$, $n \geq 2$ and $\mu(\nu)$.

In the same way: if we assume that $F(0^+) < y_n(\nu)$, $n \geq 2$ we will find $\overline{x_n}(\mu,\nu)$ such that $-\mu + \overline{x_n}^2 = y_n(\nu)$ that is $\overline{x_n}(\mu,\nu) = \sqrt{\mu + y_n(\nu)}$.
If we look for the condition $F(0^-) = \overline{x_n}(\mu,\nu)$ we get a curve $\overline{\mu_n}(\nu) = \nu^2 - y_n(\nu)$. This curve is tangent to $\mu_n(\nu)$ at $\nu = 0$ and transversally intersect $\mu_j(\nu)$, $j \geq n+1$.

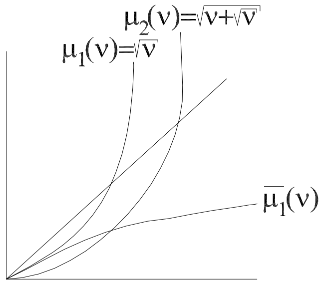

Figure 6:

We note that $(\mu,\nu) \in \{(\overline{\mu_n}(\nu), \nu); \quad \nu \geq 0\}$ imply $b(F_{(\mu,\nu)}) = \underline{10_{n+1}}$.

(f) In a similar way if we assume $F(0^-) > x_1(\mu)$ we will find $\overline{y_1}(\mu,\nu)$ such that $\nu - \overline{y_1}^2 = x_1(\mu)$ that is: $\overline{y_1}(\mu,\nu) = -\sqrt{\nu - x_1(\mu)}$.

The condition $F(0^+) = \overline{y_1}(\mu,\nu)$ imply the existence of a curve, $\overline{\nu_1}(\mu)$, such that $\overline{\nu_1}(\mu) = \mu^2 + x_1(\mu) = \mu^2 + \sqrt{\mu}$, $\mu \geq 0$. This curve(see figure 7) is tangent to $\nu_1(\mu)$ at $\nu = 0$ and transversally intersect $\nu_n(\mu)$; $n \geq 2$ and $\nu(\mu)$.
In a similar way, if we assume that $F(0^-) > x_n(\mu)$, $n \geq 2$ we will find $\overline{y_n}(\mu,\nu)$ such that $\nu - \overline{y_n}^2 = x_n(\mu)$, that is $\overline{y_n}(\mu,\nu) = \sqrt{\nu - x_n(\mu)}$.
If we ask for the condition $F(0^+) = \overline{y_n}(\mu,\nu)$ we get the curve $\overline{\nu_n}(\mu) = \mu^2 + x_n(\mu)$, $\mu \geq 0$. This curve is tangent to $\nu_n(\mu)$ at $\nu = 0$ and transversally intersect $\nu_j(\mu)$; $j \geq n+1$.

We note that $(\mu,\nu) \in \{(\mu, \overline{\nu_n}(\mu)); \quad \mu > 0\}$ imply $a(F_{(\mu,\nu)}) = \underline{01_{n+1}}$.

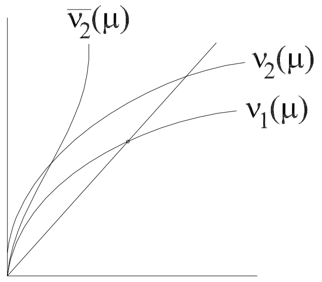

Figure 7:

Note 1 *In this section we have proved :*

1) *That associated to any sequence* $a \in A_0 = \{\underline{0_n 1};\quad \underline{01_n}/n \in \mathbb{N}\}$*, we have two curves,* $C(a)$ *and* $C(b(a))$ *, such that* $(\mu,\nu) \in C(a)$ *imply* $a(F_{(\mu,\nu)}) = a$ *and* $(\mu,\nu) \in C(b(a))$ *imply* $b(F_{(\mu,\nu)}) = b(a)$*.*

   Moreover, $C(a)$ *and* $C(b(a))$ *transversally intersect at the point* $\{P(a,b)\} = C_a \cap C(b(a))$*.*

2) *For* $(\mu,\nu) \in C(a)$ *we have that* 0 *is a fixed point of* $F^{\#per(a)}_{(\mu,\nu)}$ *that attracts a small right neighborhood of it.*

3) *For* $(\mu,\nu) \in C(b(a))$ *we observe that* $x = \nu$ *is a fixed point of* $F^{\#per(b(a))}_{(\mu,\nu)}$ *that attracts a small left neighborhood of it.*

4.2.4 Transversality of the bifurcation curves.

Let us denote

$$\varphi_1(\mu) = \sqrt{\mu}, \varphi_2(\mu) = \sqrt{\mu + \varphi_1(\mu)}, \cdots, \varphi_{n+1}(\mu) = \sqrt{\mu + \varphi_n(\mu)}$$

and

$$\psi_1(\nu) = \sqrt{\nu}\psi_2(\nu) = \sqrt{\nu + \psi_2(\nu)}; \cdots; \psi_{n+1}, (\nu) = \sqrt{\nu_n + \psi_n(\nu)}.$$

Define $G_n(\mu,\nu) = -\mu + \varphi_n(\nu)$ and $H_n(\mu,\nu) = \nu - \varphi_n(\mu)$.

As we saw, in 4.2.3, we have that $G_n^{-1}(0) = C_{\underline{0_n 1}}$ and $H_n^{-1}(0) = C_{\underline{1_n 0}}$.

Let

$$L_n(\mu,\nu) = -\mu + [G_n(\mu,\nu)]^2 =$$

$$= -\mu + (-\mu + \psi_n(\nu))^2 = -\mu + \mu^2 - 2\mu\psi_n(\nu) + \psi_n^2(\nu).$$

and

$$M_n(\mu,\nu) = \nu - [H_n(\mu,\nu)]^2 = \nu - [\nu - \varphi_n(\mu)]^2 =$$

$$= \nu - \nu^2 + 2\nu\varphi_n(\mu) - \varphi_n^2(\mu).$$

We have that $L_n^{-1}(0)$ is transversal to $H_m^{-1}(0)$, any $n, m \in \mathbb{N}$.

In fact, we have

$$\begin{aligned}
\frac{\partial L_n}{\partial \mu} &= -1 + 2\mu - 2\psi_n(\nu). \\
\frac{\partial L_n}{\partial \nu} &= 2\psi_n(\nu) \cdot \psi_n'(\nu) \;-\; 2\mu\psi_n'(v). \\
\frac{\partial H_n}{\partial \mu} &= -\varphi_n'(\mu). \\
\frac{\partial H_n}{\partial \nu} &= 1
\end{aligned}$$

Then

$$\frac{-\dfrac{\partial L_n}{\partial \mu}}{\dfrac{\partial L_n}{\partial \nu}} = \frac{1 + 2(\psi_n(\nu) - \mu)}{2\psi_n'(\nu)[\psi_n(\nu) - \mu]}$$

and

$$\frac{-\dfrac{\partial H_n}{\partial \mu}}{\dfrac{\partial H_n}{\partial \nu}} = \varphi_n'(\mu) < 1.$$

Since $L_n(\mu, \nu) = 0$ implies $\quad \sqrt{\mu} = \psi_n(\nu) - \mu, \quad$ then

$$\frac{\dfrac{\partial L_n}{\partial \mu}}{\dfrac{\partial L_n}{\partial \nu}} = \frac{1}{\psi_n'(\nu)} \cdot \left[1 + \frac{1}{2\sqrt{\mu}}\right] > 1.$$

Hence $L_n^{-1}(0)$ is transversal to $H_n^{-1}(0)$.

