Homework 1

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(I discussed the problem 1 with Fabian Latorre, and the problem 5 with Federico Castillo and Jose Samper. In the problem 5, I followed a post made by Adam in the forum about a hint you gave to this problem)

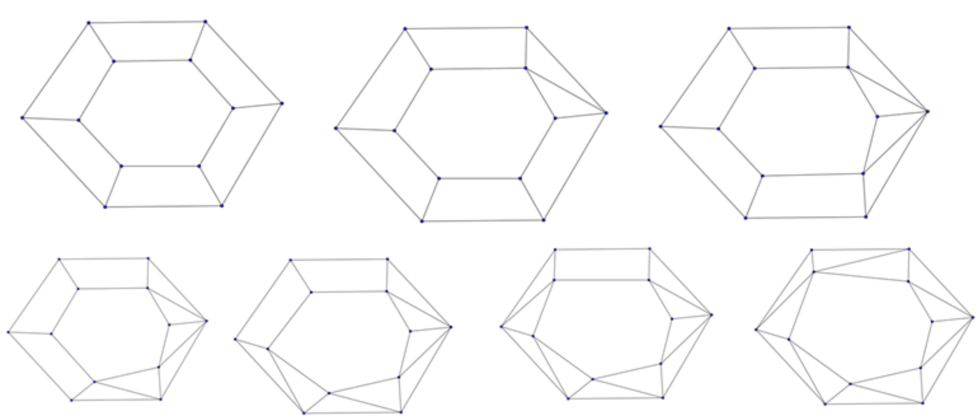
1)Lets prove that for any $3-polytope$ the inequalities $V \leq 2F-4$ and $F \leq 2V-4$ hold :

Let $v_1, v_2, ..., v_V$ be the vertexes of the polytope, and define $e_{v_1}, e_{v_1}, ..., e_{v_V}$ as the vertex degree, that is, e_{v_i} is the number of edges adjacent to v_i. Since v_i is a vertex of a $3-polytope$, we must have $e_{v_i} \geq 3$, for all i. We can observe that each edge corresponds to two vertexes, then we get $2E = e_{v_1} + e_{v_1} + ... + e_{v_V} \geq 3V \Rightarrow E \geq \frac{3}{2}V$. Introducing the previous inequality in Euler's formula $(V - E + F = 2)$, we get $V - \frac{3}{2}V + F \geq 2 \Rightarrow 2F - 4 \geq V$, as we wanted to prove.

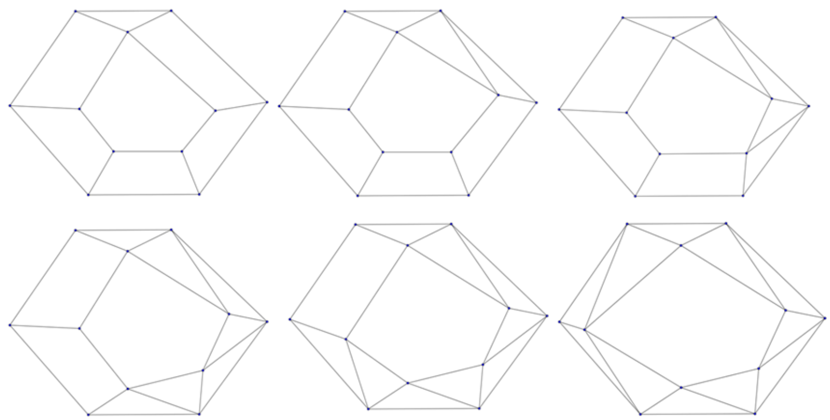
To get the other inequality, let $f_1, ..., f_F$ to be the faces of the polytope, and define e_{f_i} as the number of edges that border the face f_i. We can easily observe that $e_{f_i} \geq 3$ for all i. In this case each edge belongs to exactly two faces, so $2E = e_{f_1} + e_{f_1} + ... + e_{f_F} \geq 3F \Rightarrow E \geq \frac{3}{2}F$. If we introduce the previos inequality in Euler's formula we get $V - \frac{3}{2}F + F \geq 2 \Rightarrow 2V - 4 \geq F$, as we wanted to prove.

