2D Computer Graphics

Diego Nehab

Summer 2020

IMPA

More on Bézier curves

Reparameterization

Given a Bézier curve segment $\gamma^n(t)$, with control points $\{p_0, \ldots, p_n\}$, and a *reparameterization* $t \mapsto (1-u)r + us$, how can we obtain the control points $\{q_0, \ldots, q_n\}$ for the curve segment piece $\gamma^n_{[r,s]}(u)$?

Given a Bézier curve segment $\gamma^n(t)$, with control points $\{p_0, \ldots, p_n\}$, and a *reparameterization* $t \mapsto (1-u)r + us$, how can we obtain the control points $\{q_0, \ldots, q_n\}$ for the curve segment piece $\gamma^n_{[r,s]}(u)$?

$$\gamma^n_{[r,s]}(u) = \gamma^n\big((1-u)r + us\big) = \gamma^n(a + bu), \quad \text{for} \quad a = r, \quad b = s - r$$

Reparameterization

Given a Bézier curve segment $\gamma^n(t)$, with control points $\{p_0, \ldots, p_n\}$, and a *reparameterization* $t \mapsto (1-u)r + us$, how can we obtain the control points $\{q_0, \ldots, q_n\}$ for the curve segment piece $\gamma^n_{[r,s]}(u)$?

$$\gamma^n_{[r,s]}(u) = \gamma^n\big((1-u)r + us\big) = \gamma^n(a + bu), \quad \text{for} \quad a = r, \quad b = s - r$$

$$\gamma^n(a+bu) = \begin{bmatrix} p_0 & \cdots & p_n \end{bmatrix} B_n(a+bu) = \mathsf{C}^{\mathsf{B}}\,\mathsf{B_n}\,P_n(a+bu)$$

Given a Bézier curve segment $\gamma^n(t)$, with control points $\{p_0, \ldots, p_n\}$, and a *reparameterization* $t \mapsto (1-u)r + us$, how can we obtain the control points $\{q_0, \ldots, q_n\}$ for the curve segment piece $\gamma^n_{[r,s]}(u)$?

$$\gamma^n_{[r,s]}(u) = \gamma^n\big((1-u)r + us\big) = \gamma^n(a+bu), \quad \text{for} \quad a = r, \quad b = s - r$$

$$\gamma^n(a+bu) = \begin{bmatrix} p_0 & \cdots & p_n \end{bmatrix} B_n(a+bu) = \mathsf{C}^{\mathsf{B}}\,\mathsf{B_n}\,P_n(a+bu)$$

$$P_n(a+bu) = \begin{bmatrix} 1 \\ a+bu \\ a^2 + 2ab\,u + b^2\,u^2 \\ \vdots \\ \binom{n}{0}a^n + \cdots + \binom{n}{i}a^{n-i}b^i\,u^i + \cdots \binom{n}{n}b^n\,u^n \end{bmatrix}$$

Reparameterization

$$P_n(a+bu) = \begin{bmatrix} 1 & & & & \\ a & b & & & \\ a^2 & 2ab & b^2 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \binom{n}{0}a^n & \binom{n}{1}a^{n-1}b & \binom{n}{2}a^{n-2}b^2 & \cdots & \binom{n}{n}b^n \end{bmatrix} \begin{bmatrix} 1 \\ u \\ u^2 \\ \vdots \\ u^n \end{bmatrix}$$

Reparameterization

$$P_n(a+bu) = \begin{bmatrix} 1 & & & & \\ a & b & & & \\ a^2 & 2ab & b^2 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \binom{n}{0}a^n & \binom{n}{1}a^{n-1}b & \binom{n}{2}a^{n-2}b^2 & \cdots & \binom{n}{n}b^n \end{bmatrix} \begin{bmatrix} 1 \\ u \\ u^2 \\ \vdots \\ u^n \end{bmatrix}$$

$$= \mathsf{M}_{\mathsf{a,b}}\, P_n(u)$$

Reparameterization

$$P_n(a+bu) = \begin{bmatrix} 1 & & & & \\ a & b & & & \\ a^2 & 2ab & b^2 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \binom{n}{0}a^n & \binom{n}{1}a^{n-1}b & \binom{n}{2}a^{n-2}b^2 & \cdots & \binom{n}{n}b^n \end{bmatrix} \begin{bmatrix} 1 \\ u \\ u^2 \\ \vdots \\ u^n \end{bmatrix}$$

$$= \mathsf{M_{a,b}}\, P_n(u)$$

$$\gamma^n_{[r,s]}(u) = \gamma^n(a+bu)$$

Reparameterization

$$P_n(a+bu) = \begin{bmatrix} 1 & & & & \\ a & b & & & \\ a^2 & 2ab & b^2 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \binom{n}{0}a^n & \binom{n}{1}a^{n-1}b & \binom{n}{2}a^{n-2}b^2 & \cdots & \binom{n}{n}b^n \end{bmatrix} \begin{bmatrix} 1 \\ u \\ u^2 \\ \vdots \\ u^n \end{bmatrix}$$