Let us now see that $M_m^{-1}(0)$ is transversal to $L_n^{-1}(0)$. In fact

$$\frac{-\dfrac{\partial M_m}{\partial \mu}}{\dfrac{\partial M_m}{\partial \nu}} = \frac{2\varphi_m'(\mu)[\varphi_m(\mu) - \nu]}{1 + 2[\varphi_m(\mu) - \nu]} = \frac{2\varphi_m'(\mu) \cdot \sqrt{\nu}}{1 + 2\sqrt{\nu}}.$$

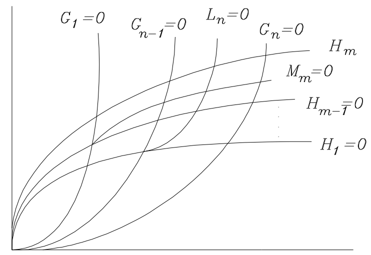

Figure 8:

And, consequently $M_m^{-1}(0)$ is transversal to $L_n^{-1}(0)$ for any $n, m \in \mathbb{N}$(see figure 8)

Remark We note that $L_n^{-1}(0)$ "begin" at the intersection $G_{n-1}^{-1}(0) \cap H_1^{-1}(0)$and it is tangent to $H_1 = 0$. In the same way, if $N_n(\mu, \nu) = F(L_n)$, then N_n is transversal to $H_n^{-1}(0)$and $M_n^{-1}(0)$ any $m \geq 0$, and $N_n^{-1}(0)$ "begin" in the intersection $G_{n-1}^{-1}(0) \cap H_2(0)$, and so on.

4.2.5 Primary saddle-nodes.

In this section we will prove

Proposition 5 *Associated with any $a \in A_0$ and $b = b(a)$ there is a curve $SN(a, b) \subset \mathbb{R}^2$ such that:*

(i) there are values $\nu_{a,b}$ and $\mu_{a,b}$ such that $F^{\#per(a)}_{(\mu_{a,b},\nu_{a,b})}$ has a fixed point with derivative equal to 1 and $(\mu_{a,b}, \nu_{a,b}) \in SN(a, b)$;

(ii) $SN(a, b) \setminus \{(\mu_{a,b}, \nu_{a,b})\} = SN(a) \cup SN(b)$ are two curves which satisfies:

(a) $SN(a)$ transversally intersects $C(b)$ at a point $\{(\overline{\mu}_{a,b}, \overline{\nu}_{a,b})\} = SN(a) \cap C(b)$. $SN(b)$ transversally intersects $C(a)$ at a point $\{(\tilde{\mu}_{a,b}, \tilde{\nu}_{a,b})\} = SN(b) \cap C(a)$;

(b) For any $(\mu, \nu) \in SN(a)$ there is a saddle-node periodic point which correspond to an unfolding of the fixed point of $F^{\#per(a)}_{(\mu,\lambda)}$, $(\mu, \nu) \in C(a)$.

(c) For any $(\mu, \nu) \in SN(b)$ there is a saddle-node periodic point which correspond to an unfolding of the fixed point of $F^{\#per(b)}_{(\mu,\nu)}$, $(\mu, \nu) \in C(b)$.

Proof: Initially, let us consider $a = \underline{01}$. In this situation the curve $C_a = C_{\underline{01}} = \{(\mu\,,\,\mu^2)\,,\,\mu \geq 0\}$ represent the fact that $F(0^+) = y_1(\nu)$,hence $F(F(0^+)) = F(y_1(\nu)) = 0$ that is, "0^+" is a periodic point whose period is $\#(a)$. Also, for $b = \underline{10}$ the curve $C_b = C_{\underline{01}} = \{(\mu\,,\sqrt{\mu})\,,\,\mu \geq 0\}$ represent the fact that $F(0^-) = x_1(\mu)$ and, then, $F(F(0^-)) = F(x_1(\mu)) = 0$, that is "$0^-$" is a periodic point (of the map F) whose period is $\#(b)$.

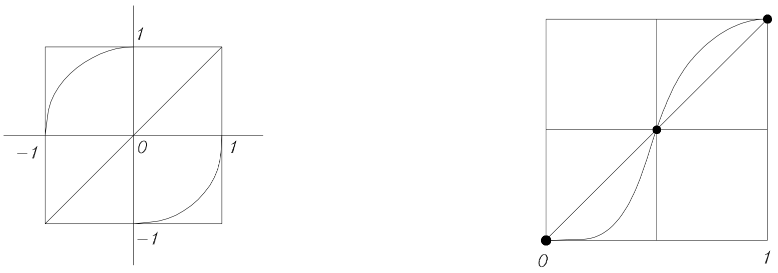

Figure 9:

Let us consider the point $(1\,,\,1) \in C_{\underline{01}} \cap C_{\underline{10}}$. The graph of the map $F_{(1,1)}$ is as in left hand side of figure 9 and the graph of the restriction of the map $F^2_{(1,1)}$ is as in the right hand side of figure 9.

Now, let $H = \{(\mu\,,\,\nu)\,;\,\mu \geq 0\,,\,\nu \geq 0\}$. We have that $H \backslash C_{\underline{01}} = \{(\mu\,,\,\nu)\,,\,,\mu^2 < \nu\} \cup \{(\mu\,,\,\nu)\,,\,\mu^2 > \nu\} = A_1 \cup A_2$ and $H \backslash C_{\underline{10}} = \{(\mu\,,\,\nu)\,,\,\sqrt{\mu} < \nu\} \cup \{(\mu\,,\nu)\,,\,\sqrt{\mu} > \nu\} = B_1 \cup B_2$. So, we have the following components of the set $H \setminus (C_a \cup C_b) = A_1 \cap B_1 \cup A_1 \cap B_2 \cup A_2 \cap B_1 \cup A_2 \cap B_2$ where $A_1 \cap B_1 = \{(\mu\,,\nu)\,,\,\nu > \mu^2$ and $\nu > \sqrt{\mu}\}$, $A_1 \cap B_2 = \{(\mu\,,\,\nu)\,,\,\sqrt{\mu} > \nu > \mu^2\}$, $A_2 \cap B_1 = \{(\mu\,,\,\nu)\,,\,\mu^2 > \nu > \sqrt{\mu}\}$ and $A_2 \cap B_2 = \{(\mu\,,\,\nu)\,,\,\mu^2 > \nu$ and $\sqrt{\mu} > \nu\}$. The next figures displays the first return map, to the interval $[0\,,\nu]$ for $(\mu\,,\,\nu) \in A_i \cap B_j\,, i = j = 1, 2$ and located near the point $(1, 1)$.

Figure 10: $A_1 \cap B_1$ $A_1 \cap B_2$.

Figure 11: $A_2 \cap B_1$ $A_2 \cap B_2$.

Now, it is clear that we can find, in the interior of the set $A_1 \cap B_2$, a point $(\mu_{a,b}\,,\,\nu_{a,b})$ such that the first return map to the interval $[0\,,\,\nu]$ looks like in figure 11.

