2D Computer Graphics

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IMPA

Differential geometry

Inside-outside test

Ideal model is *implicit*

Given a region $\Omega \subset \mathbb{R}^2$, define the *indicator function* $\mathbb{1}_\Omega : \mathbb{R}^2 \to \{0, 1\}$

$$\mathbb{1}_\Omega(p) = \begin{cases} 1, & p \in \Omega \\ 0, & p \notin \Omega \end{cases}$$

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Alternatively, using the *Iverson bracket*, $[p \in \Omega] = \mathbb{1}_\Omega(p)$, where $[\mathit{true}] = 1$ and $[\mathit{false}] = 0$

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Basis of CSG (constructive solid geometry)

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How do we define the boundary?

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A subset $S \subset \mathbb{R}^2$ can be parametrized in many different ways

$$\beta\colon [a, b] \to \mathbb{R}^2 \quad \beta(t) = \big(r\cos(\omega t + \phi), r\sin(\omega t + \phi)\big), \quad b - a \geq \frac{2\pi}{\omega}$$

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Are individual SVG *segments* regular?

The arc-length of a curve segment $\alpha : [a, b] \to \mathbb{R}^2$ is

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- Koch snowflake

Reparameterization

A curve $\beta : J \to \mathbb{R}^2$ is a *reparameterization* of α if there is a monotonic differentiable function $h : J \to I$ such that $\beta = \alpha \circ h$

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Arc-length reparameterization

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We get

$$s(t) = \int_c^t \left|\beta'(t)\right| dt = t - c$$

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$$\begin{aligned} \alpha(t) &= (at^2, 2at) \quad \Rightarrow \\ \left|\alpha'(t)\right| &= 2a\sqrt{1+t^2} \\ \int_0^t \left|\alpha'(t)\right| dt &= at\sqrt{t^2+1} + a\log\left(\sqrt{t^2+1}+t\right) \end{aligned}$$

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Standard ellipse

$$\beta(t) = (a\cos t, b\sin t) \quad \Rightarrow$$

$$\left|\beta'(t)\right| = b\sqrt{1 - m\sin^2(t)}, \quad m = 1 - \tfrac{a^2}{b^2}$$

$$\int_0^t \left|\beta'(t)\right| dt = \text{Elliptic integral of the second kind}$$

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If α is a general parameterization with $\beta = \alpha \circ u$, we obtain

$$\begin{aligned}
\kappa(t) &= \left|\alpha(u)' \times \alpha(u)''\right| \\
&= \left|\alpha'(u)\, u' \times \left(\alpha''(u)(u')^2 + \alpha'(u)\, u''\right)\right| \\
&= \left|\alpha'(u)\, u' \times \alpha''(u)\, (u')^2\right| \\
&= \left|\alpha'(u) \times \alpha''(u)\right| |u'|^3 \\
&= \frac{\left|\alpha'(u) \times \alpha''(u)\right|}{\left|\alpha'(u)\right|^3}
\end{aligned}$$

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An inflection is a point where the curvature vanishes

I.e. where the 1st and 2nd derivatives are collinear

Stroking

A different way of specifying the interior

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$$\ell(t) = \left[(1-u)\, p_1(t) + u\, p_2(t), 0 < u < 1\right] \tag{1}$$

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How would you compute the arc length? [Jüttler, 1997]

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Show offset and evolute curves

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