Parallel Tridiagonal Solvers

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IMPA

Tridiagonal Systems. Motivation

Serial Solution to Tridiagonal Systems

Parallel solution to linear recurrences

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   - x_1 is given.
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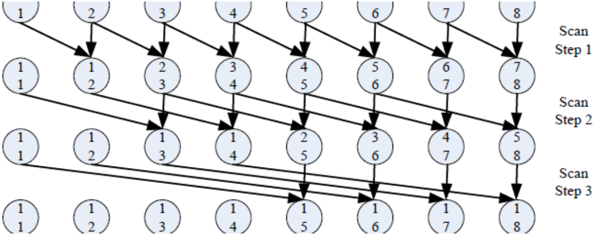

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Memory per Thread:	1	2	4	6

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$$\begin{pmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & a_3 & b_3 & c_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & & a_n & b_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{n-1} \\ d_n \end{pmatrix}$$

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The initial conditions, x_1 and x_2, are missing!!

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Instead of initial conditions we have boundary conditions!!

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4) Compute $C_i = A_i A_{i-1} \ldots A_1$ for $i = 1 \ldots n$ using **Scan!!**.

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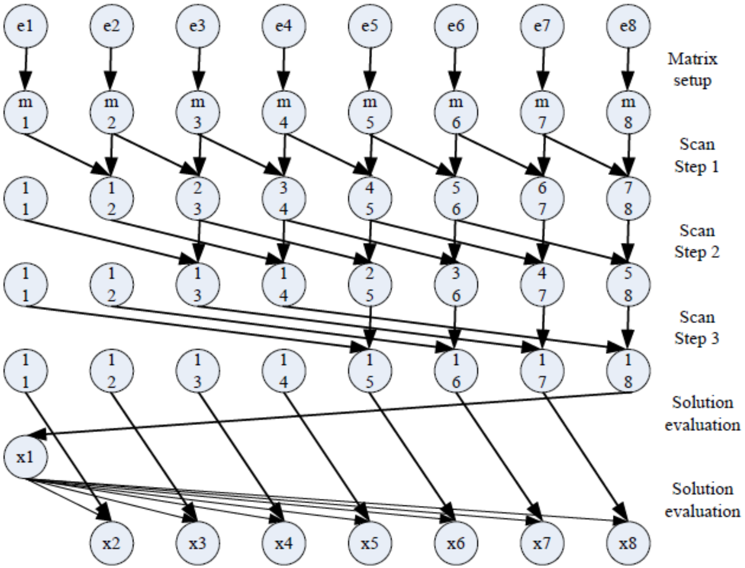

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- This could improve PCR which takes 12 Floaps/Thread/Step, where 2 of such flops are divisions.

Hybrid Algorithms

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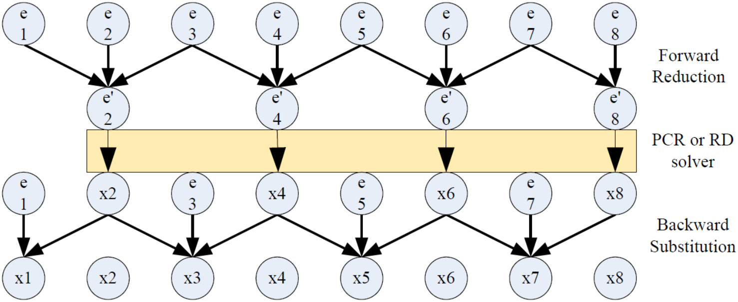

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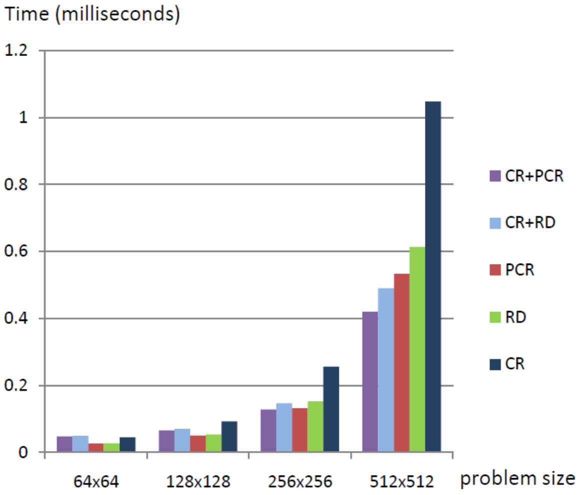
- Transformations are done in-place to save shared memory. This produces bank conflicts in CR at the last steps of forward reduction and first steps of backward substitution.

