Homework 3

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Problem 1(I worked in this problem with Ana Maria Botero):

Let P be a d-polytope which is $\left(\lfloor \frac{d}{2} \rfloor + 1\right)$ - neighborly, lets prove that it is a simplex:

i) First suppose that P has t > d + 1 vertices. Since P is a d-dimensional polytope, we can affirm that the vertices of P are contained in a d-dimensional affine subspace, even more, we can choose $v_1, v_2, ..., v_{d+1} \in V(P)$, such that $P \subseteq Aff\{v_1, v_2, ..., v_{d+1}\}$, the affine subspace generated by $v_1, v_2, ..., v_{d+1}$. Let $v_{d+2} \in P \setminus \{v_1, v_2, ..., v_{d+1}\}$. By the previous observation we get that $v_{d+2} \in Aff\{v_1, v_2, ..., v_{d+1}\}$, so we can write $v_{d+2} = \lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_{d+1} v_{d+1}$, for some λ 's such that $\lambda_1 + \lambda_2 + ... + \lambda_{d+1} = 1$. WLOG, assume that $\lambda_1, \lambda_2, ..., \lambda_k, k < d+1$, are all the negative values in the set $\{\lambda_1, \lambda_2, ..., \lambda_{d+1}\}$, then $1 - (\lambda_1 + \lambda_2 + ... + \lambda_k) = \lambda_{k+1} + \lambda_{k+2} + ... + \lambda_{d+1} \ge 1$. Arranging terms we get $v_{d+2} - \lambda_1 v_1 - \lambda_2 v_2 - \dots - \lambda_k v_k = \lambda_{k+1} v_{k+1} + \lambda_{k+2} v_{k+2} + \dots + \lambda_{d+1} v_{d+1}$, dividing both sides by $C = 1 - (\lambda_1 + \lambda_2 + \dots + \lambda_k) = \lambda_{k+1} + \lambda_{k+2} + \dots + \lambda_{d+1}$, we get a convex combination in both sides. Therefore, we conclude that there exist $q \in P$ such that $q \in conv\{v_1, v_2, ..., v_k, v_{d+2}\}$ and $q \in conv\{v_{k+1}, v_{k+2}, ..., v_{d+1}\}$. Observe that at least one of this sets contains at most $\left(\lfloor \frac{d}{2} \rfloor + 1\right)$ vertices, so such set will be a face by the $\left(\lfloor \frac{d}{2} \rfloor + 1\right)$ - neighborly hypothesis. WLOG, assume that $F := conv\{v_{k+1}, v_{k+2}, \dots, v_{d+1}\}$ is a face of P. Let c be the direction which is maximized in $F(i.e., v_{d+1})$ $P_c = F$, and suppose $c^t x = M$ for all $x \in F$. Since $q \in F$, we know that $c^t q = M$. On the other hand, since $q \in conv\{v_1, v_2, ..., v_k, v_{d+2}\}$ and $c^t v_i < M$ for all $v_i \in \{v_1, v_2, ..., v_k, v_{d+2}\}$ we get $c^t q < M$. This contradiction proves that P can not have more than d+1 vertices and still being $\left(\left|\frac{d}{2}\right|+1\right)$ - neighborly.

ii)Now suppose that P has exactly d + 1 vetices (if it has less than d + 1, it would not be d-dimensional). Since P is d-dimensional these vertices must be d-affinely independent (i.e., there is no d - 1 dimensional affine subspace that contains all of them). Lets prove that it is a simplex (in the sense that any subset of vertices form a face) and for this reason it is $\left(\lfloor \frac{d}{2} \rfloor + 1\right)$ - neighborly. Observe that if d + 1 vetices are d-affinely independent then any subset of this set with k + 1 vertices must be k-affinely independent, i.e., there is no k - 1 dimensional affine subspace that contains all of them (otherwise, if we have k + 1 vertices that are not k-affinely independent and we add the other vertices, the affine space we get is the dimension at most d - 1). Fix a set of vertices $\{v_1, v_2, ..., v_k\}$ and lets prove that they form a face. Lets do it by induction: From the set $\{v_1, v_2, ..., v_{d+1}\}$ remove v_{d+1} and observe that $conv(\{v_1, v_2, ..., v_d\})$ form a face of P, since the d-1 affine space that $conv(\{v_1, v_2, ..., v_d\})$ doesnt contain v_{d+1} . Now remove v_d from $\{v_1, v_2, ..., v_d\}$ and observe that $conv(\{v_1, v_2, ..., v_d\})$, since the d-2 affine space that contain $\{v_1, v_2, ..., v_{d-1}\}$ is a face of $conv(\{v_1, v_2, ..., v_d\})$, since the d-2 affine space that $conv(\{v_1, v_2, ..., v_{d-1}\})$ is face of P. Going in this way (constructing a chain of facets) we get that $conv(\{v_1, v_2, ..., v_k\})$ is a face of P as I wanted to prove.