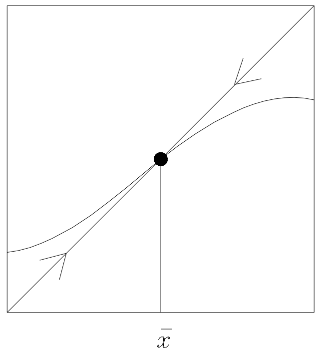

Figure 12:

Let us now consider the family $F_{(\mu\,,\mu^2+\rho)}\,,\rho\in\mathbb{R}$. Clearly, there is a unique $\rho_{a,b}$ such that, for $\mu=\mu_{a,b}$ we have $\nu_{a,b}=\mu^2+\rho$. Now it is not hard to prove, for the one-parameter family $F_{(\mu\,,\mu^2+\rho)}\,,\rho$ fixed, the following

Lemma 10 *(i) Assume $\rho<\rho_{a,b}$ and near $\rho_{a,b}$. In this case the first return map has the following generic unfolds :*

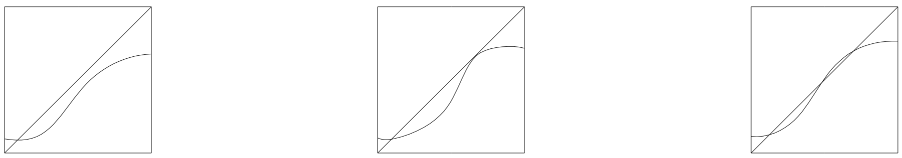

Figure 13:

(ii) Assume $\rho>\rho_{a,b}$ and near $\rho_{a,b}$. In this case the first return map has the following generic unfold :

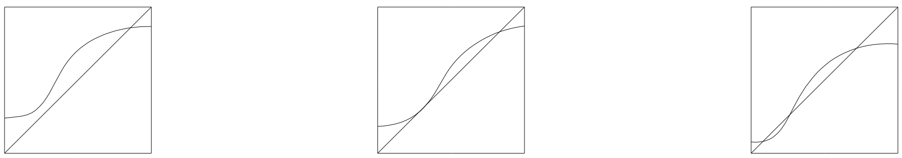

Figure 14:

(iii) The curves $SN(b)=\mu^{10}_{sn}(\rho)$ and $SN(a)=\mu^{01}_{sn}(\rho)$ are given in figure 15,

(iv) The curve $SN(b)$ intersects the curve $\{(\mu\,,\mu^2+\rho)\,,\,\mu\geq 0\}\,,\rho<\rho_{a,b}$ fixed , in a unique point.

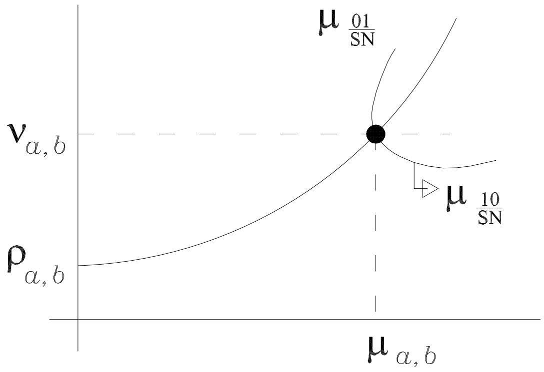

Figure 15:

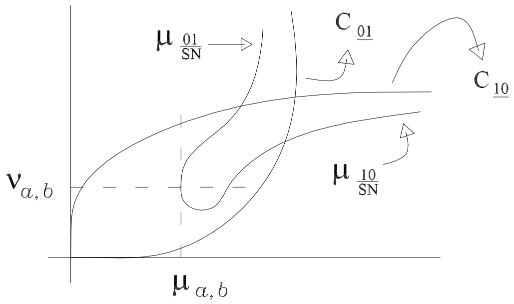

Figure 16:

(v) The curve $SN(a)$ intersects the curve $\{(\mu\,,\,\mu^2+\rho)\,,\,\mu\geq 0\}\,,\rho>\rho_{a,b}$ fixed , in a unique point(see figure 16). $\square$

This complete the proof in the case $a = \underline{01}$. The other cases follows in a similar way.

Figure 17 displays these essential facts concerning the curve $SN(a,b)$.$\square$

Remark 2 *Given $a \in A_0$, $b = b(a)$ we define $L_a = \{(\mu,\nu); a(F_{(\mu,\nu)}) = a\}$ and $R_b = \{(\mu,\nu);\quad b(F_{(\mu,\nu)}) = b\}$.*

It is not hard to prove that the sets L_a and R_b looks like as in figure 18

4.2.6 Let us consider

$$A_1 = \{m(a_1,a_2);\quad a_1,a_2 \in A_0 \,\text{are consecutive sequences}\,\} \cup A_0.$$

It is not hard to prove(see 4.2.4) that associated with $a = m(a_1,a_2)$, $b = b(a)$ we can find two curves $C(a), C(b)$, tangent to $C(a_2)$ and $C(b(a_2))$ respectively, such that:

$$(\mu,\nu)\in C(a)\;\;\text{imply}\;\;a(F_{(\mu,\lambda)}) = a\,\text{and}\,(\mu,\nu)\in C(b)\;\;\text{imply}\;\;b(F_{(\mu,\lambda)}) = b.$$

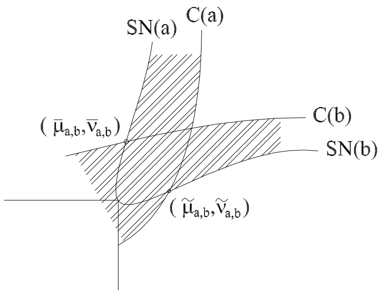

Figure 17:

Figure 18:

Also, there are saddle-nodes associated to $C(a)$ and $C(b)$ as in 4.2.5. The geometry of the sets R_b and L_a is similar to the geometry of the sets described in the remark in 4.2.5.

Similarly, we consider the set

$$A_{n+1} = A_n \cup \{m(a_1, a_2); a_1 < a_2 \text{ are consecutive sequences in} A_n\}.$$

As before associated with $a \in A_{n+1}$, $a = m(a_1, a_2)$,we are able to find the curves $C(a)$, $C(b(a))$ and $SN(a, b)$(see figure 19).

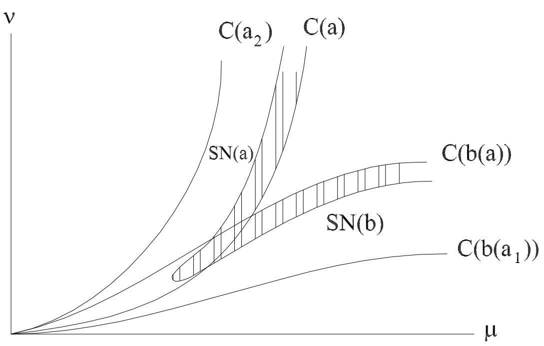

Figure 19:

4.2.6 Let $A_\infty = \bigcup_{n=0}^{\infty} A_n$. Assume $a \in \overline{A_\infty}$ is a sequence which is not periodic nor eventually periodic. Let $b = \sup\{\sigma^k(a); \quad k \in \mathbb{N}\}$ and $a_n \in A_\infty$, $a_n \to a$. let $b_n = b(a_n)$. It is clear that we have $b_n \to b$.

In this situation we can find values of the parameter (μ_a, ν_a) and curves $C(a)$, $C(b)$ such that: $C(a) = \{(\mu, \nu); \quad \nu = \varphi_a(\mu)\}$; $C(b) = \{(\mu, \nu); \quad \nu = \psi_b(\mu)$ where $\varphi_a : (0, \infty) \to \mathbb{R}$, $\psi_b : (0, \infty) \to \mathbb{R}$ satisfies

- $\varphi_a(\mu) = \psi_b(\mu)$, $0 < \mu \leq \mu_a$.
- The maps $\varphi_a|_{(\mu_a, \infty)}$ and $\varphi_b|_{(\mu_a, \infty)}$ are tangent at μ_a.

