Homework 4

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Problem 1 (I worked with Federico Castillo):

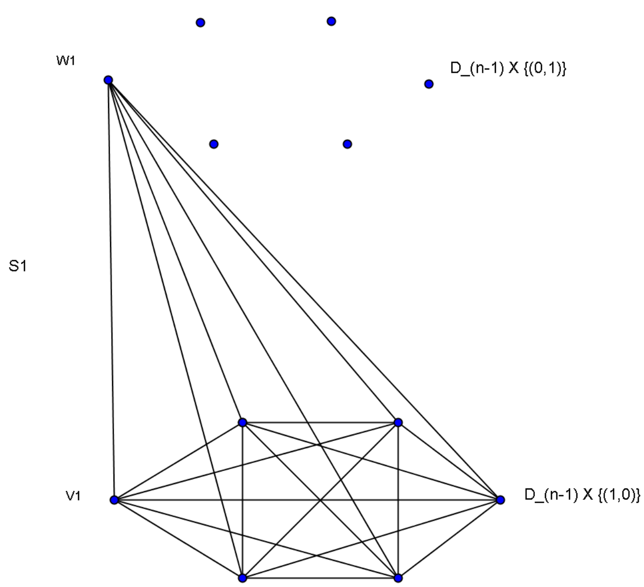
Lets prove that the polytope $P_{n-1} = \Delta_{n-1} \times \Delta_1$:

a)Has $n!$ triangulations
b)These triangulations are induced by the triangulations of $\Delta_{n-2} \times \Delta_1$ and the selection of a vertex of P_{n-1}.
c) These triangulations are regular.

Define $\underline{\Delta_{n-1}} = \Delta_{n-1} \times (1,0)$ and $\overline{\Delta_{n-1}} = \Delta_{n-1} \times (0,1)$. Let $\{v_1, v_2, ..., v_n\} = V(\underline{\Delta_{n-1}})$ and $\{w_1, w_2, ..., w_n\} = V(\overline{\Delta_{n-1}})$, and suppouse these sets of vertices are pairwise related.

For any $S : \{S_1, S_2, ..., S_n\}$ triangulation of P_{n-1}, the following statements holds:

i) $\underline{\Delta_{n-1}}$ **is a facet of exactly one** S_i:



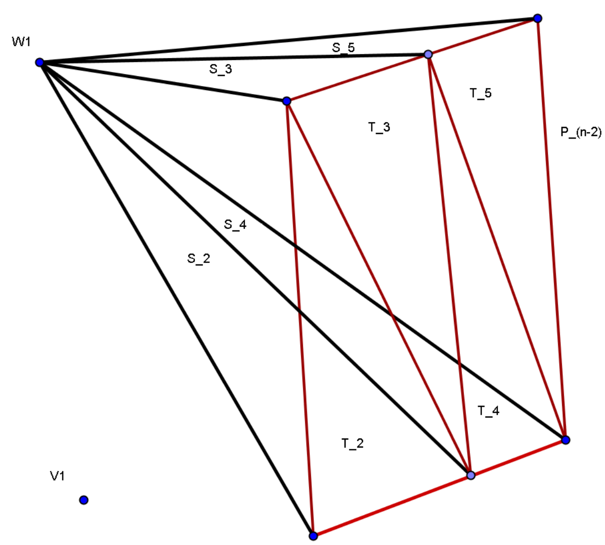
Let p be any point in the "interior" of $\underline{\Delta_{n-1}}$ (I mean, p is not contained in any proper face of $\underline{\Delta_{n-1}}$). Observe that p can be written only as a convex combination of **all** the vertex in $\underline{\Delta_{n-1}}$. Since p must be contained in some S_i (because $P_{n-1} = \cup S_i$), such S_i must contain all $\underline{\Delta_{n-1}}$. WLOG asume that $\underline{\Delta_{n-1}} \subseteq S_1$. Since S_1 is a n-dimensional simplex and $\underline{\Delta_{n-1}}$ is a $n-1$-dimensional simplex, there exists w_i vertex of $\overline{\Delta_{n-1}}$ which is also vertex of S_1; WLOG assume that w_1 is such vertix. Therefore we get that $S_1 = conv(\underline{\Delta_{n-1}} \cup \{w_1\})$. Now suppouse that

$\underline{\Delta_{n-1}} \subseteq S_k \neq S_1$. Then $S_k = conv(\underline{\Delta_{n-1}} \cup \{w_j\})$, where $w_j \neq w_1$. Observe that the point $q = \frac{1}{2n}w_1 + \frac{1}{2n}v_1 + \frac{1}{n}v_2 + \frac{1}{n}v_3 + ... + \frac{1}{n}v_n = \frac{1}{n}v_1 + \frac{1}{n}v_2 + ... + \frac{1}{2n}w_j + \frac{1}{2n}v_j + ...\frac{1}{n}v_n$,is such that $q \in Int(S_1) \cap Int(S_k)$. Therefore $S_1 \cap S_k$ would not be a face, and S would not be a triangulation. So we conclude that $\underline{\Delta_{n-1}}$ is only contained at S_1.

ii) w_1 **is vertex of all** S_i's: Observe that for any S_i, at least one between v_1, w_1 must belong to S_i (otherwise, S_i would not be n-dimensional).Therefore, it will be sufficient to show that S_1 is the only division which contains v_1. To prove this, observe that S_1 contains all the adjacent edges of v_1, so the cone centered at v_1 and generated by these edges cover all P_{n-1}. Then, any ray between v_1 and any vertex $w_j \in \overline{\Delta_{n-1}} \setminus w_1$, must interesct the interior of S_1. This implies that for any $w_j \in \overline{\Delta_{n-1}} \setminus w_1$, the vertices w_j and v_1 can not lie in any division. Therefore we conclude that S_1 is the only division that contains v_1, so w_1 is vertex of all S_i's.