Problem 2 (I worked in this problem with Federico Castillo)

Let G be a graph with V vertices and ST(G) its spanning tree polytope

a) To prove that every x_T is a vertex of ST(G), lets start proving that every spanning tree of G has exactly V - 1 edges: Let H be any conected subgraph of G that contains all the vertex of G. Select any edge of H, call it e_1 , and let v_1, v_2 be the vertices which are joined by e_1 . Since H is conected and contains all the vertex of G, there must be another edge $e_2 \in H$ that joins v_1 or v_2 to a different vertex v_3 . We can apply the previous argument again to get another edge $e_3 \in H$ that joins one of the vertices v_1, v_2 or v_3 to a different vertex v_4 . Applying this argument V - 1 times we finally get that the set $\{e_1, e_2, ..., e_{V-1}\}$ conects all the vertex of G and there is no proper subset of it that has this property (since each time we add an edge, preserving the connectedness, we can add at most a vertex). Therefore H must have at least V - 1 edges. Now suppose H has more than V - 1 edges. Let $e \in H \setminus \{e_1, e_2, ..., e_{V-1}\}$ and suppose e joins v_i and v_j , since we know that there is a path between v_i and v_j through the edges $\{e_1, e_2, ..., e_{V-1}\}$, then we conclude that H has cycles. Therefore any connected graph of G that contains all the vertices and no cycles (i.e a spanning tree), must have exactly V - 1 edges.

By the previous argument we get that χ_T has exactly V - 1 1's for every spannig tree T of G. Now observe that $V - 1 = \chi_T^t \chi_T > \chi_T^t \chi_R$, whenever R is a spanning tree of G different to T. Therefore χ_T is a vertex of $convex(\{\chi_R : R \text{ is a spanning tree of } G\}) = ST(G)$ for every spanning tree T of G.

b) \Leftarrow : Suppose that T and T' are such that $T = T' - e \cup f$ where e - T' and $f \in T' - T$ are edges of G. Then χ_T and $\chi_{T'}$ differ only at the components associated to e and f. Let $c \in \mathbb{R}^E$ be defined as follows: $(c)_i = 1$ if $(\chi_T)_i = (\chi_{T'})_i = 1$, $(c)_i = 0$ if exactly one between $(\chi_T)_i, (\chi_{T'})_i$ is 1, and $(c)_i = -1$ otherwise. By the construction of c and knowing that all $\chi_R \in ST(G)$ are vectors with V - 1 1's (and the rest 0's), we can easily check that $c^t \chi_T = c^t \chi_T = V - 2$ and $c^t \chi_R < V - 2$ for all R different to T and T'. Then we conclude that χ_T and $\chi_{T'}$ are adjacent vertices of SP(G).

 \Rightarrow I am going to prove that if T' and T differ in at least two edges, then χ_T and $\chi_{T'}$ can not be adjacent in ST(G).

Suppose $T' \cap T = \{e_1, e_2, ..., e_k\}$ with k < V - 2. Let $\{g_{k+1}, g_{k+2}, ..., g_{V-1}\} = T \setminus T'$ and $\{g'_{k+1}, g'_{k+2}, ..., g'_{V-1}\} = T' \setminus T$. Whenever we eliminate an edge from a spanning tree, we get two connected components of G. If we eliminate another edge, we divide one of the previous connected components getting two connected components from it. Therefore, we can prove inductively that if we eliminate t edges of a spanning tree we get t + 1 connected components of G. Then the set $T' \cap T = \{e_1, e_2, ..., e_k\}$, corresponds to V - k connected components of G (since it is obtained by eliminating V - 1 - k edges from T or T'). Name these V - k connected components as $D_1, D_2, ..., D_{V-k}$. Now I will consider two cases:

i) Suppose that there exists D_i and D_j which are "directly conected" in T and T', i.e, there exists $g_h \in \{g_{k+1}, g_{k+2}, ..., g_{V-1}\}$, and $g'_l \in \{g'_{k+1}, g'_{k+2}, ..., g'_{V-1}\}$ such that g_h connects D_i and D_j and g'_l also do it. Then observe that we can replace g_h by g'_l in T, and it remains as a spanning tree of G (since both edges play the same role as connectors of the same connected components), and we can also replace g'_l by g_h in T', and it remains as a spanning tree of G. Then we get that $T_1 := T - g_h \cup g'_l$ and $T'_1 = T' - g'_l \cup g_h$ are spanning trees. WLOG, let g_h be associated to the first

coordinate in ST(G) and g'_l be associated with the second coordinate. Suppose c is a direction which is maximized only at χ_T and $\chi_{T'}$. Observe that we must have $c^t \chi_T - c^t \chi_{T_1} = c_1 - c_2 > 0$ and $c^t \chi_{T'} - c^t \chi_{T'_1} = c_2 - c_1 > 0$, that is a contradition. Then we conclude that in this case χ_T and $\chi_{T'}$ can not be adjacent.

ii)Suppose there are no D_i and D_j which are "directly conected" in both T and T'. The previous situation can be interpreted in the following way: if we consider the connected components $D_1, D_2, ..., D_{V-k}$ as vertices of a graph (call it H) then the set of edges $T_H = T \setminus T'$ and $T'_H = T' \setminus T$ represent disjoint spaning trees in H:



Since the sets T_H and T'_H are disjoint and contains V - 1 - k edges each one, we get that the set $T_H \cup T'_H$ contains 2(V - 1 - k) edges. Define e_{D_i} the number of edges of $T_H \cup T'_H$ adjacent to D_i , we know that $e_{D_i} \ge 2$ (since at each D_i there is at least one adjacent edge from T_H and one from T'_H), and $e_{D_1} + e_{D_2} + \ldots + e_{D_{V-k}} = 4(V - 1 - k)$. By the previous results we get that there exists some i such that $2 \le e_{D_i} \le 3$:

* If $e_{D_i} = 2$, this means that at D_i it arrives exactly one edge from T_H (say g_h) and one from T'_H (say g'_l). Observe that $T_H - g_h \cup g'_l$ and $T'_H - g'_l \cup g_h$ are spannig trees in H (since $T_H - g_h$ and $T'_H - g'_l \cup g_h$ are "connectively equivalent"), and similarly we get that $T - g_h \cup g'_l$ and $T' - g'_l \cup g_h$ are spannig trees of G. Now, if we define $T_1 := T - g_h \cup g'_l$ and $T'_1 = T' - g'_l \cup g_h$ as in part i) and apply the same arguments, we show that T and T' are not adjacent.

* If $e_{D_i} = 3$, we can assume, WLOG, that at D_i it arrives exactly two edges from T_H (say g_h and g_j) and one from T'_H (say g'_l). Since T_H is a tree in H, we observe that $T_H - g_h - g_j$ has three connected components, say C_h, C_j and the vertex D_i , similarly $T'_H - g'_l$ has two connected components, say C_l and the vertex D_i . Let g'_l joins D_i with D_m , and assume, WLOG, that $D_m \in C_h$, then $T_H - g_h \cup g'_l$ is an aspannig tree of H and $T'_H - g'_l \cup g_h$ is also a spannig tree of H. Therefore $T - g_h \cup g'_l$ and $T' - g'_l \cup g_h$ are spannig trees of G, and applying the same arguments of the part i), we show that T and T' are not adjacent.



c)Let T and T' be any two spannig trees of G. Suppose $T = \{e_1, e_2, ..., e_{V-1}\}$ and $T' = \{l_1, l_2, ..., l_{V-1}\}$ are the edges of T'. WLOG asume that T and T' have exactly k common edges and $e_1 = l_1, e_2 = l_2, ..., e_k = l_k$. Since T is an spanning tree, $T - e_{k+1}$ has exactly 2 connected components C_0 and C_1 which are joint through e_{k+1} . Now observe that $(T - e_{k+1}) \cup \{l_{k+1}, l_{k+2}, ..., l_{V-1}\}$ is connected since $T' \subset (T - e_{k+1}) \cup \{l_{k+1}, l_{k+2}, ..., l_{V-1}\}$, therefore there must be $l_i, k+1 \leq i \leq V-1$, that joins C_0 and C_1 . For such l_i , we get that $T_1 := T - e_{k+1} \cup l_i$ is a tree, observe that T_1 and T' have exactly k + 1 common edges. By the result in b) we get that χ_T and χ_{T_1} are adjacents in ST(G). We can apply the previos argument, now using T_1 instead of T, to get a new tree T_2 such that χ_{T_1} and χ_{T_2} are adjacents in ST(G), and T_2 and T' have exactly k + 2 common edges. Since T and T' differ in at most V - 1 edges, we can get from T to T' in at most V - 1 steps. This implies that we can get from χ_T to $\chi_{T'}$ in at most V - 1 steps, going through adjacent vertices in each step. Therefore we conclude that diameter of ST(G) is less than V.

Problem 3: (I worked in this problem with Ana Maria Botero and Federico Castillo)

Let P be a d-politope with n facets and assume n < 2d.

a) In order to prove that any two vertices lie in a common facet, it will be sufficient to show that each vertex belongs to at least d facets. If we prove this condition, then by the box principle (and using that there are n < 2d facets in P), we get that for any two vertices in P there is at least one common facet. We can prove that each vertex in a d-polytope belongs to at least d facets using the Shuffling Flags result of the previous homework: Let v be any vertex of P and let \mathcal{F} be any flag such that v = F(0). We know by the previous homework that $T^{-s}(F(d-1)) \neq T^{-r}(F(d-1))$ for $0 \leq r < s \leq d$ (this is not exactly what we proved, but it is equivalent since T is bijective), also we know that $F(0) \subset T^{-k}(F(k)) \subset T^{-k}(F(d-1))$ for k = 0, 1, ..., d-1 (just by definition of T), so we can conclude that v belongs to d different facets. This completes the proof.