Moreover

(i) the curves $C(a_n)$ converges to $C(a)$ and the curves $C(b_n)$ converges to $C(b)$;

(ii) the intersections $C(a_n) \cap C(b_n)$ converges to $\{(\mu_a, \nu_a)\}$;

(iii) $(\mu, \nu) \in C(a)$ imply $a(F_{(\mu,\lambda)}) = a$ and $(\mu, \nu) \in C(b)$ imply $b(F_{(\mu,\lambda)}) = b$.

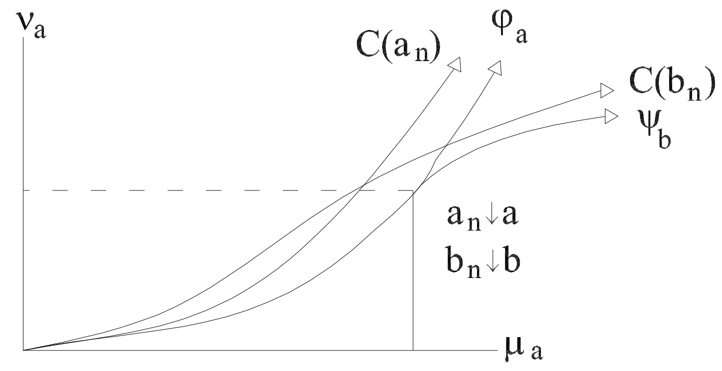

Figure 20:

Maybe, the unique point which is not easy to agree in this result is related with the coincidence of the curves $C(a)$ and $C(b)$ along $\{(\mu, \varphi_a(\mu)); \ 0 < \mu \leq \mu_a\}$, but this is an easy consequence of the next result.

Let $r > 0$ be a small number. Let $B_r^+ = \{(\mu, \nu) \in \mathbb{R}^2, \ \sqrt{\mu^2 + \nu^2} \leq r, \ \mu \geq 0, \quad \nu \geq 0\}$. Let $m(A)$ denotes the Lebesgue measure of the set $A \subset \mathbb{R}^2$.

Lemma 11 $m(\{(\mu, \nu) \in B_r^+; \quad F_{(\mu,\nu)} \notin MS_0\}) = 0$.

Proof: In fact, the proof can be done in a similar way as was done in [19] or in [30].

Remark 3 *1) In fact, Arnold ([3] and Boyland ([5]), obtained a similar result for the structure of the irrational Arnold's Tongues of the canonical family* $f_{\lambda,b}(x) = x + \lambda + \dfrac{b}{2\pi}\sin(2\pi x)$.

2) A similar result(as in lemma 13) was proved by Swiatek in [34] with respect to parameterized families of homeomorphisms of the circle.

4.3 Secondary Bifurcations

We call the bifurcations in this section secondary because this kind of changes in the dynamic cannot appear for injective maps in DM_0 . Also, it is not possible to understand this scenario only by using rotation numbers as it is possible to see by comparing our results with those of [5].

4.3.1 Let $B_\infty = \{b(a); \quad a \in A_\infty\}$. Let us now consider $a \in A_\infty$, $b \in B_\infty$ such that $C(a) \cap C(b) \neq \emptyset$.It is not hard to prove that (see section 4.2.3) that:

a) $C_a \cap C_b = \{P(a,b)\}$ is a unique point.

b) define $A(a,b) = \{c \in \Sigma_0\,; \quad c = \underline{a_-b_+a^n};\ n \in \mathbb{N}$ or $c = \underline{a_-b^m b_+};\ m \in \mathbb{N}\}$ and $B(a,b) = \{d \in \Sigma_1;\ d = \underline{b_+a^n a_-}, \quad n \in \mathbb{N}$ or $d = \underline{b_+a_-b^m};\ m \in \mathbb{N}\}$.

((i) Associated with $c = \underline{a_-b_+}$ there is a curve C_c, tangent to C_b; $P(a,b) \in \overline{C_c}$ and such that $(\mu,\nu) \in C_c$ imply $a(F_{(\mu,\nu)}) = c$;

((ii) Associated with $d = \underline{b_+a}$ there is a curve C_d, tangent to C_a; $P(a,b) \in \overline{C_d}$ and such that $(\mu,\nu) \in C_d$ imply $b(F_{(\mu,\nu)}) = d$;

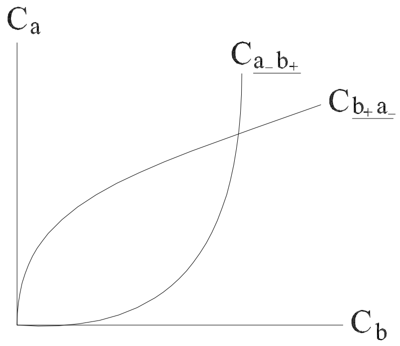

Figure 21: (b)(i) and (b)(ii)

(iii) Associated with $c = \underline{a_-b_+a^n}$, $n \geq 1$, there is a curve C_c, tangent to C_a; $P(a,b) \in \overline{C_c}$ and such that $(\mu,\nu) \in C_c$ imply $a(F_{(\mu,\nu)}) = c$;

(iv) Associated with $d = \underline{b_+a^n a_-}$, $n \geq 1$, there is a curve C_d, tangent to C_a; $P(a,b) \in \overline{C_c}$ and such that $(\mu,\nu) \in C_d$ imply $b(F_{(\mu,\lambda)}) = d$;

(v) Associated with $c = \underline{a_-b^n b_+}$, $n \geq 1$, there is a curve C_c, tangent to C_b; $P(a,b) \in \overline{C_c}$ and such that $(\mu,\lambda) \in C_c$ imply $a(F_{(\mu,\lambda)}) = c$;

(vi) Associated with $d = \underline{b_+a_-b^n}$, $n \geq 1$, there is a curve,C_d, tangent to C_b; $P(a,b) \in \overline{C_d}$ and such that $(\mu,\lambda) \in C_d$ imply $b(F_{(\mu,\lambda)}) = d$;

In fact, let denote by $\xi_b : [\mu_b, \infty[\to \mathbb{R}$and $\xi_a : [\mu_a, \infty[\to \mathbb{R}$ the maps that satisfies:

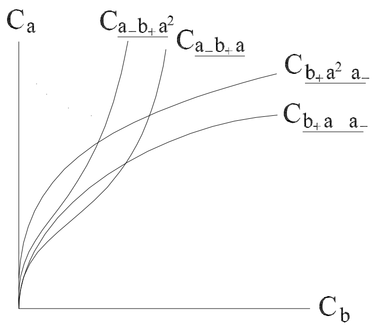

Figure 22: (b)(iii) and (b)(iv)

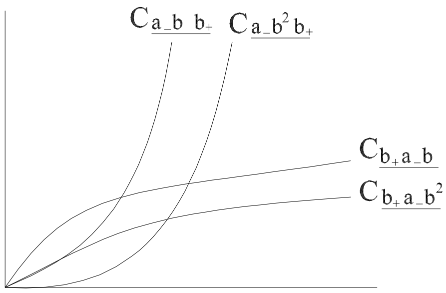

Figure 23: (b)(v) and (b)(vi)

(i) $\{(\mu; \xi_b(\mu)\}; \quad \mu \geq \mu_b\}$ is the curve which represents the fact that $x = v$ is a periodic point with itinerary b;

(ii) $\{(\mu; \xi_a(\mu)\}; \quad \mu > \mu_a\}$ is the curve which represents the fact that $x = -\mu$ is a periodic point with itinerary a.