Now define $P_{n-2} = conv(V(P_{n-1}) \setminus \{v_1, w_1\})$ (observe that P_{n-2} is combinatorially equivalen to $\Delta_{n-2} \times \Delta_1$).

iii) **The divisions** $S_2, ..., S_n$ **induces a triangulation of** P_{n-2}:



Define $T_i = conv(V(S_i) \setminus w_1)$ for all $i = 2, 3, ..., n$. Observe that $\cup_{i=2}^{n} T_i = (\cup_{i=2}^{n} S_i) \cap P_{n-2} = P_{n-2}$ (because $S_1, S_2 ..., S_n$ is a triangulation of P_{n-1} and S_1 does not contains "interior" points of P_{n-2}). Also observe that $T_i \cap T_j = (S_i \cap S_j) \cap P_{n-2}$ is a face of both T_i, T_j, because S_i and S_j are divisions of a triangulation. Therefore $T : \{T_2, T_3, ..., T_n\}$ is a triangulation of P_{n-2}.

We have proven that any triangulation of $P_{n-1} = \Delta_{n-1} \times \Delta_1$ induces a triangulation of P_{n-2} (which is combinatorially equivalent to $\Delta_{n-2} \times \Delta_1$). We can get a converse result in the following way:

Fix a vertex $w_1 \in \overline{\Delta_{n-1}}$ and a triangulation $T' : \{T'_2, ..., T'_n\}$ of $\Delta_{n-2} \times \Delta_1$. Identify the traingulation T' of $\Delta_{n-2} \times \Delta_1$ with a triangulation $T : \{T_2, ..., T_n\}$ of P_{n-2} in a natural way (this can be done by the combatorially equivalence). Define a triangulation $S : \{S_1, S_2, ..., S_n\}$of $\Delta_{n-1} \times \Delta_1$, induced by w_1 and T, in the folloing way:

i)$S_1 = conv(\underline{\Delta_{n-1}} \cup \{w_1\})$. This makes that S_1 is the only division which contains v_1
ii) For $2 \leq i \leq n$, let $S_i = conv(T_i \cup \{w_1\})$.

Now, we are able to identify a bijection between $\{$Triangulations of $\Delta_{n-1} \times \Delta_1\}$ and $\{$Vertices of $\overline{\Delta_{n-1}}\} \times \{$Triangulations of $\Delta_{n-2} \times \Delta_1\}$. Applying induction to the previous result we get: $|\{$ Triangulations of $\Delta_{n-1} \times \Delta_1\}| = n!$

Since we have shown that any triangulation of $\Delta_{n-1} \times \Delta_1$ can be constructed from a triangulation of $\Delta_{n-2} \times \Delta_1$, **I tried** to prove by induction that all the triangulations of $\Delta_{n-1} \times \Delta_1$ are regular:

(*the following proof is not complete*)

i) The base case $n = 2$ is easy to check.

ii) Suppose he have proved the case $n-1$, i.e, that all the triangulations of $\Delta_{n-2} \times \Delta_1$ are regular. Then for any triangulation T' of $\Delta_{n-2} \times \Delta_1$ there exists a function $h' : V(\Delta_{n-2} \times \Delta_1) \to \mathbb{R}$ such that the projection of the lower faces of $Q' = conv(\{(u_i, h'(u_i)) : u_i \in V(\Delta_{n-2} \times \Delta_1)\})$ induces T'.

Now suppouse $S : \{S_1, S_2, ..., S_n\}$ is a triangulation of $\Delta_{n-1} \times \Delta_1$ with vertix at w_1 and such that $\{S_2, S_3, ..., S_n\}$ induces a triangulation $T : \{T_2, T_3, ..., T_n\}$ at P_{n-2}. Since P_{n-2} is combinatorially equivalent to $\Delta_{n-2} \times \Delta_1$, we can apply the induction hypothesis to get $h : V(P_{n-2}) \to \mathbb{R}$ such that $Q = conv(\{(u_i, h'(u_i)) : u_i \in V(P_{n-2})\})$ induces T. Now we must define $h(w_1)$, $h(v_1)$ in such a way we get the triangulation S. Consider $v_1 = (1\ 0\ 0...\ 1\ 0)$ and $w_1 = (1\ 0\ 0...\ 0\ 1)$, and $(v_i)_1 = (w_i)_1 = 0$ for all $i \neq 1$. Let $F_1, F_2, ..., F_{n-1}$ be the lower facets of Q (these facets are associated to $T_2, T_3, ..., T_n$) and let c_i be the direction which maximimizes F_i (i.e, $Q_{c_i} = F_i$ and the last coordinate of c_i is negative). **Now I would try to construct a set of directions** $d_1, d_2, ..., d_n$, which depends on $c_1, ..., c_{n-1}$, $h(w_1), h(v_1)$ and such that:

i)d_i is maximized at $conv((w_1, h(w_1)) \cup F_i)$ for $i = 1, ..., n-1$.
ii)d_n is maximized at $conv((w_1, h(w_1)) \cup (v_1, h(v_1)) \cup (v_2, h(v_2))... \cup (v_n, h(v_n)))$
iii) The last component of each d_i is negative for $i = 1, ..., n$.

Then the lower facects of $conv((w_1, h(w_1)) \cup (v_1, h(v_1)) \cup Q)$ wolud be $conv((w_1, h(w_1)) \cup F_i)$ for $i = 1, ..., n-1$ and $conv((w_1, h(w_1)) \cup (v_1, h(v_1)) \cup (v_2, h(v_2))... \cup (v_n, h(v_n)))$. In that case the triangulation induced by projection of the lower facets $conv((w_1, h(w_1)) \cup (v_1, h(v_1)) \cup Q)$ would be S.