b)Let v, u be any vertices in P. By the result in a) we know that $u, v \in F$ for some F facet of P. We know that F is a d-1-polytope, and now we are going to prove that it has at most n-1 facets. Lets argue by contradiction: suppose F has t > n-1 facets, since each of these tfacets of F belong to at least 2 facets of P (because the t facets of F are d-2 dimensional faces of P), we get by the box principle (using that t > n-1) that there is F' facet of P (distinct to F) and G_1, G_2 facets of F such that G_1, G_2 are also facets of F'. The previous situation contradicts the existence of a unique join for G_1, G_2 in the poset of P. Then, we conclude that F has at most n-1 facets. Since F is d-1-polytope with n-k facets (for some $k \ge 1$), and $u, v \in F$ we get that Distance $(u, v) \le \Delta(d-1, n-k) \le max_{k\ge 1}\Delta(d-1, n-k)$. The previous result holds for any $u, v \in P$, so we can afirm $\Delta(d, n) \le max_{k\ge 1}\Delta(d-1, n-k)$

Now we lets check that $\Delta(d-1,m) \leq \Delta(d-1,m+1)$ for any value of m at which $\Delta(d-1,m)$ is well defined: Let R be a d-1 polytope with m facets. Define R' in the following way: Fix a vertex v_0 of R, remove from R a vertex polytope VP_0 associated to v_0 , and define $R' := R \setminus VP_0$:



The idea is to check that $Diam(R') \ge Diam(R)$. Let c be the direction which is maximized at v_0 and give to the edges of R the orientation induced by c. The vertices adjacent to v_0 , are those that go to v_0 in a single step, let $v_1, ..., v_s$ be such vertices. If $c^t v_0 = M$ we can get R' by taking $R' = R \cap c^t x \le M - \epsilon$ for some $\epsilon > 0$. Observe that in each edge $[v_i, v_0]$ there exists a unique point,

say v_{0i} , such that $c^t v_{0i} = M - \epsilon$. We can prove that the set $\{v_{01}, v_{02}, ..., v_{0s}\}$ are all the vertex in $V(R') \setminus V(R)$, they form a new facet, and each v_{0i} is adjacent to v_i . Therefore what we have done is adding a new facet by eliminating v_0 and adding new vertices $\{v_{01}, v_{02}, ..., v_{0s}\}$. Observe that $\{v_1, v_2, ..., v_s\}$ are all the vertices in R' that are adjacents to any vertex in $\{v_{01}, v_{02}, ..., v_{0s}\}$ (because of the directed graph induced by c), and also observe that the adjacency relations for the rest of vertices of R is the same at R'. Then any path in R' can be replicated in R when we identify the set $\{v_{01}, v_{02}, ..., v_{0s}\}$ with v_0 . So we get $Diam(R') \ge Diam(R)$, and since this is true for any d-1 dimensional polytope with m facets we get $\Delta(d-1, m) \le \Delta(d-1, m+1)$.

By the previos result we conclude that $\max_{k\geq 1}\Delta(d-1,n-k) = \Delta(d-1,n-1)$, so $\Delta(d,n) \leq \Delta(d-1,n-1)$

c)Asume n < 2d and observe that $n-k < 2(d-k) \iff k < 2d-n$, so we have n-k < 2(d-k) for k = 0, 1, ., 2d - n - 1. Since n < 2d the result in b) implies $\Delta(d, n) \leq \Delta(d - 1, n - 1)$. Assume that we have proved $\Delta(d, n) \leq \Delta(d-k, n-k) \leq \Delta(d-k-1, n-k-1)$ until some $0 \leq k < 2d-n-1$. Since k < 2d-n-1 I get that that k+1 < 2d-n so n-(k+1) < 2(d-(k+1)). Therefore I can use b) again to get $\Delta(d-(k+1), n-(k+1)) \leq \Delta(d-(k+2), n-(k+2))$ then by the hipothesis asumption I get $\Delta(d, n) \leq \Delta(d-(k+1), n-(k+1)) \leq \Delta(d-(k+2), n-(k+2))$. Observe I can do this until k+1 = 2d-n-1, in that case we get $\Delta(d, n) \leq \Delta(d-(2d-n), n-(2d-n)) = \Delta((n-d), 2(n-d))$ that is what we wanted to prove.

Problem 4:

b)Let P be a d-polytope with n > 2d facets. Lets construct a d+1-polytope Q with n+1 facets in the following way:

i) Fix a facet F of P, and assume $\{u_1, u_2, ..., u_s\} = V(F)$. Let c_0 be the direction such that $P_{c_0} = F$ and $M := c_0^t x$ for all $x \in F$ (i.e., the linear functional is maximized in F and has value M).

ii)WLOG, asume that P "lives" in \mathbb{R}^d and embed it in an hyperplane of \mathbb{R}^{d+1} in the following way: $P \to (P, 0)$.

iii) Suppose $\{v_1, v_2, ..., v_k\} = V(P) \setminus V(F)$. Define $c_1 := (c_0, 1)$, and for all $i \in \{1, 2, ..., k\}$, define $w_i = (v_i, \zeta_i)$, such that $c_1^t w_i = M$ (since $c_0^t v_i < M$, we get $\zeta_i > 0$, for all i).