Now lets prove that for all (V, F) such that the inequalities holds, there exists a polytope with such characteristics. First I explain some constructions:

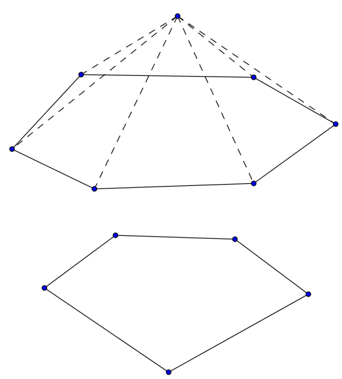
Suppouse you have two n-agons such that you can put one inside the other and you will have a correspondence of parallel sides (as in the first graph). If you move the interior n-agon to a parallel plane you will construct a polytope with $n+2$ faces (since parallel sides will be in the same face).Now, if you move slightly a vertex producing that a pair of sides are not parallel anymore (see second graph), you will get a new face (since there must be exactly a new edge from opposite vertexes of the sides). I you move slightly a second vertex getting another pair of sides not parallel you will add a new face. Continuing this process you can get a polytope with $2n+2$ faces.



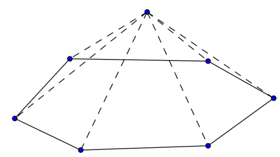
We can apply a similiar treatment two the case of an n-agon and a $n-1$ agon. We can start with a polytope with $n+2$ faces, and moving slightly a vertex in each step (to loose the parallel condition of a pair of sides) we can finally get one of $2n+1$.

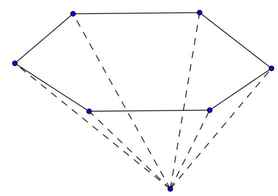


Lets start with the case V is even. Suppouse $V = 2n$. Then it is posible to construct polytopes with $n+2 \rightarrow 2n+2$ faces just using two parallel n-agonal faces (as in the first construction). Now pick one of the vertexes from one n-agon and put it over the other:



By the second construction, you can get using the n-agonal and $n-1$ agonal faces, polytopes with $n+2 \rightarrow 2n+1$ faces, now if you add a vertex over the n-agonal face you will be adding $n-1$ faces more (the one in the top is covered). Therefore we have got polytopes with $2n+1 \rightarrow 3n$ faces in this way.
Now take two $n-1$ agonal faces and two vertexes, as in the graph:

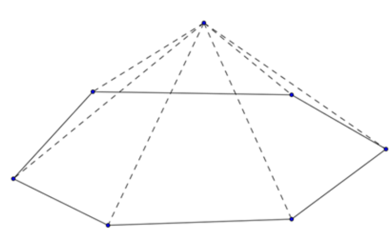


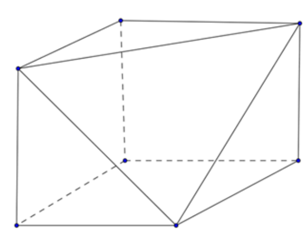


Using just the two $n-1$ agonal faces as in the first construction, we can get polytopes from $n+1 \rightarrow 2n$ faces, and since the two vertexes will add $2n-4$ faces more, in this way you can construct polytopes with $3n-3 \rightarrow 4n-4$ faces. Since $V = 2n$, the inequalities $V \leq 2F - 4$ and $F \leq 2V - 4$, imply $n+2 \leq F \leq 4n-4$, so we have constructed polytopes for all the posible pairs (V, F) when V is even.

The case $V = 2n - 1$ is almost the same. From the construction of an n-agonal and an $n-1$ agonal faces you will get polytopes with $n+2 \rightarrow 2n+1$ faces. If you consider two $n-1$ agonal faces an a vertex in the top, you will get $2n \rightarrow 3n-1$ faces; and if you take a $n-1$ agonal face, a $n-2$ agonal face and two vertexes (one over one below), you can get polytopes with $3n-4 \rightarrow 4n-6$ faces. In this case the inequalities implies $n+2 \leq F \leq 4n-6$, so all the pair (V, F), with V odd can be constructed.

2)We present two polytopes that shares the same number of vertixes, faces and edges but are combinatorially different:





Both polytopes has 7 vertexes, 7 faces, and 12 edges. However the polytope in the left has a face with 6 edges while the polytope in the right doesn't have any face of this type.

3)Lets prove that convex$\{+1, -1\}^d = \{x \in R^d : -1 \leq x_i \leq 1$, for all $1 \leq i \leq d\}$:

I will check first convex$\{+1, -1\}^d \subseteq \{x \in R^d : -1 \leq (x)_i \leq 1$, for all $1 \leq i \leq d\}$: Let $v_1, v_2, ..., v_{2^d}$ be any order of the points in the set $\{+1, -1\}^d$. Let $x \in$ convex$\{+1, -1\}^d$, so we write $x = \lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_{2^d} v_{2^d}$, where $\lambda_1 + \lambda_2 + ... + \lambda_{2^d} = 1$, and $\lambda_j \geq 0$ for all j. Observe that $(x)_i = (\lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_{2^d} v_{2^d})_i = \lambda_1 (v_1)_i + \lambda_2 (v_2)_i + ... + \lambda_{2^d} (v_{2^d})_i$; we know that $(v_k)_i \in \{+1, -1\}$ for $1 \leq k \leq 2^d$, so its also true that $-1 \leq (v_k)_i \leq 1$. Since $\lambda_k \geq 0$, using the previous inequality, we get $-\lambda_k \leq \lambda_k (v_k)_i \leq \lambda_k$, for all k. Addindg this inequalities we get $-(\lambda_1 + \lambda_2 + ... + \lambda_{2^d}) \leq (x)_i \leq (\lambda_1 + \lambda_2 + ... + \lambda_{2^d}) \Rightarrow -1 \leq (x)_i \leq 1$. Since the previous result holds for $1 \leq i \leq d$, we conclude $x \in \{x \in R^d : -1 \leq (x)_i \leq 1, \text{ for all } 1 \leq i \leq d\}$, so we have completed this part of the proof.

Now lets check $\{x \in R^d : -1 \leq (x)_i \leq 1, \text{ for all } 1 \leq i \leq d\} \subseteq \text{convex}\{+1, -1\}^d$:

Apply induction on d. The base case $d = 1$ is obvious, since $[-1, +1] = convex\{+1, -1\}$. Assume that the result its true for $d - 1$. Let $v_1, v_2, ..., v_{2^{d-1}}$ be any order of the points in the set $\{+1, -1\}^{d-1}$, now define $u_1 = (v_1, -1), u_2 = (v_2, -1), ..., u_{2^{d-1}} = (v_{2^{d-1}}, -1)$, and $w_1 = (v_1, +1), w_2 = (v_2, +1), ..., w_{2^{d-1}} = (v_{2^{d-1}}, +1)$, so $u_1, u_2, ..., u_{2^{d-1}}, w_1, w_2 ..., w_{2^{d-1}}$ are all the points of the set $\{+1, -1\}^d$. Let $x \in \{x \in R^d : -1 \leq (x)_i \leq 1, \text{ for all } 1 \leq i \leq d\}$, then $x = (\hat{x}, x_d)$, where $\hat{x} \in \{x \in R^{d-1} : -1 \leq (x)_i \leq 1\}$, and $x_d \in [-1, +1]$. By induction hypothesis $\hat{x} \in \text{convex}\{+1, -1\}^{d-1}$ so $\hat{x} = \lambda_1 v_1 + \lambda_2 v_2 ... + \lambda_{2^{d-1}} v_{2^{d-1}}$, a convex combination. Since $x_d \in [-1, +1]$, we know that $x_d = -1\mu + 1\lambda$ a convex combination. Now observe that $(\lambda_k \mu) u_k + (\lambda_k \lambda) w_k = \lambda_k (\mu (v_k, -1) + \lambda (v_k, +1)) = \lambda_k (v_k, x_d)$, for $1 \leq k \leq 2^{d-1}$, so we get that:

$$
\begin{aligned}
&(\lambda_1 \mu) u_1 + (\lambda_1 \lambda) w_1 + (\lambda_2 \mu) u_2 + (\lambda_2 \lambda) w_2 + ... + (\lambda_{2^{d-1}} \mu) u_{2^{d-1}} + (\lambda_{2^{d-1}} \lambda) w_{2^{d-1}} \\
&= \lambda_1 (v_1, x_d) + \lambda_2 (v_2, x_d) + ... + \lambda_{2^{d-1}} (v_{2^{d-1}}, x_d) \\
&= \big(\lambda_1 v_1 + \lambda_2 v_2 ... + \lambda_{2^{d-1}} v_{2^{d-1}}, (\lambda_1 + \lambda_2 + ... + \lambda_{2^{d-1}}) x_d\big) \\
&= (\hat{x}, x_d) = x.
\end{aligned}
$$

Since $(\lambda_1 \mu) + (\lambda_1 \lambda) + (\lambda_2 \mu) + (\lambda_2 \lambda) + ... + (\lambda_{2^{d-1}} \mu) + (\lambda_{2^{d-1}} \lambda) = (\mu + \lambda)(\lambda_1 + \lambda_2 + ... + \lambda_{2^{d-1}}) = 1$,and all of them are nonnegative, we conclude that $x \in \text{convex}\{+1, -1\}^d$. This complete the proof.