Problem 2:

I will prove **inductively** that for any positive integer d we have:

$$\sum_{t\geq 0}(t+1)^d z^t = \frac{A(d,1)z^0+...+A(d,d)z^{d-1}}{(1-z)^{d+1}}$$

Where $A(d,k)$ is defined by:
i)$A(1,1)=1$
ii)$A(d,k)=0$, if $k\leq 0$ or $k\geq d+1$
iii) $A(d,k)=(d-k+1)A(d-1,k-1)+kA(d-1,k)$

Case $d=1$:
Observe that $\sum_{t\geq 0}(t+1)z^t = (\sum_{t\geq 0} z^{t+1})' = (\sum_{t\geq 1} z^t)' = (\frac{1}{1-z}-1)' = \frac{1}{(1-z)^2}$. Then we conclude that the condition holds for $d=1$.

Assume we have the desired result for the case $d-1$, i.e, we have

$$\sum_{t\geq 0}(t+1)^{d-1}z^t = \frac{A(d-1,1)z^0+...+A(d-1,d-1)z^{d-2}}{(1-z)^d}$$

Now lets prove that we can get the case d:

$$\sum_{t\geq 0}(t+1)^d z^t = (\sum_{t\geq 0}(t+1)^{d-1}z^{t+1})' = (z\sum_{t\geq 1}(t+1)^{d-1}z^t)' = \sum_{t\geq 0}(t+1)^{d-1}z^t + z(\sum_{t\geq 0}(t+1)^{d-1}z^t)' =$$

$$\frac{A(d-1,1)z^0+...+A(d-1,d-1)z^{d-2}}{(1-z)^d} + z\Big(\frac{A(d-1,1)z^0+...+A(d-1,d-1)}{z^{d-2}}\Big)' =$$

$$\frac{A(d-1,1)z^0+...+A(d-1,d-1)z^{d-2}}{(1-z)^d} +$$

$$\frac{A(d-1,2)z^1+2A(d-1,3)z^2+...+(d-2)A(d-1,d-1)z^{d-2}}{(1-z)^d} +$$

$$\frac{dA(d-1,1)z^1+dA(d-1,2)z^2+...+dA(d-1,d-1)z^{d-1}}{(1-z)^{d+1}} =$$

$$\frac{A(d-1,1)z^0+\big((d-1)A(d-1,1)+2A(d-1,2)\big)z^1+\big((d-2)A(d-1,2)+3A(d-1,3)\big)z^2+...}{(1-z)^{d+1}}$$

$$\frac{...+\big((d-k+1)A(d-1,k-1)+kA(d-1,k)\big)z^{k-1}+...+A(d-1,d-1)z^{d-1}}{(1-z)^{d+1}} =$$

$$\frac{A(d,1)z^0+A(d,2)z^1...+A(d,k)z^{k-1}+....+A(d,d)z^{d-1}}{(1-z)^{d+1}}$$

By the way we define $A(d,k)$, we have $A(d,1)=dA(d-1,0)+A(d-1,1)=A(d-1,1)$ and $A(d,d)=A(d-1,d-1)+dA(d-1,d)=A(d-1,d-1)$ so we were allowed to change $A(d-1,1)$

by $A(d,1)$, and $A(d-1,d-1)$ by $A(d,d)$, as we did in the last equation.

Finally we conclude:

$$\sum_{t\geq 0}(t+1)^d z^t = \frac{A(d,1)z^0 + ... + A(d,d)z^{d-1}}{(1-z)^{d+1}}$$

for all positive integers d.

Problem 3:

Lets prove that $E(d,k)$, the number of permutations of $[d]$ having exactly $k-1$ descents, $1 \leq k \leq d$, satisfies the formula :

$$E(d,k) = (d-k+1)E(d-1,k-1) + kE(d-1,k)$$

Let p_d be any permutation of $[d]$ and consider it as a row vector. If we remove d from the vector p_d, we get a vector associated to a **unique** permutation of $[d-1]$. Similarly, if we have a permutation p_{d-1} of $[d-1]$, and we consider it as a row vetor, we can get d diferent permutations of $[d]$, by inserting d to p_{d-1}, at any of d possible positions (before the first coordinate, between the fisrt and the second coordinate,..., after the last coordinate).Therefore we get a bijection between $\{$ permutations of $[d]\}$ and $\{$ permutations of $[d-1]\} \times \{d$ positions to insert $d\}$

Let $p_{d-1} = (a_1, a_2, ..., a_{d-1})$ be any permutation of $[d-1]$, and assume p_{d-1} has m descents. Lets consider 3 cases:

i) $p_d := (d, a_1, a_2, ..., a_{d-1})$ will have $m+1$ descents since $d > a_1$ and the other order relations remain the same.

ii)$p_d := (a_1, a_2, ..., a_i, d, a_{i+1}.., a_{d-1})$. If we introduce d at a descent position (i,e $a_i > a_{i+1}$), the number of total descents remain m. If we introduce d at a non-descent position (i,e $a_i < a_{i+1}$), the number of total descents increases by 1, so we get $m+1$ descents.

iii) $p_d := (a_1, a_2, ..., a_{d-1}, d)$. Since $a_{d-1} < d$, the number of descents is m.

By the previous observations we conclude that we can get p_d, a permutation of $[d]$ with $k-1$ descents, in any of the following ways:

i)Fix p_{d-1} a a permutation of $[d-1]$ with $k-2$ descents. Insert d before the first coordinate of p_{d-1}, or in any of the $\big(d-2-(k-2)\big)$ non-desecent positions of p_{d-1}. So we can get p_d from p_{d-1} by inserting d at $d-k+1$ possible positions.

ii) Fix p_{d-1} a a permutation of $[d-1]$ with $k-1$ descents. Insert d after the last coordinate of p_{d-1}, or in any of the $k-1$ descents positions of p_{d-1}. So we can get p_d from p_{d-1} by inserting d at k possible positions.