Let denote $C_a \cap C_b = \{(\overline{\mu_b}, \overline{\nu_b})\}$.

Now, associated with $c_n = \underline{b_+a^na_-}$, $n \geq 0$ we can obtain maps $\xi_n : [\overline{\mu_b}, \infty[\to \mathbb{R}$ such that

(i) $\xi_n'(\overline{\mu_b}) = \xi_a'(\overline{\mu_b}) - \xi_b'(\mu_b)$; $\xi_n(\overline{\mu_b}) = 0$

(ii) $\{(\mu; \xi_b(\mu) + \xi_n(\mu)); \quad \mu \geq \overline{\mu_b}\}$ is the curve which represents the fact that $x = v$ is a periodic point with itinerary c_n.

In a similar way, associated with $d_m = \underline{b_+a_-b^m}$, $m \geq 1$, there are maps $\phi_m : [\overline{\mu_b}, \infty[\to \mathbb{R}$ such that

(i) $\phi_m(\overline{\mu_b}) = 0$; $\phi_m(\overline{\mu_b}) = 0$.

(ii) $\{(\mu, \xi_b(\mu) + \phi_m(\mu)); \quad \mu \geq \overline{\mu_b}\}$ is the curve which represents the fact that $x = v$ is a periodic point with itinerary d_m.

(c) Associated with $c = a_-\underline{b}$ there is a curve C_c, tangent to $SN(b)$ at the point $P_c \in C_c$, transversal to C_b at the point Q_c, such that $(\mu, \lambda) \in C_c$ imply $a(F_{(\mu,\lambda)}) = c$.

Moreover let L_c be the open rectangle bounded by $C_a, SN(b)$, C_c and C_b. We have $L_c \subset R_b$ and $(\mu, \nu) \in L_c$ imply $a(F_{(\mu,\lambda)}) = c$. Also, $(\mu, \lambda) \in \overline{L_c} \cap [C_b \cup SN(b) \setminus C_a]$ imply $a(F_{(\mu,\lambda)}) = c$;

(d) Associated with $d = b_+a_-\underline{b}$ there is a curve C_d, tangent to $C_{a-\underline{b}}$ at the point $Q_{a-\underline{b}} \in \overline{C_d} \cap C_{a-\underline{b}}$ and such that $(\mu, \nu) \in C_d$ imply $b(F_{(\mu,\lambda)}) = d$;

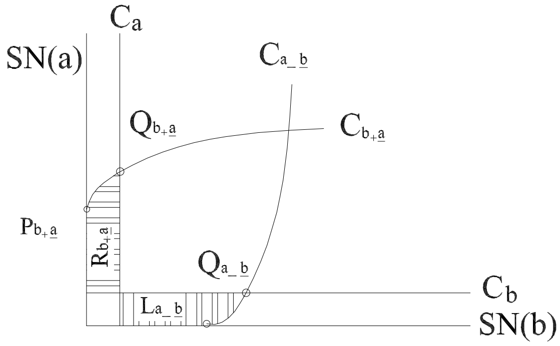

Figure 24: (c) and (e)

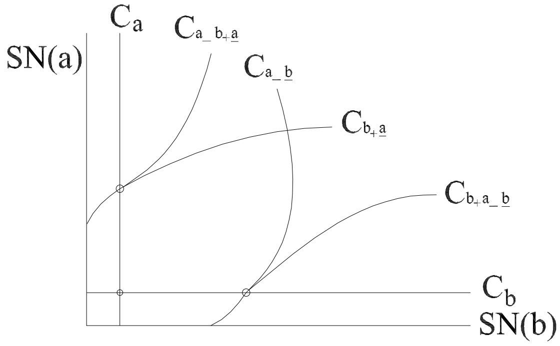

Figure 25: (d) and (f)

(e) Associated with $d = b_+\underline{a}$ there is a curve, C_d, tangent to $SN(a)$ at the point $P_d \in C_d$, transversal to C_a at the point Q_d such that $(\mu, \nu) \in C_d$ imply $b(F_{(\mu,\lambda)}) = d$. Moreover, let R_d be the open rectangle bounded by C_a, $SN(a)$, C_b and C_d. We have $R_d \subset L_a$ and $(\mu, \nu) \in R_d$ imply $b(F_{(\mu,\lambda)}) = d$. Also,

$$(\mu, \lambda) \in \overline{R_d} \cap [C_a \cup SN(a) \setminus C_b] \text{ imply } b(F_{(\mu,\lambda)}) = d;$$

(f) Associated with $c = a_-b_+\underline{a}$ there is a curve C_c, tangent to $C_{b+\underline{a}}$at the point $Q_{b+\underline{a}} \in \overline{C_c} \cap C_{b+\underline{a}}$, and such that $(\mu, \nu) \in C_c$ imply $a(F_{(\mu,\lambda)}) = c$.

Figures (21),(22),(23),(24) and (25) displays the properties in 4.3.1.

Remark 4 *(i) Let* $\gamma(a, a_-b_+\underline{a})$ *be the curve which start in* $P(a,b)$*, follows* C_a *up to* $Q_{b_+\underline{a}}$ *and then follows* $C_{a_-b_+\underline{a}}$*. We have that the sequence of curves*

$$\{C_{\underline{a_-b_+a^m}};\ m \in \mathbb{N}\} \quad \textit{converges to} \quad \gamma(a, a_-b_+\underline{a});$$

(ii) Let $\gamma(a, b_+\underline{a})$ *be the curve which start in* $P(a,b)$*, follows* C_a *up to* $Q_{b_+\underline{a}}$ *and then follows* $C_{b_+\underline{a}}$*. We have that the sequence of curves* $\{C_{\underline{b_+a^m a_-}},\ m \in \mathbb{N}\}$ *converges to* $\gamma(a, b_+\underline{a})$*;*

(iii) Let $\gamma(b, a_-\underline{b})$ *be the curve which start in* $P(a,b)$*, follows* C_b *up to* $Q_{a_-\underline{b}}$ *and then follows* C_{a_-b}*. We have that the sequence of curves* $\{C_{\underline{a_-b^m b_+}};\ m \in \mathbb{N}\}$ *converges to* $\gamma(b, a_-\underline{b})$*;*

(iv) Let $\gamma(b, b_+a_-\underline{b})$ *be the curve which start in* $P(a,b)$*, follows* C_b *up to* $Q_{a_+\underline{b}}$ *and then follows* $C_{b_+a_-\underline{b}}$*. We have that the sequence of curves* $\{C_{\underline{b_+a_-b^m}},\ m \in \mathbb{N}\}$ *converges to* $\gamma(b, b_+a_-\underline{b})$*;*

(v) The same construction that we did here can be done for any $a \in A^{\infty}_{\infty}$*,* $b \in B^{\infty}_{\infty}$ *such that* $C_a \cap C_b \neq \emptyset$*.*

4.3.2 We will say that $f \in DM_0$ satisfy the <u>axiom A</u> if:

(i) All the attracting periodic point of f are hyperbolic and

(ii) The complement of the stable manifolds of the attracting periodic orbits, in $[f(0^+), f(0^-)]$, is an hyperbolic set.

Lemma 12 *Let* $a \in \mathcal{A}^{\infty}_{\infty}$ *and* $b \in \beta^{\infty}_{\infty}$ *such that* $C_a \cap C_b \neq \emptyset$*. There is a neighborhood* $U(a,b) \subset \mathbb{R}^2$ *of the point* $P(a,b)$ *such that*

$$m(\{(\mu,\lambda) \in U(a,b);\ F_{(\mu,\nu)} \ \textit{ do not satisfy axiomA}\}) = 0.$$

That is; the measure of the bifurcation in $U(a,b)$ *in zero.*

Proof This result follows as similar results in [19] or [30].