iv) Define $Q := conv((P, 0) \cup \{w_1, w_2, ..., w_k\}).$



Lets prove that Q is d + 1 -polytope with n + 1 facets :

Clearly Q is a d + 1 dimensional polytope becauese (P, 0) is a d-polytope and each w_i is not contained in the d-dimensional affine subspace defined by (P, 0) (since Q "lives" in \mathbb{R}^{d+1} , we know that its dimension is not higher to d + 1). Now lets see why Q has n + 1 facets. To do this I divide the problem in 3 cases, according the direction of maximization:

i) Suppose $c \in \mathbb{R}^{d+1}$ is a direction that points downward (i.e, the last coordinate is negative). Such direction (interpreted as a functional) can only be maximized in vertices of (P, 0) (remember that each $w_i = (v_i, \zeta_i)$, with $\zeta_i > 0$), so if Q_c is a facet it must be contained in (P, 0). However (P, 0)is itself a facet (which is maximized by the functional given by (0, 0, ..., 0, -1)) and since any facet cannot be contained in any other facet, we conclude that the only facet that is associated with a direction c that points downward is the facet (P, 0).

ii) Suppose $c \in \mathbb{R}^{d+1}$ is a direction with last coordinate 0, so we can write $c = (\hat{c}, 0)$. Observe that the vertices of Q that are maximized by c, are those associated to the vertices of P maximized by \hat{c} , i.e, $(v_i, 0), w_i$ are maximized by $c \iff v_i$ is maximized by \hat{c} , and $(u_j, 0)$ is maximized by c $\iff u_j$ is maximized by \hat{c} . In order to have Q_c as a facet of Q, $P_{\hat{c}}$ must be facet of P, and Q_c must contains vertices outside (P, 0) (i.e, it must contains at least one w_i), otherwise the dimension of Q_c would be lower to d. The previos conditions are satisfied iff we take take $c = (\hat{c}, 0)$, in such a way that $P_{\hat{c}} = G$, where G is any facet of P distinct to the fixed facet F (since $v_i \in G$ for some i, then $w_i \in Q_c$, and $Q_c \subset conv((G, 0) \cup w_i)$, which is d dimensional). Observe that if we take $c = (c_0, 0)$ were c_0 is the direction such that $P_{c_0} = F$, then we get $Q_c = (F, 0)$ which is a d -1-dimensional face of Q, an for this reason is not a facet. Therefore we conclude that given $c \in \mathbb{R}^{d+1}$, a direction with last coordinate 0, we can generate exactly n - 1 facets that are associated to the facets of Pdifferent to F. iii)Suppose $c \in \mathbb{R}^{d+1}$ is a direction that points upward. Then the points that are maximized by c belong to the set $(F,0) \cup \{w_1, w_2, ..., w_k\}$ (since for such c we have $c^t w_i > c^t v_i$). Therefore if Q_c is a facet it must be contained in $conv((F,0) \cup \{w_1, w_2, ..., w_k\})$. However the previous set is by construction a facet of Q which is maximized by the vector $c_1 := (c_0, 1)$ (since $Proy(conv((F,0) \cup \{w_1, w_2, ..., w_k\})) = (P, 0)$, and AffDim((P,0)) = d, we get that $AffDim(conv((F,0) \cup \{w_1, w_2, ..., w_k\})) \ge d$, and we conclude that is is exactly d since it belong to an hyperplane of \mathbb{R}^{d+1}). Therefore the only facet that can be associated to a vector c pointing

upward is $conv((F, 0) \cup \{w_1, w_2, ..., w_k\})$.

We have completed the proof that Q has n+1 facets.Now lets check which vertices are adjacent in Q:

i) $(v_i, 0)$ is adjacent to $(v_k, 0) \iff v_i$ is adjacent to v_k (take $c = (\hat{c}, -1)$, where \hat{c} is the direction maximized by v_i, v_k).

ii) w_i is adjacent to $w_k \iff v_i$ is adjacent to v_k (take $c = c_1 + (\hat{c}, 0)$, where \hat{c} is the direction maximized by v_i, v_k).

iii) w_i and $(v_k, 0)$ are adjacent $\iff i = k$: If i = k then $c = (v_i, 0)$ is maximized only in w_i and $(v_i, 0)$, therefore they are adjacent. If $i \neq k$ and c is a any direction such that $c^t w_i = c^t(v_k, 0)$, we observe that $(c)_{d+1} > 0$ implies $c^t w_k > c^t(v_k, 0)$, $(c)_{d+1} < 0$ implies $c^t(v_i, 0) > c^t w_i$, and $(c)_{d+1} = 0$ implies $c^t w_k = c^t(v_k, 0) = c^t(v_i, 0) = c^t w_i$, so we conclude that w_i and $(v_k, 0)$ are not adjacent whenever $i \neq k$.

iv) $(u_i, 0)$ is adjacent to $(u_l, 0) \iff u_i$ is adjacent to u_l (is analogous to case i))

- $v(v_i, 0)$ is adjacent to $(u_i, 0) \iff v_i$ is adjacent to u_i (is analogous to case i)).
- vi) w_i is adjacent to $(u_i, 0) \iff v_i$ is adjacent to u_i (is analogous to case ii)).