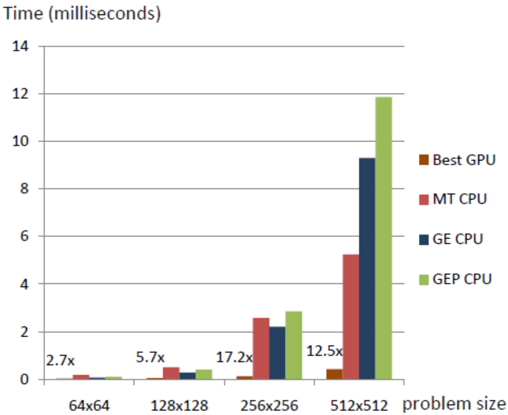
Performance Results:Parallel Algorithms in GPU



Hardware Specifications:

- GPU: GTX 280, 30 SM's, 8 cores per SM, 16kb shared memory. CUDA 2.0.

GPU vs CPU (Ignoring Transfer Time)



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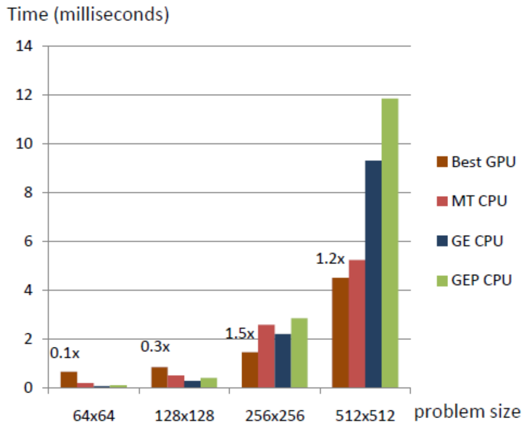

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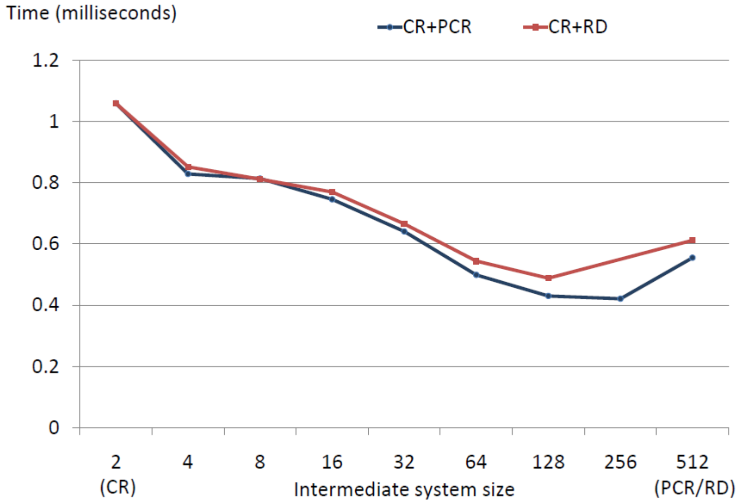
Performance Analysis

- "We use a differental method to measure the time for each part of the algorithm. We first comment out the whole code and uncomment it incrementally in program order and measure excecution time."
- "To estimate shared memory access time, we replaced shared memory accesses with register accesses, and calculate the shared memory access time as the difference between this program and the original program."

Hybrid Algorithms Performance



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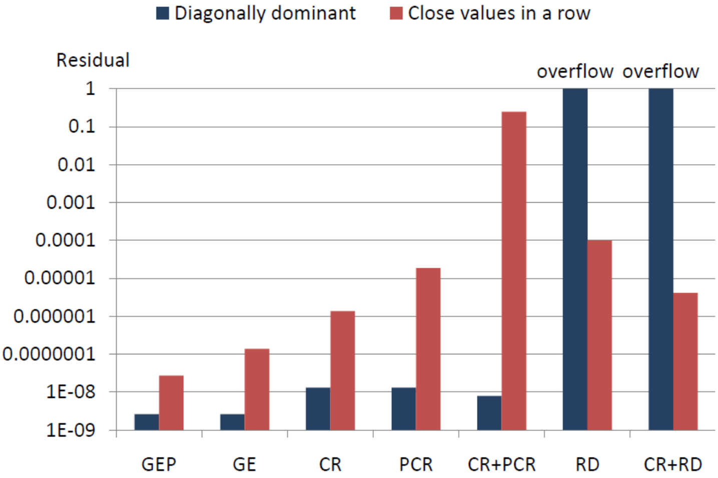
- Conclusion: Using CR for the first iterations reduce the per step time required by PCR or RD to solve the intermediate system.

PCR and RD are preferred to solve the intermediate system since they require less steps and are free of bank conflicts.

Accuracy Experiments

$||Ax - b||$ error in Tridiagonal Systems



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