4.4 Doubling Periods

Let us consider $a \in A^{\infty}_{\infty}$ and $b \in \beta^{\infty}_{\infty}$ such that $C_a \cap C_b \neq \emptyset$.

We will use $a_0 = a$, $b_0 = b$. Let us construct the following sequences:

$$a_{n+1} = \underline{(a_n)_-(b_n)_+}\,; \text{ and } b_{n+1} = \underline{(b_n)_+(a_n)_-}.$$

We have

$$a_0 > a_1 > a_2 > \ldots > \lim_{n\to\infty} a_n = c$$
$$b_0 < b_1 < b_2 < \ldots < \lim_{n\to\infty} b_n = d$$

We will call the sequence (a_n, b_n) the doubling period sequence with starting point (a_0, b_0). The pair (c,d) will be called the <u>target</u> of the doubling period with starting point (a_0, b_0).

Lemma 13 *Associated with c (resp. d) there is a curve C_c (resp. C_d) which is the limit of the curves C_{a_n} (resp. C_{b_n}). Moreover $C_c \cap C_d$ contains a unique point $P(c,d)$.*

For $(\mu,\lambda) \in C_c$ we have $a(F_{(\mu,\lambda)}) = c$ and for $(\mu,\lambda) \in C_d$ we have $b(F_{(\mu,\lambda)}) = d$.

4.5 Tails

Lemma 14 *a) Assume $a \in A^\infty_\infty$ and $d \in Max_2$ is a non-periodic sequence such that $C_a \cap C_d \neq \emptyset$.*

There is a curve C_{a_-d}, which "born" in $\{P(a,d)\} = C_a \cap C_d$, tangent to C_d such that $P(a,d) \in \overline{C_{a_-d}}$ and $(\mu,\lambda) \in C_{a_-d}$ imply $a(F_{(\mu,\lambda)}) = a_-d$.

b) Assume $b \in \beta^\infty_\infty$ and $c \in Max_2$ is a non-periodic sequence such that $C_c \cap C_b \neq \emptyset$.

There is a curve C_{b_+c}, which "born" in $\{P(c,d)\} = C_c \cap C_b$, tangent to C_c at this point such that $P(c,d) \in \overline{C_{b_+c}}$ and $(\mu,\lambda) \in C_{b_+c}$ imply $b(F_{(\mu,d)}) = b_+c$.

We let the proof of this lemma to the reader.

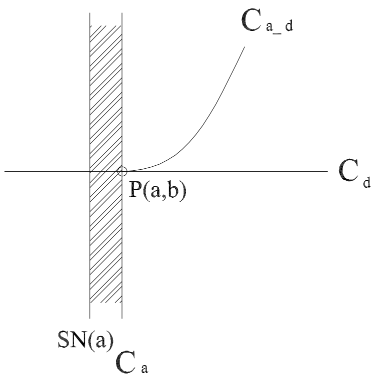

Figure 26:

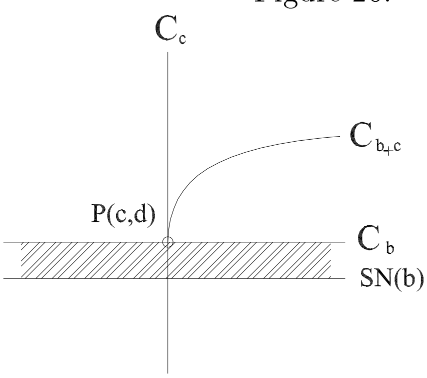

Figure 27:

The figures (26)and (27) displays these facts.

4.6 Entropy zero

In this section we will make (following [21]) some remarks concerning the set

$$EZ = \{(\mu,\lambda);\ h(F_{(\mu,\lambda)}) = 0\}.$$

Here, $h(f)$ means the topological entropy of the map $f \in DM_0$.

a) Assume $a \in A^\infty_\infty$, $b = b(a)$. We have

$$\mathrm{int}(L_a \cap R_b) \subset \mathrm{int}(EZ)$$

In fact, in this situation the set $\Sigma_{a,b}$ is formed by the iterations $\sigma^i(a)$, $i = 1,\cdots,\#per(a)-1$ and their preimages under the map σ.

b) Assume $a \in A^\infty_\infty$, $b = b(a)$ and $\{P(a,b)\} = C_a \cap C_b$ then $P(a,b) \in \mathrm{int}(EZ)$.

In fact, there is a neighborhood of the point $P(a,b)$, which is covered by the union $L_a \cup L_{a-\underline{b}} \cup \{L_c; c \in A(a,b) \cup B(a,b)\} \cup \{C_c; c \in \overline{A(a,b) \cup B(a,b)}\}$. (here $A(a,b)$ and $B(a,b)$ where defined in 4.3.1. It is clear that for any $c \in \overline{A(a,b) \cup B(a,b)}\}$ we have that the set $\Sigma_{c,d}$ is formed by the iterations $\sigma^i(c)$ $i = 1,\cdots,\#per(c)-1$ and their preimages under the map σ, here $d = \sup\{\sigma^i(c); i \in \mathbb{N}\}$.

c) Assume $a \in A^\infty_\infty$, $b = b(a)$ then

$$int(L_{a-\underline{b}}) \cup R_{b+\underline{a}}) \subset \mathrm{int}(EZ).$$

In fact, if $(\mu,\nu) \in \mathrm{int}(L_{a-\underline{b}})$ we have that $a(\mu,\nu) = a$ and $b(\mu,\nu) = \sup\{\sigma^i(a)\ i \in \mathbb{N}\}$.

d) Assume $a \in A^\infty_\infty$, $b = b(a)$ then

$$\overline{L_a} \cap C_{b+\underline{a}} \subset EZ \setminus \mathrm{int}(EZ)$$

and

$$\overline{R_b} \cap C_{a-\underline{b}} \subset EZ \setminus \mathrm{int}(EZ)$$

In fact, given any $(\mu,\nu) \in \overline{L_a} \cap C_{b+\underline{a}}$ we have that there is a small neighborhood $V(\mu,\nu)$ of the point (μ,ν) which is divided by the curve $C_{b+\underline{a}}$ in two disjoint pieces, one of them, included in $\mathrm{int}(EZ)$ and the other included in the interior of the positive entropy set.

All the next results, in this section, follows in a similar way.

e) Let $a \in A^\infty_\infty$, $b = b(a)$ and $(P(a,b), Q_{a-\underline{b}}) \subset C_b$ be the open interval whose boundary points are $P(a,b)$ and $Q_{a-\underline{b}}$ then

$$(P(a,b), Q_{a-b}) \subset \mathrm{int}(EZ).$$

f) Let $a \in A^\infty_\infty$, $b = b(a)$ and $(P(a,b), Q_{b+\underline{a}}) \subset C_a$ be the open interval whose boundary points are $P(a,b)$ and $Q_{b+\underline{a}}$ then $(P(a,b), Q_{b+\underline{a}}) \subset \mathrm{int}(EZ)$.