Finally we conclude that $E(d,k) = (d-k+1)E(d-1,k-1) + kE(d-1,k)$, for all $1 \leq k \leq d$.

Since $E(1,1) = 1$; $E(d,k) = 0$, if $k \leq 0$ or $k \geq d+1$; and $E(d,k) = (d-k+1)E(d-1,k-1) + kE(d-1,k)$, for all $1 \leq k \leq d$; we conclude that the numbers $E(d,k)$ and $A(d,k)$ shares exactly the same recursive definition, so we get $E(d,k) = A(d,k)$ for all $1 \leq k \leq d$.

Therefore, if we want to prove $A(d,k) = A(d,d+1-k)$, it will be sufficinet to show $E(d,k) = E(d,d+1-k)$, in other words, that the number of permutations of $[d]$ with $k-1$ descents is equal to the number of permutation with $d-k$ descents. Define the function $R : [d] \to [d]$ that reflects any permutation, for instance if $p = (1342) \Rightarrow R(p) = (2431)$.Let p_d be any permutation of $[d]$ and obvserve that: p_d has $k-1$ descents $\iff$ p_d has $d-k$ ascents $\iff$ $R(p_d)$ has $d-k$ descents. Then R is a bijection between permutation of $k-1$ descents and those of $d-k$ descents, so $E(d,k) = E(d,d+1-k)$.

Problem 4 :

Given $f : \mathbb{N} \to \mathbb{N}$ such that $\displaystyle\sum_{t\geq 0} f(t)z^t = \frac{g(z)}{(1-z)^{d+1}}$, lets prove that the following are equivalent:

i) f is a polynomila of degree d
ii)g is a polynomial of degree at most d such that $g(1) \neq 0$.

i)$\Rightarrow$ ii): Observe that the set $\{1, (t+1), (t+1)^2, ..., (t+1)^d\}$ is a basis for the set of polynomials of degree at most d (this is true because each polynomial $1, (t+1), (t+1)^2, ..., (t+1)^d$ has a different degree). Since f is a polynomial of deggre d, we can write

$$f = a_d(t+1)^d + a_{d-1}(t+1)^{d-1} + ... + a_1(t+1) + a_0$$

with $a_d \neq 0$. By the result in Problem 2, we know that

$$\sum_{t\geq 0}(t+1)^m z^t = \frac{A(m,1)z^0 + ... + A(m,m)z^{m-1}}{(1-z)^{m+1}}$$

Then

$$\sum_{t\geq 0} f(t)z^t = \sum_{t\geq 0}(a_d(t+1)^d + a_{d-1}(t+1)^{d-1} + ... + a_1(t+1) + a_0)z^t =$$

$$\sum_{t\geq 0}\sum_{m=0}^{d} a_m(t+1)^m z^t = \frac{1}{1-z} + \sum_{m=1}^{d}\sum_{t\geq 0} a_m(t+1)^m z^t = \frac{1}{1-z} + \sum_{m=1}^{d} \frac{A(m,1)z^0 + ... + A(m,m)z^{m-1}}{(1-z)^{m+1}}$$

Defining $P_m(z) := A(m,1)z^0 + ... + A(m,m)z^{m-1}$ for $m = 1, ..., d$ (polynomial of degree $m-1$), and replacing $\frac{A(m,1)z^0+...+A(m,m)z^{m-1}}{(1-z)^{m+1}}$ by $\frac{P_m(z)(1-z)^{d-m}}{(1-z)^{d+1}}$ we get that

$$\sum_{t\geq 0} f(t)z^t = \frac{(1-z)^d}{(1-z)^{d+1}} + \sum_{m=1}^{d} \frac{P_m(z)(1-z)^{d-m}}{(1-z)^{d+1}} = \frac{(1-z)^d + \sum_{m=1}^{d} P_m(z)(1-z)^{d-m}}{(1-z)^{d+1}}$$

Now define $g(z) := (1-z)^d + \sum_{m=1}^{d} P_m(z)(1-z)^{d-m}$, and observe that $g(z)$ is a polynomial of degree d, since $(1-z)^d$ is of degree d, and $P_m(z)(1-z)^{d-m}$ is of degree $d-1$ for $m = 1, ..., d$. Observe that $g(1) = (1-1)^d + \sum_{m=1}^{d} P_m(1)(1-1)^{d-m} = P_d(1) = A(d,1) + A(d,2) + ... + A(d,d) \neq 0$ (since $A(d,k) > 0$ for $1 \leq k \leq d$) , so we get $g(1) \neq 0$. Finally we conclude that $\sum_{t \geq 0} f(t)z^t = \frac{g(z)}{(1-z)^{d+1}}$, where g is a polynomial with the desired properties.

ii)$\Rightarrow$ i): Lets start checking that the set

$$\{(1-z)^d, (1-z)^{d-1}P_1(z), (1-z)^{d-2}P_2(z), ..., (1-z)P_{d-1}(z), P_d(z)\}$$

is basis for the set of polynomial of degree at most d. Since this set has $d+1$ polynomias it will be sufficient to show that the set is linearly indepemdent. Suppouse

$$\alpha_0(1-z)^d + \alpha_1(1-z)^{d-1}P_1(z) + ... + \alpha_d P_d(z) = 0$$

, and let k be the greatest index such that $\alpha_k \neq 0$, then

$$\alpha_0(1-z)^d + \alpha_1(1-z)^{d-1}P_1(z) + ... + \alpha_{k-1}(1-z)^{d-k+1}P_{k-1}(z) = -\alpha_k(1-z)^{d-k}P_d(k)$$

Observe that $P_k(1) = A(k,1) + A(k,2) + ... + A(k,k) \neq 0$, so $d-k$ is the multiplicity of 1 as a root of the right side. However the multiplicity of 1 as a root of the left side is at least $d-k+1$, this contridiction let us to conclude that

$$\{(1-z)^d, (1-z)^{d-1}P_1(z), (1-z)^{d-2}P_2(z), ..., (1-z)P_{d-1}(z), P_d(z)\}$$