4)The given set of inequalities that define the polygon can be written in the form:

$$
\begin{pmatrix} 9 - 4x_2(A) \\ 2 - \frac{1}{2}x_2(B) \\ 3x_2 - \frac{17}{2}(C) \\ 1 - \frac{1}{6}x_2(D) \end{pmatrix} \leq x_1 \leq \begin{pmatrix} 2x_2(X) \\ 4(Y) \\ \frac{11}{2} - \frac{1}{2}x_2(Z) \end{pmatrix}
$$

In order to identify for which values of x_2 there is a value of x_1 that satisfies all the system , we can start by identifiying for which values of x_2 there is a value of x_1 that satisfies each pair of inequalities (taking one from the left column and one from the right column). In the following table I summarize the information obtained by solving each pair of inequalities:

	A	B	C	D
X	$x_2 \geq \frac{3}{2}$	$x_2 \geq \frac{4}{5}$	$x_2 \leq \frac{17}{2}$	$x_2 \geq \frac{6}{13}$
Y	$x_2 \geq \frac{5}{4}$	$x_2 \geq -4$	$x_2 \leq \frac{25}{6}$	$x_2 \geq -18$
Z	$x_2 \geq 1$	$2 \geq \frac{11}{2}$	$x_2 \leq 4$	$6 \geq 11$

Then we conclude that there is x_1 that satisfies all the inequalities simultaneously if and only if

$x_2 \in [\frac{3}{2}, 4]$. Therefore $proj_1(P) = [\frac{3}{2}, 4]$.

5)a)Lets start with the following important propositions:

i)Let $P = convex\{p_1, p_2, ..., p_n\}$ be a d-polytope, for any point x define $v_i = p_i - x$ for $1 \leq i \leq n$, so v_i is the vector from the point x to the vertex p_i.
Then: $x \in \text{Int}(P) \iff cone\{v_1, v_2, ..., v_n\} = R^d(*)$.

Proof:

Suppouse $x \in \text{Int}(P)$, then there exists $\epsilon > 0$ such that $x \pm \epsilon e_i \in P$ for $1 \leq i \leq d$. Observe that $P = convex\{p_1, p_2, ..., p_n\} = x + convex\{p_1 - x, p_2 - x, ..., p_n - x\} = x + convex\{v_1, v_2, ..., v_n\}$, so we get $\{\pm\epsilon e_i\} \subset convex\{v_1, v_2, ..., v_n\} \subset cone\{v_1, v_2, ..., v_n\}$.Therefore $cone\{\pm\epsilon e_i\} \subset cone\{v_1, v_2, ..., v_n\}$. Since $cone\{\pm\epsilon e_i\} = R^d$ we conclude $cone\{v_1, v_2, ..., v_n\} = R^d$.

Now suppouse that $cone\{v_1, v_2, ..., v_n\} = R^d$. Then we get by Caratheodorys Theorem, that for each $1 \leq i \leq d$, $e_i \in cone\{v_1', v_2', ..., v_d'\}$ for some $\{v_1', v_2', ..., v_d'\} \subseteq \{v_1, v_2, ..., v_n\}$. Then we get that for some $r_i > 0$, $r_i e_i \in convex\{v_1', v_2', ..., v_d'\} \subseteq convex\{v_1, v_2, ..., v_n\}$(if $e_i = s_1 v_1' + ... + s_d v_d'$, just take $r_i = \frac{1}{s_1 + ... + s_d}$). In this way we can get some $\epsilon > 0$ such that $\{\pm\epsilon e_i, 1 \leq i \leq d\} \subseteq convex\{v_1, v_2, ..., v_n\}$, then$\pm\epsilon e_i, 1 \leq i \leq d\} \subseteq convex\{x + v_1, x + v_2, ..., x + v_n\} = P$. Then $convex\{x \pm \epsilon e_i, 1 \leq i \leq d\} \subseteq P$, therefore x must be an interior point of P.

ii) Given $\{v_1, v_2, ..., v_n\} \subset R^d$ and $\{v_1, v_2, ..., v_d\}$ linearly independent,
Then: $cone\{v_1, v_2, ..., v_n\} = R^d \iff \text{Int}(-cone\{v_1, v_2, ..., v_d\}) \cap cone\{v_{d+1}, v_{d+2}, ..., v_n\} \neq \emptyset(**)$

Proof:

Suppouse $cone\{v_1, v_2, ..., v_n\} = R^d$, and let $w \in \text{Int}(-cone\{v_1, v_2, ..., v_d\})$. By Caratheodorys Theorem the exists $\{t_1, t_2, ..., t_d\} \subseteq \{v_1, v_2, ..., v_n\}$ such that $w \in cone\{t_1, t_2, ..., t_d\}$. Then $w = -\lambda_1 v_1 - \lambda_2 v_2 ... - \lambda_d v_d = \mu_1 t_1 + \mu_2 t_2 + ... + \mu_d t_d$, for some λ, μ's positive. Now substract in both sides of the equation all those t_i's that belongs to $\{v_1, v_2, ..., v_d\}$.This will produce that the left side will be in $\text{Int}(-cone\{v_1, v_2, ..., v_d\})$ and the right one in $cone\{v_{d+1}, v_{d+2}, ..., v_n\}$, therefore this element will belong to $\text{Int}(-cone\{v_1, v_2, ..., v_d\}) \cap cone\{v_{d+1}, v_{d+2}, ..., v_n\}$.

For the other direction suppouse $w \in \text{Int}(-cone\{v_1, v_2, ..., v_d\}) \cap cone\{v_{d+1}, v_{d+2}, ..., v_n\}$. Then $-w = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n$ with $\alpha_i > 0$, for all i. Now take any $y \in R^d$, and write it as $y = \beta_1 v_1 + \beta_2 v_2 + ... + \beta_n v_n$ (we can do this because $\{v_1, v_2, ..., v_d\}$ is linearly independent). Then we can find $M > 0$ such that $M\alpha_i + \beta_i > 0$ for all i. Therefore $y - Mw \in cone\{v_1, v_2, ..., v_d\}$, and we conclude that $y \in cone\{v_1, v_2, ..., v_d, w\}$ for all $y \in R^d$. Since $w \in cone\{v_{d+1}, v_{d+2}, ..., v_n\}$, we finally assert that $cone\{v_1, v_2, ..., v_n\} = R^d$.

As a corollary of the previous proposition I get:

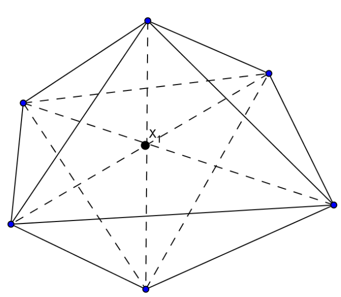
iii) If $cone\{v_1, v_2, ..., v_n\} = R^d$,$n \geq 2d$, then there exists $\{v_1', v_2', ..., v_{2d}'\} \subseteq \{v_1, v_2, ..., v_n\}$, such that $cone\{v'_1, v'_2, ..., v'_{2d}\} = R^d(***)$

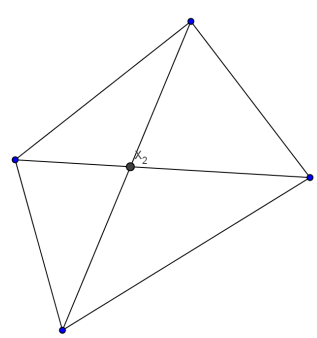
Proof:

Since $cone\{v_1, v_2, ..., v_n\} = R^d$, we can find d linearly independent vectors in $\{v_1, v_2, ..., v_n\}$, WLOG let $\{v_1, v_2, ..., v_d\}$ be linearly independent. By the previous proposition there exists $w \in \text{Int}(-cone\{v_1, v_2, ..., v_d\}) \cap cone\{v_{d+1}, v_{d+2}, ..., v_n\}$, and by Caratheodorys Theorem we can find $\{v'_{d+1}, v'_{d+2}, ..., v'_{2d}\} \subseteq \{v_{d+1}, v_{d+2}, ..., v_n\}$ such that $w \in cone\{v'_{d+1}, v'_{d+2}, ..., v'_{2d}\}$. Therefore $w \in \text{Int}(-cone\{v_1, v_2, ..., v_d\}) \cap cone\{v'_{d+1}, v'_{d+2}, ..., v'_{2d}\}$, so by the previous proposition we get $cone\{v_1, v_2, ..., v_d, v'_{d+1}, v'_{d+2}, ..., v'_{2d}\} = R^d$.