g) Also, we have

$$\{(\mu,0);\ 0<\mu<1\}\subset \operatorname{int}(EZ)$$

and

$$\{(0,\nu);\ 0<\nu<1\}\subset \operatorname{int}(EZ).$$

Moreover:

$$(1,0)\in [EZ\setminus \operatorname{int}(EZ)]$$

and

$$(0,1)\in [EZ\setminus \operatorname{int}(EZ)].$$

h) Let $a\in A_\infty^\infty$, $b=b(a)$. For any (μ,ν) such that

$$a(F_{(\mu,\lambda)})\leq a_{-}\underline{b} \quad \text{and} \quad b(F_{(\mu,\lambda)})>b$$

or

$$a(F_{(\mu,\lambda)})<a_{-}\underline{b} \quad \text{and} \quad b(F_{(\mu,\lambda)})\geq b$$

we have that $(\mu,\lambda)\in \operatorname{int}\{(\mu,\lambda);\ h(F_{(\mu,\lambda)})>0\}$

i) Let $a\in A_\infty^\infty$, $b=b(a)$. For any (μ,ν) such that

$$a(F_{(\mu,\lambda)})\leq a \quad \text{and} \quad b(F_{(\mu,\lambda)})>b_{+}\underline{a}$$

or

$$a(F_{(\mu,\lambda)})<a \quad \text{and} \quad b(F_{(\mu,\lambda)})\geq b_{+}\underline{a}$$

We have that $(\mu,\lambda)\in \operatorname{int}\{(\mu,\lambda);\ h(F_{(\mu,\lambda)})\neq 0\}$.

j) Let $a\in A_\infty^\infty$, $b=b(a)$. Denote by $B(a,b)\in\mathbb{R}^2$ the point where the curves C_a and C_b born. Let $SN(a,b)=C_a\cap SN(b)$; $\overline{SN}(a,b)=C_b\cap SN(a)$. We have that the intervals $[B(a,b),SN(a,b)[\subset C_a$ and $[B(a,b),\overline{SN}(a,b)[$ are contained in $\operatorname{int}(\{(\mu,\lambda);\ h(F_{(\mu,\lambda)})=0\})$.

k) Let

$$C_{\underline{0}}=\left\{\left(\frac{1+\sqrt{1+4\nu}}{2},\nu\right);\quad \nu\geq 0\right\}$$

and

$$C_{\underline{1}}=\left\{\left(\mu,\frac{1+\sqrt{1+4\mu}}{2}\right);\quad \mu\geq 0\right\}.$$

We have,

$$(C_{\underline{0}}\setminus\{(1,0)\})\cup(C_{\underline{1}}\setminus\{(0,1)\})\subset \operatorname{int}(\{(\mu,z);\ h(F_{(\mu,\lambda)})\neq 0\}).$$

l) Let

$$L_{\underline{0}}=\{(\mu,\nu);\ \mu\geq \frac{1+\sqrt{1+4\nu}}{2},\ \nu\geq 0\},$$

$$R_{\underline{1}}=\{(\mu,\nu);\ \nu\geq \frac{1+\sqrt{1+4\nu}}{2},\ \mu\geq 0\},$$

$$L_{0\underline{1}} \;=\; \{(\mu\,,\nu)\,,\, a(\mu\,,\nu) \;=\; 0\underline{1}\,\},$$

and

$$R_{1\underline{0}} \;=\; \{(\mu\,,\nu)\,,\, b(\mu\,,\nu) \;=\; 1\underline{0}\,\}.$$

We have

$$\mathrm{int}(L_{\underline{0}} \setminus R_{1\underline{0}}) \cup \mathrm{int}(R_{\underline{1}} \setminus L_{0\underline{1}}) \subset \mathrm{int}(\{(\mu,\nu);\;\; h(F_{(\mu,\lambda)}) \neq 0\}).$$

¿From all of the above results it is clear that EZ is an arc-connected set. We do not know if it is locally connected.

4.7 Non-zero measure for the Bifurcation

The result in this subsection is a consequence of a result proved by Rovella in [32]. We established it here only for the sake of completness.

Let $a \in Min_2$ and $b \in Max_2$ be eventually periodic sequences such that $C_a \cap C_b \neq \emptyset$. Let $C_a \cap C_b = \{P(a,b)\}$ and $U(a,b)$ be any neighborhood of $P(a,b)$. We have

Lemma 15 $m(\{(\mu,\nu) \in U(a,b);\;\; F_{(\mu,\nu)}$ *is not* $AxA\}) > 0$.

As a consequence of this result we get

Corollary 2 *Let* $(\mu,\nu) \in (EZ \setminus int(EZ))$ *and* $U(\mu,\lambda)$ *be any neighborhood of* (μ,λ). *We have*

$$m(\{(\alpha,\beta) \in U(\mu,\lambda);\;\; F_{(\alpha,\beta)} \text{ is not } Ax.A\}) > 0.$$

Proof: (μ,λ) can be approximated by a sequences (μ_n,ν_n) as in the Lemma 17.

4.8 Tales in $L_{\underline{0}}$ and $R_{\underline{1}}$

a) Associated to $b = 1\underline{0}$ there is a curve $C_{1\underline{0}} \subset L_{\underline{0}}$ such that $(1,0) \in \overline{C_{1\underline{0}}}$;this curve is tangent to $C_{\underline{0}}$ at $(1,0)$ and $(\mu,\nu) \in C_{1\underline{0}}$ imply $b(F_{(\mu,\lambda)}) = 1\underline{0}$.

b) Associated to $a = 0\underline{1}$ there is a curve $C_{0\underline{1}} \subset R_{\underline{1}}$ that satisfy $(0,1) \in \overline{C_{0\underline{1}}}$; it is tangent to $C_{\underline{1}}$ and $(0,1)$; $(\mu,\nu) \in C_{0\underline{1}}$ imply $a(F_{(\mu,\lambda)}) = 0\underline{1}$;

c) Let $\varphi_{1\underline{0}} :]1,\infty[\to \mathbb{R}$ denote the map that satisfy $\mathrm{Graph}(\varphi_{1\underline{0}}) = C_{1\underline{0}}$.
We have

$$\{(\mu,\nu);\;\; b(F_{(\mu,\lambda)})\} = 1\underline{0}\} = \{(\mu,\nu);\;\; 0 < \nu \leq \varphi_{1\underline{0}}(\mu);\;\; \mu > 1\}.$$

d) Let $\varphi_{0\underline{1}} :]1,\infty[\to \mathbb{R}$ denote the map that satisfy $\mathrm{Graph}(\varphi_{0\underline{1}}) = C_{0\underline{1}}$.

We have

$$\{(\mu,\nu);\ a(F_{(\mu,\lambda)}) = 0\underline{1}\} = \{(\mu,\nu);\ 0 < \mu \leq \varphi_{0\underline{1}}(\nu), \quad \nu > 1\}.$$

e) Let $b \in B_\infty^\infty$ and $\{P(\underline{0},b)\} = C_{\underline{0}} \cap C_b$.

There is a curve $C_{b+\underline{0}} \subset L_{\underline{0}}$ such that $P(\underline{0},b) \in \overline{C_{b+\underline{0}}}$; $C_{b+\underline{0}}$ is tangent to $C_{\underline{0}}$ at $P(\underline{0},b)$ and $(\mu,\lambda) \in C_{b+\underline{0}}$ imply $b(F_{(\mu,\lambda)}) = b_+\underline{0}$.

f) Let $P(\underline{0},b) = (\mu_{\underline{0},b}, \nu_{u0,b})$ and $\varphi_{b+\underline{0}} :]\mu_{\underline{0},b}, \infty[\to \mathbb{R}$ be the map that satisfy Graph$(\varphi_{b+\underline{0}}) = C_{b+\underline{0}}$.

Let $\varphi_b : [\mu_{\underline{0}}, \infty[\to \mathbb{R}$ be the map that satisfy Graph$(\varphi_b) = L_{\underline{0}} \cap C_b$.