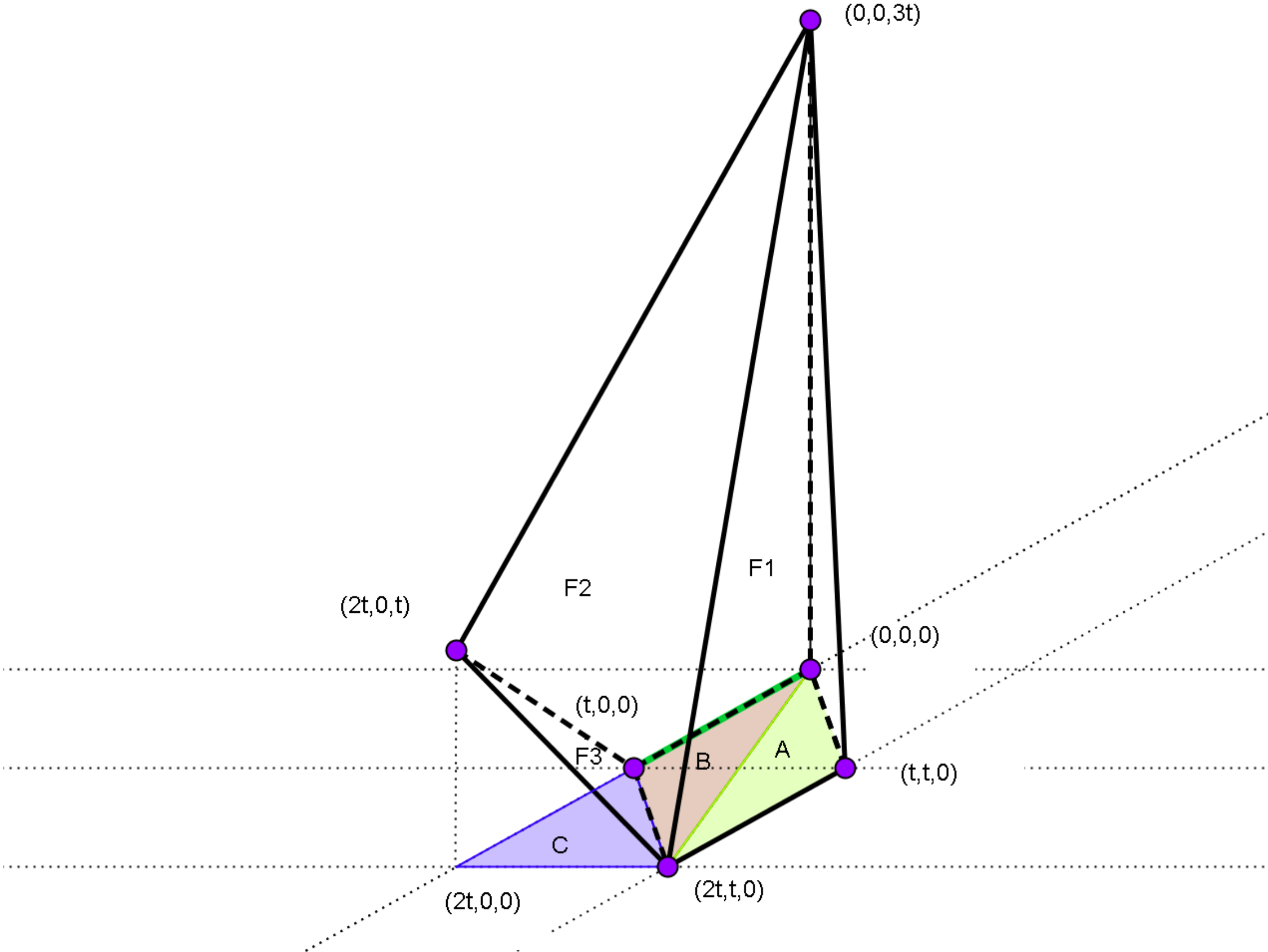
is basis for the set of polynomial of degree at most d.

Now write $g(z) = a_0(1-z)^d + a_1(1-z)^{d-1}P_1(z) + ... + a_{d-1}(1-z)P_{d-1}(z) + a_d P_d(z)$. Since $g(1) = a_d P_d(1)$ and $P_d(1) \neq 0$, we get that the condition $g(1) \neq 0$ implies $a_d \neq 0$.

Therefore $\frac{g(z)}{(1-z)^{d+1}} = a_0\frac{1}{1-z} + a_1\frac{P_1(z)}{(1-z)^2} + ... + a_d\frac{P_d(z)}{(1-z)^{d+1}}$, where $a_d \neq 0$. Since each $\frac{P_m(z)}{(1-z)^{m+1}} = \sum_{t \geq 0}(t+1)^m z^t$, then $\frac{g(z)}{(1-z)^{d+1}} = \sum_{t \geq 0}\left(a_0 + a_1(t+1) + a_2(t+1)^2 + ... + a_d(t+1)^d\right)z^t$. Finally we get that $f(t) = a_0 + a_1(1+t) + a_2(1+t)^2 + ... + a_d(1+t)^d$, with $a_d \neq 0$, i.e., f is a polynomial of degree d as we wanted to prove.