Observe that P and the facet $F_1 := (P, 0) = conv(((F, 0) \cup \{(v_1, 0), (v_2, 0), ..., (v_k, 0)\}))$, have an identical edge-verticies composition if we associate $(v_i, 0)$ with v_i for all i. Also, observe that the facets F_1 and $F_2 = conv((F, 0) \cup \{w_1, w_2, ..., w_k\})$ have an identical edge-verticies composition if we associate $(v_i, 0)$ with w_i for all i, even more, the only way to go from F_1 to F_2 is from $(v_i, 0)$ to w_i or throughout (F, 0) (and vice versa). By the previous observations, we can show that for every path between $(v_i, 0)$ and $(v_j, 0)$ in Q there exists a shorter or equal path between v_i and v_j in P. Let S be the path in Q, lets construct S' in the following way: if you move from $(v_h, 0)$ to $(v_l, 0)$ in S, move from v_h to v_l in S'; if you move from $(v_h, 0)$ to $(u_s, 0)$ (or vice versa), move from v_h to u_s (or vice versa); if you move from $(v_h, 0)$ to w_h (or vice versa) stay in the same place (i.e v_h); if you move from w_h to $(u_s, 0)$ (or vice versa), move from v_h to u_s (or vice versa). By the previous construction if you start at $(v_i, 0)$ and end at $(v_j, 0)$ in S, then S' will start at v_i and end at v_k , and will do it in at most the same amount of movements. A similar result holds if you start at $(u_i, 0)$ and end at $(u_j, 0)$ or if you move between $(u_i, 0)$ and $(v_j, 0)$. Since $V(P) = \{u_1, u_2, ..., u_s, v_1, v_2, ..., v_k\}$, the previous results indicates that $diam(Q) \ge diam(P)$. Since P is any d-polytope with n facets and Q is a d+1-polytope with n+1 facets built from P, we conclude that $\Delta(d,n) \leq \Delta(d+1,n+1)$ as we wanted to prove.

c)The induction prove is analogous to the one done in 3.c:

Asume n > 2d and observe that $n + k > 2(d + k) \iff k < n - 2d$, so we have n + k < 2(d + k)for k = 0, 1, ., n - 2d - 1. Since n > 2d the result in b) implies $\Delta(d, n) \leq \Delta(d + 1, n + 1)$. Assume that we have proved $\Delta(d, n) \leq \Delta(d + k, n + k) \leq \Delta(d + k + 1, n + k + 1)$ until some $0 \leq k < n - 2d - 1$. Since k < n - 2d - 1 I get that that k + 1 < n - 2d so n + (k + 1) < 2(d + (k + 1)), therefore I can use b) again to get $\Delta(d + (k + 1), n + (k + 1)) \leq \Delta(d + (k + 2), n + (k + 2))$ then by the hipothesis asumption I get $\Delta(d, n)\Delta(d + (k + 1), n + (k + 1)) \leq \Delta(d + (k + 2), n + (k + 2))$. Observe I can do this until k + 1 = 2d - n - 1, in that case we get $\Delta(d, n) \leq \Delta(d + (n - 2d), n + (n - 2d)) = \Delta(n - d), 2(n - d))$ that is what we wanted to prove.

Problem 5 (I worked in this problem with Federico Castillo and Diego Cifuentes):

Let $m_1 = (a_{11}, a_{12}, ..., a_{1n}), m_2 = (a_{21}, a_{22}, ..., a_{2n}), ..., m_n = (a_{n1}, a_{n2}, ..., a_{nn})$ be the vectors of preferences of the males, i.e., a_{ij} represents the amount of money that the male *i* is willing to pay (or must we pay) to date with the female *j*. Similarly, define $f_1 = (b_{11}, b_{12}, ..., b_{1n}), f_2 = (b_{21}, b_{22}, ..., b_{2n}), ..., f_n = (b_{n1}, b_{n2}, ..., b_{nn})$ the vectors of preferences of the females, i.e., b_{ij} represents the amount of money that the female *i* is willing to pay (or must we pay) to date with the male *j*. Using the previos notation, and knowing that we must arrange *n* disjoint dates with the objetive

of maximize profit, what we are looking for is: $\max_{\sigma \in S_n} \sum_{i=1}^n a_{i\sigma(i)} + b_{\sigma(i)i}$. Let $c \in \mathbb{R}^{n^2}$,