Clearly $\varphi_b(\mu) < \varphi_{b+\underline{0}}(\mu)$, $\mu > \mu_{\underline{0},b}$.

We have $\{(\mu,\nu); b(F_{(\mu,\lambda)}) = b_+\underline{0}\} = \{(\mu,\nu);\ \varphi_b(\mu) < \nu \leq \varphi_{b+\underline{0}}(\mu)\}$.

g) Let $a \in A_\infty^\infty$ and $\{P(a,\underline{1})\} = C_a \cap C_{\underline{1}}$. There is a curve $C_{a-\underline{1}} \subset R_{\underline{1}}$ such that $P(a,\underline{1}) \in \overline{C_{a-\underline{1}}}$; $C_{a-\underline{1}}$ is tangent to $C_{\underline{1}}$ at $P(a,\underline{1})$ and $(\mu,\lambda) \in C_{a-\underline{1}}$ imply $a(F_{(\mu,\lambda)}) = a_-\underline{1}$.

h) Let $P(a,\underline{1}) = (\mu_{a,\underline{1}}, \nu_{a,\underline{1}})$ and $\varphi_{a-\underline{1}} :]\nu_{a,\underline{1}}, \infty[\to \mathbb{R}$, $\varphi_a : [\nu_{a,\underline{1}}, \infty[\to \mathbb{R}$ be the maps that satisfies:

$$\text{Graph}(\varphi_{a-\underline{1}}) = C_{a-\underline{1}} \quad \text{and} \quad \text{Graph}(\varphi_a) = R_{\underline{1}} \cap C_a.$$

Clearly $\varphi_a(\nu) < \varphi_{a-\underline{1}}(\nu)$,for any $\nu > \nu_{a-,\underline{1}}$.

We have $\{(\mu,\nu);\ a(F_{(\mu,\nu)}) = a_-\underline{1}\} = \{(\mu,\nu);\ \varphi_a(\nu) < \mu \leq \varphi_{a-\underline{1}}(\nu), \quad \nu > \nu_{a,\underline{1}}\}$.

4.9 Bifurcations in $R_{\underline{1}} \cup L_{\underline{0}}$

In the same way as was proved in [19] or as in [30] we can prove.

Lemma 16 *$m(\{(\mu,\nu) \in R_{\underline{1}} \cup L_{\underline{0}};\ F_{(\mu,\nu)}$ do not satisfy Ax.A$\}) = 0$.*

4.10 Density of Axiom A Systems.

A consequence of our's and Rovella's result is:

Lemma 17 *$\{(\mu,\nu);\ F_{(\mu,\nu)}$ satisfy Ax.A$\}$ is a dense set in $\mathbb{R}_0^+ \times \mathbb{R}_0^+$.*

4.11 The Global Picture

Taking together all the results in this section the global picture of the bifurcation diagramme is as in figure 28.

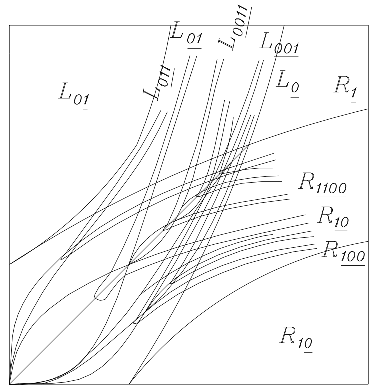

Figure 28:

5 The general case

Let us consider the five parameter family of maps $F_{(\alpha,c_1,c_2,\mu,\nu)}\colon (R\setminus\{0\})\to\mathbb{R}$, $\mu\geq 0$, $\nu\geq 0$ $\alpha>1$, $c_1>0$, $c_2>0$ in DM_0 given by:

$$F_{((\alpha,c_1,c_2,\mu,\nu)}(x)=\begin{cases} -\mu+c_1x^{\alpha}, & x>0 \\ \nu-c_2(-x)^{\alpha}, & x<0 \end{cases}$$

We have to observe that all the previous results, for the quadratic family, are (still) true when we fix generic values of the parameter α, c_1 and c_2. That is, the bifurcation diagramme associated to any generic two parameter family $F_{(\mu,\nu)}=F_{(\alpha,c_1,c_2,\mu,\nu)}$is,up to an homeomorphism, the same as the given for the quadratic family.

6 Application to Geometric Vector fields

Assume $X_0\colon U\subset\mathbb{R}^3\to\mathbb{R}^3$ is a three dimensional vector field that satisfies:

a) $0\in U$ is an hyperbolic singularity of the vector field X, whose eigenvalues satisfies $-\lambda_1<-\lambda_2<0<\lambda_3$;

b) The components, γ_1 and γ_2, of the set $(W_0^u\setminus\{0\})$ satisfies $\gamma_1\subset W_0^s$, $\gamma_2\subset W_0^s$;

c) There is a transversal section, $\Sigma\subset U$, such that: Σ is transversal to W_0^s and $\Sigma\setminus W_0^s=\Sigma_1\cup\Sigma_2$ are two disjoint sets in Σ that satisfies:

$$\pi_1(\Sigma_1)\subset\Sigma_1 \quad \text{and} \quad \pi_2(\Sigma_2)\subset\Sigma_2$$

where $\pi_i\colon\Sigma_i\to\Sigma$, $i=1,2$ is the first return map associated to the respective cross section.

Let $\mathcal{U}_0 \subset \mathfrak{X}^r(U, \mathbb{R}^3)$ be a neighborhood of the vector field X_0 in $\mathfrak{X}^r(U, \mathbb{R}^3) = \{X : U \subset \mathbb{R}^3 \to \mathbb{R}^3; \quad X \text{ is } C^r\}$ with the usual C^r-topology.

In the above conditions there are codimension one submanifold, $\mathcal{N}_1, \mathcal{N}_2 \subset \mathcal{U}_0$; such that:

a) $Y \in \mathcal{N}_i$ if and only if $\gamma_i(Y) \subset W^s_{\sigma(Y)}$ here $\sigma(Y)$ is the hyperbolic singularity, near 0, that correspond to the vector field Y. Also, $(W_{\sigma(Y)} \setminus \{\sigma(Y)\}) = \gamma_1(Y) \cup \gamma_2(Y)$, where $\gamma_i(Y)$ is the natural extension of the component γ_i;

b) $(\mathcal{U}_0 \setminus \mathcal{N}_i) = \mathcal{U}_1^i \cup \mathcal{U}_2^i$ where $Y \in \mathcal{U}_1^1$ if and only if $\pi_1(\Sigma_1) \subset \Sigma_1$ and $Y \in \mathcal{U}_2^2$ if and only if $\pi_2(\Sigma_2) \subset \Sigma_2$ and

c) $\mathcal{N}_1$ is transversal to $\mathcal{N}_2$.

Finally we will assume that Σ can be foliated by one-dimensional submanifolds $\{\mathcal{F}_x; \quad x \in \Sigma\}$ such that

(i) $x \in W^s_Y \cap \Sigma$ imply $\mathcal{F}_x = W^s_Y \cap \Sigma$ and

(ii) $\pi_i(\mathcal{F}_x) \subset \mathcal{F}_{\pi_i(x)}$, $x \in \Sigma_i$.

In this situation the interesting part of the bifurcation theory, for elements in $\mathcal{U}_0$, is located in $\mathcal{U}_1^2 \cap \mathcal{U}_2^1$. In this set we can apply the results in Sections 3,4 and 5 to obtain, bis a bis, similar results for generic two parameter families of vector fields in $\mathcal{U}_0$.

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