Now we can prove the desired result: Suppouse $P = convex\{p_1, p_2, ..., p_n\}$, $x \in \text{Int}(P)$, and define $v_i = p_i - x$ for $1 \leq i \leq n$. By prop $(*)$ $cone\{v_1, v_2, ..., v_n\} = R^d$. Now by prop $(***)$, there exists $\{v'_1, v'_2, ..., v'_{2d}\} \subseteq \{v_1, v_2, ..., v_n\}$, such that $cone\{v'_1, v'_2, ..., v'_{2d}\} = R^d$. Applying again prop $(*)$, we can conclude that $x \in \text{Int}(convex\{p'_1, p'_2, ..., p'_{2d}\})$, where $p'_1, p'_2, ..., p'_{2d}$, are the vertixes of P associated to the vectors $v'_1, v'_2, ..., v'_{2d}$. Therefore the subset $V = \{p'_1, p'_2, ..., p'_{2d}\}$ of $2d$ vertices of P is such that x is in the interior of the convex hull of V.

b)The d-polytopes P that are similar to the cross polytopes, in the sense that we have d pairs of opposite vertixes whose diagonals intersect in a point x, are posible combinations (P, x), for which $2d$ vertexes are strictly needed to make x to be in the interior of the convex hull. I drew a example of this in dimension 2 and 3.





In c) I prove that for these (P, x) it is true that $2d$ points are needed, and that these (P, x) are the only ones for which it is true.

c)I am going to prove that the only d-polytopes P and points x for which $2d$ vertexes are needed in V, are those polytopes that are similar to the d-cross polytope and x is the point of intersection of the diagonals between opposite vertexes. More specifically, these polytopes are of the form $P = convex\{\lambda_1 u_1 + x, -\mu_1 u_1 + x, \lambda_2 u_2 + x, -\mu_2 u_2 + x, ..., \lambda_d u_d + x, -\mu_d u_d + x\}$, where $\{u_1, u_2, ..., u_d\}$ is linearly independent,λ's and μ's are positives, and x is the "center" of the polytope.

Let P a d-polytope, x an interior point, p_i's the vertexes, and v_i's the vectors form x to the vertexes. As we discussed previously we can assume that $\{v_1, v_2, ..., v_d\}$ is linearly indenpedent. We can get the following observation:

($\star$) if we can find $\{t_1, t_2, ..., t_s\} \subseteq \{v_1, v_2, ..., v_n\}, s < d$, such that
$\text{Int}(-cone\{v_1, v_2, ..., v_d\}) \cap cone\{t_1, t_2, ..., t_s\} \neq \emptyset$, then $cone\{v_1, v_2, ..., v_d, t_1, t_2, ..., t_s\} = R^d$ (by $**$). Therefore $x \in \text{Int}(convex\{p_1, p_2, ..., p_d, p'_1, p'_2, ..., p'_s\})$ (by $*$), which imply that x is in the interior of the convex hull of less than $2d$ vertexes.

Let $w \in \text{Int}(-cone\{v_1, v_2, ..., v_d\})$, since $cone\{v_1, v_2, ..., v_n\} = R^d$, we can affirm by Caratheodorys Theorem that $w \in cone\{t_1, t_2, ..., t_d\} \subseteq \{v_1, v_2, ..., v_n\}$. If $dim(cone\{t_1, t_2, ..., t_d\}) < d$, we can apply Caratheodorys theorem again to find $\{t'_1, t'_2, ..., t'_s\} \subseteq \{v_1, v_2, ..., v_n\}, s < d$, such that $w \in cone\{t'_1, t'_2, ..., t'_s\}$. Then $w \in \text{Int}(-cone\{v_1, v_2, ..., v_d\}) \cap cone\{t'_1, t'_2, ..., t'_s\}$, and by ($\star$) x would be in the interior of the convex hull of less than $2d$ vertexes. Therefore we will asume that $dim(cone\{t_1, t_2, ..., t_d\}) = d$, and $w \in \text{Int}(cone\{t_1, t_2, ..., t_d\})$.

Now I will prove that $-cone\{v_1, v_2, ..., v_d\} = cone\{t_1, t_2, ..., t_d\}$, is a necesary condition for x to satisfy the particular conditions of the problem. Suppouse $-cone\{v_1, v_2, ..., v_d\} \neq cone\{t_1, t_2, ..., t_d\}$, so $\text{Int}(-cone\{v_1, v_2, ..., v_d\}) \neq \text{Int}(cone\{t_1, t_2, ..., t_d\})$. WLOG take
$u \in \text{Int}(-cone\{v_1, v_2, ..., v_d\}) \setminus \text{Int}(cone\{t_1, t_2, ..., t_d\})$. Since $u \notin \text{Int}(cone\{t_1, t_2, ..., t_d\})$ and
$w \in \text{Int}(cone\{t_1, t_2, ..., t_d\})$, we can find r in the segment $[u, w]$ such that $r \in \text{Frontier}(cone\{t_1, t_2, ..., t_d\})$, therefore $r \in cone\{t'_1, t'_2, ..., t'_{d-1}\}$, for some $\{t'_1, t'_2, ..., t'_{d-1}\} \subset \{t_1, t_2, ..., t_d\}$.
Since $u, w \in \text{Int}(-cone\{v_1, v_2, ..., v_d\})$, we get that $r \in \text{Int}(-cone\{v_1, v_2, ..., v_d\})$, so we conclude that $r \in \text{Int}(-cone\{v_1, v_2, ..., v_d\}) \cap cone\{t'_1, t'_2, ..., t'_{d-1}\})$, and by ($\star$) x would be in the interior of the convex hull of less than $2d$ vertexes. (If take $u \in \text{Int}(cone\{t_1, t_2, ..., t_d\}) \setminus \text{Int}(-cone\{v_1, v_2, ..., v_d\})$, we get some $r \in \text{Int}(-cone\{t_1, t_2, ..., t_d\}) \cap cone\{v'_1, v'_2, ..., v'_{d-1}\})$, which is again ($\star$)).