Problem 5(I worked with Jose Samper and Fabian Latorre):

The image of $P_t = conv\{(0,0,0), (0,0,3t), (t,0,0), (t,t,0), (2t,t,0)(2t,0,t)\}$ is the following :



And the inequality description of this polytope, P_t is:

$$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 3 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \leq \begin{pmatrix} t \\ 3t \\ 3t \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Consider a division of the base of the polytope P_t in triangles A, B, C as shown in the figure. The face above the triangle A is F_1 which is defined by $3y + z = 3t$. The face above the triangle B is F_2 which is defined by $x + y + z = 3t$. And the face above the triangle C is F_3 which is defined by $x - y + z = t$, also, we can see that F_2 is over F_3. Lets consider the amount of points over a point $(x,y,0)$ in each of these regions:

i) For the face A, the amuont of points over $(x,y,0)$ is $3t-3y+1$ ($(x,y,0)$ is counted). Observe that $A=\{(x,y,0):0\leq y\leq t, y\leq x\leq 2y\}$, so the amuont of points in and over A is $\sum_{y=0}^{t}\sum_{x=y}^{2y}3t-3y+1$

ii) For the face B, the amuont of points over $(x,y,0)$ is $3t-x-y+1$ ($(x,y,0)$ is counted). Observe that $B\setminus A=\{(x,y,0):0\leq y\leq t, 2y+1\leq x\leq t+y\}$, so the amuont of points in and over $B\setminus A$ is $\sum_{y=0}^{t}\sum_{x=2y+1}^{t+y}3t-x-y+1$

iii) For the face C the amuont of points over $(x,y,0)$ is $(3t-x-y+1)-(x-y-t)=4t-2x+1$ ($(x,y,0)$ is counted). Observe that $C\setminus B=\{(x,y,0):0\leq y\leq t, t+y+1\leq x\leq 2t\}$ so the amount of points in and over $C\setminus B$ is $\sum_{y=0}^{t}\sum_{x=t+y+1}^{2t}4t-2x+1$

Therefore the Ehrhart polynomial of of this polytope is

$$P(t)=\sum_{y=0}^{t}\sum_{x=y}^{2y}3t-3y+1\ +\sum_{y=0}^{t}\sum_{x=2y+1}^{t+y}3t-x-y+1\ +\sum_{y=0}^{t}\sum_{x=t+y+1}^{2t}4t-2x+1$$

.

Lets compute this:

$$P(t)=\sum_{y=0}^{t}(y+1)(3t-3y+1)+(t-y)(3t-y+1)+2(t-y)y-\frac{(t-y)(t-y+1)}{2}+$$

$$(t-y)(4t+1)-2(t-y)(t+y)-(t-y)(t-y+1)$$

$$=\sum_{y=0}^{t}(y+1)+(t-y)(\frac{7t-y+7}{2})$$

$$=\sum_{y=0}^{t}\frac{7}{2}t^2-4yt+\frac{1}{2}y^2+\frac{7}{2}t-\frac{5}{2}y+1$$

...

$$=(t+1)(\frac{20t^2+28t+12}{12})$$

$$\Rightarrow P(t)=\frac{5}{3}t^3+4t^2+\frac{10}{3}t+1$$

In order to find the Ehrhart series of the polytope lets start writing $P(t)$ in the base

$$\{1,(t+1),(t+1)^2,(t+1)^3:$$

$$P(t) = (t+1)(\frac{20t^2+28t+12}{12}) = \frac{1}{12}(t+1)(20(t+1)^2-12(t+1)+4) = \frac{1}{12}(20(t+1)^3-12(t+1)^2+4(t+1))$$

In the second problem we showed that $\sum_{t\geq 0}(t+1)^d z^t = \frac{A(d,1)z^0+...+A(d,d)z^{d-1}}{(1-z)^{d+1}}$.

- For $d=1$ then $\sum_{t\geq 0}(t+1)z^t = \frac{1}{(1-z)^2}$
- For $d=2$ then $\sum_{t\geq 0}(t+1)^2 z^t = \frac{1+z}{(1-z)^3}$
- For $d=3$ then $\sum_{t\geq 0}(t+1)^3 z^t = \frac{1+4z+z^2}{(1-z)^4}$

Therefore

$$\sum_{t\geq 0} P(t)z^t = \sum_{t\geq 0}[\frac{1}{12}(20(t+1)^3-12(t+1)^2+4(t+1))]z^t = (\frac{5}{3})\frac{1+4z+z^2}{(1-z)^4} - \frac{1+z}{(1-z)^3} + (\frac{1}{3})\frac{1}{(1-z)^2}$$

$$= \frac{3z^2+6z+1}{(1-z)^4}$$

So we get that $\frac{3z^2+6z+1}{(1-z)^4}$ is the value of the Ehrhart series of the polytope.

Problem 6:

Let e_i,f_j be the standard unit vectors in $\mathbb{R}^m, \mathbb{R}^n$ (resp.), $v_{ij} = e_i \times f_j$, and $\Delta_{m-1} \times \Delta_{n-1} = conv\{v_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$.

Let $\Gamma := \{$ staircase from $(1,1)$ to $(m,n)\}$, so Γ has $\binom{m+n-2}{m-1}$. For each $S \in \Gamma$, define $P_S := conv\{v_{ij} : (i,j) \in S\}$. Lets prove that $\{P_S : S \in \Gamma\}$ is a triangulation of $\Delta_{m-1} \times \Delta_{n-1}$:

i)**Lets prove that** P_S is a simplex for all $S \in \Gamma$:

To prove this is sufficient to show that the $m+n-1$ vertices of P_S are affinely independent (i.e they doesn't lie in a $m+n-3$-dimensional affine space). Name the vertices of P_S, $w_1, w_2, ..., w_{m+n-1}$, according to the order the appear in the staircase, so $w_1 = v_{11}$ and $w_{m+n-1} = v_{mn}$. Suppouse we have $\lambda_1 w_1 + \lambda_2 w_2 + ... + \lambda_{m+n-1} w_{m+n-1} = 0$ for some λ's such that $\lambda_1 + \lambda_2 + ... + \lambda_{m+n-1} = 1$. Let k be the greatest index such that $\lambda_k \neq 0$, then we can write $\lambda_1 w_1 + \lambda_2 w_2 + ... + \lambda_{k-1} w_{k-1} = -\lambda_k w_k$. If we write $w_1 = v_{a_1 b_1}, w_2 = v_{a_2 b_2}, ..., w_k = v_{a_k b_k}$,and remembering that the points $w_1, w_2, ..., w_k$ are ordered according a staircase, we can conclude that one of the following conditions must hold: $a_k > a_i$ for all $i < k$, or $b_k > b_i$ for all $i < k$. WLOG assume $a_k > a_i$ for all $i < k$. Then the a_k-th component of the vector $w_k = v_{a_k b_k}$ is 1, while the a_k-th component of the vectors $w_i = v_{a_i b_i}$ is 0 for all $i < k$.Therefore we cannot have the equality $\lambda_1 w_1 + \lambda_2 w_2 + ... + \lambda_{k-1} w_{k-1} = -\lambda_k w_k$, where $\lambda_k \neq 0$. This implies that $\lambda_1 w_1 + \lambda_2 w_2 + ... + \lambda_{m+n-1} w_{m+n-1} = 0$ only holds when $\lambda_i = 0$ for all i, so $w_1, w_2, ..., w_{m+n-1}$ are affinely independent.

ii) **Lets prove that** $\cup_{\{S\in\Gamma\}}P_S = \Delta_{m-1}\times\Delta_{n-1}$:
For any $x \in \Delta_{m-1}\times\Delta_{n-1}$ write $x = (\alpha_1, \alpha_2, ..., \alpha_m, \beta_1, \beta_2, ..., \beta_n)$. Since $(\alpha_1, \alpha_2, ..., \alpha_m) \in \Delta_{m-1}$, we get that $\alpha_i \geq 0$ for all i, and $\alpha_1+\alpha_2+...+\alpha_m = 1$. Similarly, since $(\beta_1, \beta_2, ..., \beta_n) \in \Delta_{n-1}$, we get that $\beta_j \geq 0$ for all j, and $\beta_1+\beta_2+...+\beta_n = 1$.

Define $A_1 = \alpha_1, A_2 = \alpha_1+\alpha_2, A_3 = \alpha_1+\alpha_2+\alpha_3, ..., A_m = \alpha_1+\alpha_2+\alpha_3+...+\alpha_m = 1$, and $B_1 = \beta_1, B_2 = \beta_1+\beta_2, B_3 = \beta_1+\beta_2+\beta_3, ..., B_n = \beta_1+\beta_2+\beta_3+...+\beta_n = 1$. Observe that $0 \leq A_1 \leq A_2 \leq ... \leq A_m = 1$ and $0 \leq B_1 \leq B_2 \leq ... \leq B_n = 1$. We can "mix" the previous sequences in a single ordered chain of lenght $m+n$, for instance if $A_1 = 0, A_2 = 0.6, A_3 = 0.8, A_4 = 1$, $B_1 = 0.3, B_2 = 0.5, B_3 = 1$, we get $A_1 \leq B_1 \leq B_2 \leq A_2 \leq A_3 \leq B_3 \leq A_4$. Since $0 \leq A_1 \leq A_2 \leq ... \leq A_m = 1$ and $0 \leq B_1 \leq B_2 \leq ... \leq B_m = 1$, there are exactly $\binom{m+n-2}{m-1}$ classes of chains (I mean, chains with the identical order of A_i's and B_i's, up two the order of $A_m = B_n = 1$ in the last two places of the chain), that are obtained by selecting the $m-1$ positions of $A_1, A_2, ..., A_{m-1}$ in the first $m+n-2$ places of the chain. For instance $A_1 \leq B_1 \leq B_2 \leq A_2 \leq A_3 \leq B_3 \leq A_4$ and $A_1 \leq B_1 \leq A_2 \leq B_2 \leq A_3 \leq B_3 \leq A_4$ are **diferent** classes of chains, but $A_1 \leq B_1 \leq B_2 \leq A_2 \leq A_3 \leq B_3 \leq A_4$ and $A_1 \leq B_1 \leq B_2 \leq A_2 \leq A_3 \leq A_4 \leq B_3$ are the **same** class.

Therefore the amount of classes of chains is equal to the number of staicases in Γ. Let see the relation. For a given chain construct a staircase as follows:

0) Start at $(1, 1)$.
1) If the first element of the chain is A_1 move to the east, if it is B_1 move to the north.
2) If the k-th element of the chain is of the form A move to the east, if it is of the form B move to the north.

Fix $x \in \Delta_{m-1}\times\Delta_{n-1}$, let C_x be a chain related to x, and let S_x be the staircaes induced by C_x using the previous construction. I claim that $x \in P_{S_x}$:

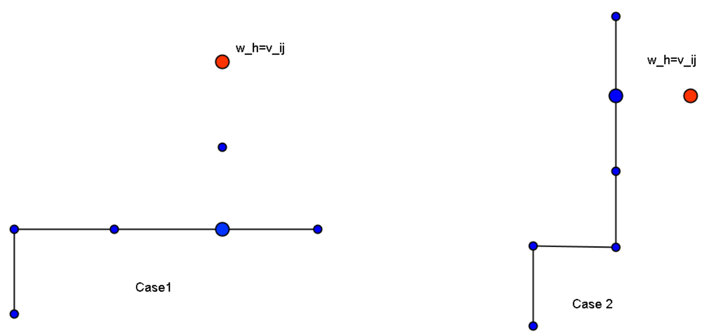
Write the chain C_x associated to x in the form $C_1 \leq C_2 \leq ... \leq C_{m+n}$ (for instance if C_x is the chain $A_1 \leq B_1 \leq B_2 \leq A_2 \leq A_3 \leq B_3 \leq A_4$, then we have $C_1 = A_1, C_2 = A_2, C_3 = B_2, ..., C_7 = A_4$). Name the vertices of P_{S_x}, $w_1, w_2, ..., w_{m+n-1}$, according to the order the appear in the staircase, so $w_1 = v_{11}$ and $w_{m+n-1} = v_{mn}$. **Now, we can check that** $x = C_1 w_1 + (C_2 - C_1)w_2 + ... + (C_{m+n-1} - C_{m+n-2})w_{m+n-1}$. This show that $x \in P_{S_x}$ since $C_1 w_1 + (C_2 - C_1)w_2 + ... + (C_{m+n-1} - C_{m+n-2})w_{m+n-1}$ is a convex combination of the vertices of P_{S_x}.