 $c = (a_{11}+b_{11}, a_{12}+b_{21}, ..., a_{1n}+b_{n1}, a_{21}+b_{12}, a_{22}+b_{22}, ..., a_{2n}+b_{n2}, ..., a_{n1}+b_{1n}, a_{n2}+b_{2n}, ..., a_{nn}+b_{nn}),$ (i.e., $(c)_{n(i-1)+j} = a_{ij} + b_{ji}$ for all $i, j \in \{1, 2, ..., n\}$). Now for each $\sigma \in S_n$ let $v_{\sigma} \in \mathbb{R}^{n^2}$ be defined as follows: $(v_{\sigma})_{n(i-1)+j} = 1$ if $\sigma(i) = j$ and $(v_{\sigma})_{n(i-1)+j} = 0$ otherwise, that is, in the first n entries I set 1 in the $\sigma(1)$ position and 0 in the rest, in the second n entries I set 1 in the $\sigma(2)$ position and 0 in the rest, ..., in the last n entries I set 1 in the $\sigma(n)$ position and 0 in the rest. Using the previous notation the problem can be written as $max_{\{v_{\sigma}:\sigma\in S_n\}}c^tv_{\sigma}$. Defining the polytope $P = convex\{v_{\sigma}: \sigma \in S_n\}(V$ -description) and knowing that the max of a linear functional in a polytope always is attained at least in a vertex, the original problem is equivalent to $max_{\{x\in P\}}c^tx$.

Now lets find the H - description of P. To simplify the situation, lets consider the vectors $x \in \mathbb{R}^{n^2}$ as $n \times n$ real matrices, where the first n coordinates of x are represented at the first row of the matrix, the second n coordinates are in the second row of the matrix, and so on. Using this representation we can observe that the vertices of P are the permutation matrices. Now I am going to prove that P (i.e the convex hull of the permutation matrices) is the set of doubly stochastic matrices. The set of doubly stochastic matrices can be viewed as the polytope in the space of $n \times n$ matrices given by the restrictions:

- i) The entries of M must be nonnegative.
- ii) The sum of the eantries at each row is 1.

iii) The sum of the eartries at each colum is 1.

The permutation matrices are vertices of this polytope, since if we regard them as vectors again, each of them is the only maximum of the linear functional associated to a vector that is equal to themselves (for instance, (010100001) is a vertex since it is the only maximum asociated with the functional given by c = (010100001)).

Now lets check that this polytope can not have more vertices:

Let M be a doubly stochastic matrix different to any permutation. Then we can find an entry m_1 of M such that $m_1 \in (0, 1)$. Let m_2 be an entry in the same row of m_1 such that $m_2 \in (0, 1)$. Let m_3 be an entry in the same column of m_2 such that $m_3 \in (0, 1)$. We can continue in this way, going through rows and columns in consecutive steps, and taking entries with values in (0, 1). Since there are finitely many entries we must repeat entries in this process, asume WLOG that m_1 is the first entry we repeat in the process, and we return to it at step n, i.e., $m_n = m_1$. Define $\epsilon = \min\{m_i, 1 - m_i : i = 1, ..., n\}$, so $0 \le m_i - \epsilon < m_i < m_i + \epsilon \le 1$ for i = 1, 2, ..., n. We must consider two cases according to the value of n:

* If n is odd , define M^+ from M by changing the entries $m_1, m_2, ..., m_{n-1}$ to $m_1 = m_1 + \epsilon$, $m_2 = m_2 - \epsilon, m_3 = m_3 + \epsilon, ..., m_{n-1} = m_{n-1} - \epsilon$, and the other entries of M remain equal. Observe that M^+ is doubly stochastic since all the entries are nonnegative (by the election of ϵ) and the sum in rows and columns is not changed. Now define M^- from M by changing the entries $m_1, m_2, ..., m_{n-1}$ to $m_1 = m_1 - \epsilon, m_2 = m_2 + \epsilon, m_3 = m_3 - \epsilon, ..., m_{n-1} = m_{n-1} - \epsilon$ and the other entries remain equal, so M^- is doubly stochastic. Since $M = \frac{1}{2}M^+ + \frac{1}{2}M^-$ we conclude that M is not a vertex of the polytope of doubly stochastic matrices.

* If n is even we get an analogous result to the previos one: setting M^+ from M by changing the entries $m_2, m_3, ..., m_{n-1}$ to $m_2 = m_2 + \epsilon, m_3 = m_3 - \epsilon, ..., m_{n-1} = m_{n-1} - \epsilon$ and M^- from M by changing the entries $m_2, ..., m_{n-1}$ to $m_2 = m_2 - \epsilon, m_3 = m_3 + \epsilon, ..., m_{n-1} = m_{n-1} + \epsilon$, we get two DS, such that $M = \frac{1}{2}M^+ + \frac{1}{2}M^-$ so we conclude that M is not a vertex of the polytope of doubly stochastic matrices.