We have proved that $-cone\{v_1, v_2, ..., v_d\} = cone\{t_1, t_2, ..., t_d\}$ is a necesary condition. Now observe that the elements in $-cone\{v_1, v_2, ..., v_d\}$ are of the form $-\lambda_1 v_1 - \lambda_2 v_1 ... - \lambda_d v_d$, with λ's nonnegatives,in particular, each t_i can be written in this way. Similarly, the elements in $cone\{t_1, t_2, ..., t_d\}$ are of the form $\mu_1 t_1 + \mu_1 t_1 + ... + \mu_d t_d$, with μ's nonnegatives, so each v_j can be written in this way. By the two previous observations and also knowing that $\{v_1, v_2, ..., v_d\}$ and $\{t_1, t_2, ..., t_d\}$ are linearly independent we can conclude (by an easy inspection) that each t_i is a negative multiple of a different v_j, therefore $\{t_1, t_2, ..., t_d\} = \{-\delta_1 v_1, -\delta_2 v_2, ..., -\delta_d v_d\}$, for δ's positives.

We have shown that it is necesary for x to be the intersection of the diagonals between d pairs of opposite vertexes of the polytope. Lets prove that this polytope can't have more vertexes: Suppose p_{2d+1} is another vertex and asume $v_{2d+1} = \alpha_1 v_1 + ... + \alpha_d v_d$, where at less two α's must be different to 0 (otherwise we would have colineality between x and two vertexes). Let F be a set defined as follows: if $\alpha_i > 0$ then $-v_i \in F$, if $\alpha_i < 0$ then $v_i \in F$, and if $\alpha_i = 0$ then $v_i, -v_i \in F$. We can check that F must have at most $2d - 2$ elements and $v_{2d+1} \in \text{Int}(-cone(F))$. Therefore

$\text{Int}(-cone(F)) \cap cone(v_{2d+1}) \neq \emptyset$, and by the initial propositions, this imply that x belongs to the interior of the convex hull of the vertexes associated to the vectors in F and p_{2d+1}, which are at most $2d-1$ vertexes.

Finally lets prove that if $P = convex\{\lambda_1 u_1 + x, -\mu_1 u_1 + x, \lambda_2 u_2 + x, -\mu_2 u_2 + x, ..., \lambda_d u_d + x, -\mu_d u_d + x\}$, where $\{u_1, u_2, ..., u_d\}$ is linearly independent,λ's and μ's are positives, then x doesnt belong to the interior of the convex hull of less than $2d$ vertexes: Just observe that
$\text{Int}(-cone\{v_1, v_2, ..., v_d\}) \cap cone\{v_{d+1}, v_{d+2}, ..., v_{2d-1}\} = \emptyset$, for any ordering of the vectors from x to the vertexes, then by the initial propositions $x \notin \text{Int}(convex\{v_1, v_2, ..., v_{2d-1}\})$.

6)a) In order to show that if $P, Q \subset R^d$ are polytopes $\Rightarrow P \cap Q \subset R^d$ is a polytope, we can use the H-description of polytopes. Since we can describe $P = \{x \in R^d : A_p x \leq z_p\}$ and $Q = \{x \in R^d : A_q x \leq z_q\}$, we can affirm that $P \cap Q = \{x \in R^d : A_p x \leq z_p$ and $A_q x \leq z_q\} = \{x \in R^d : \begin{pmatrix} A_p \\ A_q \end{pmatrix} x \leq \begin{pmatrix} z_p \\ z_q \end{pmatrix}\}$, which is a H-description of $P \cap Q$. Since P and Q are bounded, then $P \cap Q$ is also bounded , so we conclude that it is a polytope.