For example suppouse $x = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3) = (0, 0.6, 0.2, 0.2, 0.3, 0.2, 0.5)$ so $A_1 = 0, A_2 = 0.6, A_3 = 0.8, A_4 = 1$, and $B_1 = 0.3, B_2 = 0.5, B_3 = 1$. Then we can take C_x, the chain associated to x, as $A_1 \leq B_1 \leq B_2 \leq A_2 \leq A_3 \leq B_3 \leq A_4$. Using the construction of the staircase from the chain C_x,we get the following order of movements: east,north,north,east,east. This produces the vertices $v_{1,1}, v_{2,1}, v_{2,2}, v_{2,3}, v_{3,3}, v_{4,3}$. Writting the chain C_x in the form $C_1 \leq C_2 \leq ... \leq C_7$, observe that $C_1 v_{1,1} + (C_2 - C_1)v_{2,1} + ... + (C_6 - C_5)v_{4,3} =$

$$0(1000100) + 0.3(0100100) + 0.2(0100010) + 0.1(0100001) + 0.2(0010001) + 0.2(0001001) =$$

$$(0, 0.6, 0.2, 0.2, 0.3, 0.2, 0.5) = x$$

iii) **Lets prove that** $P_{S_1} \cap P_{S_2}$ is a face of both of them. Let $\{v_1, v_2, ..., v_k\} = V(P_{S_1}) \cap V(P_{S_2})$, I claim that $P_{S_1} \cap P_{S_2} = conv\{v_1, v_2, ..., v_k\}$. To prove this I will argue by contradiction. Suppouse there exists $q \in (P_{S_1} \cap P_{S_2}) \setminus conv\{v_1, v_2, ..., v_k\}$. Then, there exists $w_h \in V(P_{S_1}) \setminus V(P_{S_2})$ which is component of q, i.e,if we write q as a convex combination of the vertices of P_{S_1}, then the coefficient of w_h, say λ_h is greater than 0 (since P_{S_1} is a simplex, the point q can be written in a unique way as convex combination of the vertices of P_{S_1}). Let $w_h = v_{ij}$. Since $w_h \notin V(P_{S_2})$, there exists $\hat{j} < j$ such that $v_{i,\hat{j}}$ and $v_{i+1,\hat{j}}$ belong to $V(P_{S_2})$ (*Case 1*), **or** there exist $\hat{i} < i$ such that $v_{\hat{i},j}$ and $v_{\hat{i},j+1}$ belong to $V(P_{S_2})$(*Case 2*) . These two cases are presented below:



a) Case 1: When we write q as a point in P_{S_1} the condition $\lambda_h > 0$ (which is the coefficient of $w_h = v_{ij}$) implies $A_i > B_{j-1}$. On the other hand, when we write q as a point in P_{S_2}, the existence of $v_{i,\hat{j}}$ and $v_{i+1,\hat{j}}$ in $V(P_{S_2})$, with $\hat{j} < j$, implies $A_i \leq B_{\hat{j}} \leq B_{j-1}$. The conditions $A_i > B_{j-1}$ and $A_i \leq B_{j-1}$ lead us to a contradiction. Therefore $(P_{S_1} \cap P_{S_2}) \setminus conv\{v_1, v_2, ..., v_k\} = \emptyset$, so $P_{S_1} \cap P_{S_2} = conv\{v_1, v_2, ..., v_k\}$. Since P_{S_1} and P_{S_2} are simplexes, then $conv\{v_1, v_2, ..., v_k\}$ is a face of both of them.

b)Case 2: This case is analogous to the previous one. It leads to the contradiction ,$B_j > A_{i-1}$ and $B_j \leq A_{i-1}$.

7) Para desarrollar el proyecto voy a trabajar con Fabian Latorre. Estamos interesados en trabajar en algun topico de Optimizacion Combinatorica. En especial, nos llama la atencion trabajar en problemas relacionados a optimizacion de matchings y flujos sobre grafos (tal como el problema de las parejas de hombres y mujeres que se trato en la tarea anterior), y nuestro objetivo seria comprender (o modelar) este tipo de problemas desde el punto de vista de politopos. El libro que hemos mirado es *Combinatorial Optimization* de William J. Cook,Cunningham,Pulleyblank, y Schrijver (es un libro delgado que le mostre cuando vino a Bogota, y que es de introduccion en temas de

optimizacion combinatorica). En el capitulo de politopos de este libro, se encuentran teoremas interesantes sobre matchings y perfect matchings asociados a politopos. Si bien el contenido del libro en este tema no es muy extenso, seguramente podriamos profundizar mas con los libros amarillos de Schrijver o buscar articulos en el tema. Tambien nos ha parecido interesante el enfoque algoritmico que da este libro a los problemas, por lo cual proponer o estudiar alguna aplicacion algoritmica a la solucion de un problema podria hacer parte de nuestro proyecto.

Por otra parte, cuando usted vino a Bogota yo le comente que mi area de interes era analisis numerico y usted me hablo de unos articulos sobre splines y anillos de polinomios. Tambien me gustaria mirar estos articulos, pues me podrian ser de utilidad para mi tesis de pregrado en la cual estoy abordando problemas de interpolacion.