Therefore if M is any dobly stochastic matrix different to a permutation matrix, it can not be a vertiex of the polytope of doubly stochastic matrices. So we conclude that $conv\{M_{\sigma}: M_{\sigma} \text{ is a} permutation matrix}\} = doubly stochastic matrices.$

Returning to the vetor notation, we can transform the matrix restriction that define DS matrices, to the vector restrictions that define the polytope P in this way: $x \in P \iff :$

i) $(00...011..100...0) \cdot x \leq 1$ (where the 1's are in the positions kn + 1 to (k + 1)n for some k)

ii) $(00...0 - 1 - 1.. - 100...0) \cdot x \le -1$ (where the -1's are in the positions kn + 1 to (k + 1)n for some k)

iii) $(0...010...010...010...0) \cdot x \leq 1$ (where the 1's are in the positions kn + i to for a fixed i)

iv) $(0...0-10...0-10...0-10...0) \cdot x \leq -1$ (where the -1's are in the positions kn+i to for a fixed i)

 $v(00...00 - 100....00) \cdot x \le 0$ (where the -1 is in any position k)

The previous inequalities give the H description of P.

Problem 6:

a) In a simplex all the vertices are adjacent: Let $e_i, e_j \in \mathbb{R}^d$ be vertices of Δ_{d-1} . Define $c = e_i + e_j$, then the functional associated to c is maximized only in the vertices e_i and e_j , this prove that they are adjacent. Therefore the diameter of Δ_{d-1} is 1. Since Δ_{d-1} has d facets, in this case the expression n-d(estimation of diameter by Hirsch conjecture) is $d-(d-1) = 1 = diam(\Delta_{d-1})$, then the Hirshc conjecture holds in this case.

b) In the cube C_d , two vertices are adjacent \iff they differ in exactly one coordinate: Let v_1 and v_2 vertices that differ in exactly one coordinate, define $c \in \mathbb{R}^d$ in the following way: $(c)_i = (v_1)_i$ if $(v_1)_i = (v_2)_i$, and $(c)_i = 0$ if $(v_1)_i \neq (v_2)_i$. Then $c^t v_1 = c^t v_2 = d - 1$ and for any $v \in V(C_d) \setminus \{v_1, v_2\}$, $c^t v < d - 2$ since $(c)_j(v)_j = -1$ for at least one $j \in \{1, 2, ..., d\}$, therefore v_1 and v_2 are adjacent. Now suppose that u_1 and u_2 are vertices that differ in at least two coordinates. WLOG write $u_1 = (1 - 1\hat{u}_1)$, and $u_2 = (-11\hat{u}_2)$. If $(c)_1 > 0$ then $c^t(11\hat{u}_2) > c^t u_2$, if $(c)_1 < 0$ then $c^t(-1 - 1\hat{u}_1) > c^t u_1$, and if $(c)_1 = 0$ then $c^t(-1 - 1\hat{u}_1) = c^t u_1$, therefore there is no c that is maximized only at u_1, u_2 , so they can not be adjacent. By the previous observation, we conclude that the distance between two vertices is equivalent to the amount of coordinates they differ. Therefore the greatest distance is d, that occurs for instance, between (-1, -1, ..., -1, -1) and (1, 1, ..., 1, 1). So diameter of $C_d = d$. Since the dual of C_d is \Diamond_d we easily get that C_d has 2d facets (the number of vertices of \diamondsuit_d), therefore the expression n - d in this case is $2d - d = d = diam(C_d)$, so Hirsch conjecture holds in this case.

c) In a crosspolytope \Diamond_d all the vertices are adjacent except for the couples $(e_i, -e_i)$: observe that for $\pm e_i$ and $\pm e_k$ with $i \neq k$, if we define $c = \pm e_i + \pm e_k$, the only two vertices of \Diamond_d that maximize care $\pm e_i$ and $\pm e_k$ (in the previous sentence the sign of e_i and e_k remains the same in all expressions), therefore they are adjacent. Now suppose there is c which maximized only at e_i and $-e_i$, then $c^t e_i = c^t (-e_i)$, and this implies $c^t e_i = 0$. However for any c and $k \neq i$, $c^t e_k \geq 0$ or $c^t (-e_k) \geq 0$, so e_i and $-e_i$ are not the only points of maximization of c, and we can conclude that they aren't adjacent. Therefore the greatest distance between to vertices of \Diamond_d is 2 (for instance, going from (1, 0, 0, ..., 0)to (-1, 0, 0, ..., 0) can be done just passing through (0, 1, 0, ..., 0)), so diameter of \Diamond_d is 2. Using that \Diamond_d is dual to C_d we get that \Diamond_d has 2^d facets (amount of vertices in C_d), so n - d in this case is $2^d - d$, since $2^d - d \geq 2 = diam(\Diamond_d)$ for $d \geq 2$ (and for d = 1, we have $n - d = 2 - 1 = 1 = diam \Diamond_1$), we conclude the Hish conjecture holds in this case.

d) As we can observe in the following picture the diameter of a dodecahedron is 5, and this distance is achieved when the vertices are taken at opposite faces. Since n-d is equal to 12-3=9 and n-d=9>5=diam(dodecahedron) we get that Hirsh conjecture holds.



e) As we can observe in the following pictuares the diameter of the icosahedron is 3, and this distance is achieved when we take opposite vetices of the icosahedron. Since n - d is equal to 20 - 3 = 17 and n - d = 17 > 3 = diam(icosahedron) we get that Hirsh conjecture holds.