b)Lets prove that $P, Q \subset R^d$ polytopes $\Rightarrow P + Q \subset R^d$ is a polytope:
Let $P = convex\{p_1, p_2, ..., p_n\}$, and $Q = convex\{q_1, q_2, ..., q_m\}$, I claim that $P + Q = convex\{p_i + q_j : 1 \leq i \leq n, 1 \leq j \leq m\}$. First observe that $P + Q$ is a convex set: take $x, y \in P + Q$, we can write $x = p_x + q_x$ and $y = p_y + q_y$, where $p_x, p_y \in P$ and $q_x, q_y \in Q$; then for any convex combination of x and y we get $\lambda x + \mu y = (\lambda p_x + \mu p_y) + (\lambda q_x + \mu q_y)$, where $(\lambda p_x + \mu p_y) \in P$ and $(\lambda q_x + \mu q_y) \in Q$, so $\lambda x + \mu y \in P + Q$, then we conclude that $P + Q$ is a convex. Now by definition we check that $\{p_i + q_j : 1 \leq i \leq n, 1 \leq j \leq m\} \subset P + Q$ so we can conclude, by the convexity of $P + Q$, that $convex\{p_i + q_j : 1 \leq i \leq n, 1 \leq j \leq m\} \subset P + Q$.

Now take any $\hat{x} \in P + Q$, we can write $\hat{x} = \hat{p} + \hat{q}$, $\hat{p} = \sum_{i=1}^n \lambda_i p_i$ and $\hat{q} = \sum_{j=1}^m \beta_j q_j$ (convex combinations), therefore $\hat{x} = \sum_{i=1}^n \lambda_i p_i + \sum_{j=1}^m \beta_j q_j$. If we write the previous sum as a convex combination of $\{p_i + q_j : 1 \leq i \leq n, 1 \leq j \leq n\}$ we are done. To get this define $\mu_{ij} = \lambda_i \beta_j$, and observe that

$$\begin{aligned}
&\textstyle\sum_{i=1}^n \sum_{j=1}^m \mu_{ij}(p_i + q_j) = \\
&= \textstyle\sum_{i=1}^n \sum_{j=1}^m \mu_{ij} p_i + \sum_{i=1}^n \sum_{j=1}^m \mu_{ij} q_j \\
&= \textstyle\sum_{i=1}^n \sum_{j=1}^m \lambda_i \beta_j p_i + \sum_{i=1}^n \sum_{j=1}^m \lambda_i \beta_j q_j \\
&= \textstyle\sum_{i=1}^n \lambda_i p_i + \sum_{j=1}^m \beta_j q_j = \hat{x}
\end{aligned}$$

Since $\mu_{ij} = \lambda_i \beta_j \geq 0$ for all i, j, and $\sum_{i=1}^n \sum_{j=1}^m \mu_{ij} = (\sum_{i=1}^n \lambda_i)(\sum_{j=1}^m \beta_j) = 1$, we conclude that $\hat{x} \in convex\{p_i + q_j : 1 \leq i \leq n, 1 \leq j \leq n\}$, so $P + Q \subset convex\{p_i + q_j : 1 \leq i \leq n, 1 \leq j \leq n\}$ and this completes the proof.

c)To prove that $P \subset R^d$, $Q \subset R^e$ polytopes $\Rightarrow P \times Q \subset R^{d+e}$ is a polytope we can use the previous result. First, observe that $\phi : R^d \to R^{d+e}$, defined $\phi(x) = \begin{pmatrix} x \\ 0_e \end{pmatrix}$, sends $P \subset R^d$ to a polytope $\phi(P) = \begin{pmatrix} P \\ 0_e \end{pmatrix} \subset R^{d+e}$ (its easy to check that if $P = convex\{p_1, p_2, ..., p_n\} \Rightarrow \phi(P) =$

$convex\{\phi(p_1), \phi(p_2), ..., \phi(p_n)\}$).Similarly define $\psi : R^e \to R^{d+e}$ by $\psi(y) = \begin{pmatrix} 0_d \\ y \end{pmatrix}$,which sends the polytope $Q \subset R^e$ to a polytope $\psi(Q) = \begin{pmatrix} 0_d \\ Q \end{pmatrix} \subset R^{d+e}$. By the previous result we now that $\phi(P) + \psi(Q) \subset R^{d+e}$ is a polytope, and it is easy to check that $\phi(P) + \psi(Q) = \{z = \phi(p) + \psi(q) : p \in P, q \in Q\} = \{z = \begin{pmatrix} p \\ q \end{pmatrix} : p \in P, q \in Q\} = P \times Q$. So $P \times Q$ is a polytope.