$$= \mathsf{M_{a,b}}\, P_n(u)$$

$$\gamma^n_{[r,s]}(u) = \gamma^n(a+bu)$$

$$= \mathsf{C^B}\, \mathsf{B_n}\, P_n(a+bu)$$

Reparameterization

$$P_n(a+bu) = \begin{bmatrix} 1 & & & & \\ a & b & & & \\ a^2 & 2ab & b^2 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \binom{n}{0}a^n & \binom{n}{1}a^{n-1}b & \binom{n}{2}a^{n-2}b^2 & \cdots & \binom{n}{n}b^n \end{bmatrix} \begin{bmatrix} 1 \\ u \\ u^2 \\ \vdots \\ u^n \end{bmatrix}$$

$$= \mathsf{M_{a,b}}\, P_n(u)$$

$$\gamma^n_{[r,s]}(u) = \gamma^n(a+bu)$$

$$= \mathsf{C^B}\, \mathsf{B_n}\, P_n(a+bu)$$

$$= \mathsf{C^B}\, \mathsf{B_n}\, \mathsf{M_{a,b}}\, P_n(u)$$

Reparameterization

$$P_n(a+bu) = \begin{bmatrix} 1 & & & & \\ a & b & & & \\ a^2 & 2ab & b^2 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \binom{n}{0}a^n & \binom{n}{1}a^{n-1}b & \binom{n}{2}a^{n-2}b^2 & \cdots & \binom{n}{n}b^n \end{bmatrix} \begin{bmatrix} 1 \\ u \\ u^2 \\ \vdots \\ u^n \end{bmatrix}$$

$$= \mathsf{M_{a,b}}\, P_n(u)$$

$$\gamma^n_{[r,s]}(u) = \gamma^n(a+bu)$$

$$= \mathsf{C^B}\, \mathsf{B_n}\, P_n(a+bu)$$

$$= \mathsf{C^B}\, \mathsf{B_n}\, \mathsf{M_{a,b}}\, P_n(u)$$

$$= \mathsf{C^B}\, \mathsf{B_n}\, \mathsf{M_{a,b}}\, \mathsf{B_n}^{-1}\, B_n(u)$$

$$
\begin{aligned}
P_n(a+bu) &= \begin{bmatrix} 1 & & & & \\ a & b & & & \\ a^2 & 2ab & b^2 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \binom{n}{0}a^n & \binom{n}{1}a^{n-1}b & \binom{n}{2}a^{n-2}b^2 & \cdots & \binom{n}{n}b^n \end{bmatrix} \begin{bmatrix} 1 \\ u \\ u^2 \\ \vdots \\ u^n \end{bmatrix} \\
&= \mathsf{M_{a,b}}\, P_n(u) \\
\gamma^n_{[r,s]}(u) &= \gamma^n(a+bu) \\
&= \mathsf{C^B}\, \mathsf{B_n}\, P_n(a+bu) \\
&= \mathsf{C^B}\, \mathsf{B_n}\, \mathsf{M_{a,b}}\, P_n(u) \\
&= \mathsf{C^B}\, \mathsf{B_n}\, \mathsf{M_{a,b}}\, \mathsf{B_n^{-1}}\, B_n(u) \\
\begin{bmatrix} q_0 & \cdots & q_n \end{bmatrix} &= \mathsf{C^B}\, \mathsf{B_n}\, \mathsf{M_{a,b}}\, \mathsf{B_n^{-1}}
\end{aligned}
$$

Let $p : \mathbb{R} \to \mathbb{R}$ be a degree-n polynomial, and let $P : \mathbb{R}^n \to \mathbb{R}$

Blossoms

Let $p : \mathbb{R} \to \mathbb{R}$ be a degree-n polynomial, and let $P : \mathbb{R}^n \to \mathbb{R}$

Every polynomial p can be associated to a *blossom* P as follows

Blossoms

Let $p : \mathbb{R} \to \mathbb{R}$ be a degree-n polynomial, and let $P : \mathbb{R}^n \to \mathbb{R}$

Every polynomial p can be associated to a *blossom* P as follows

- Symmetry: $P(t_1, t_2, \ldots, t_n) = P(t_{\pi_1}, t_{\pi_2}, \ldots, t_{\pi_n})$

Blossoms

Let $p : \mathbb{R} \to \mathbb{R}$ be a degree-n polynomial, and let $P : \mathbb{R}^n \to \mathbb{R}$

Every polynomial p can be associated to a *blossom* P as follows

- Symmetry: $P(t_1, t_2, \ldots, t_n) = P(t_{\pi_1}, t_{\pi_2}, \ldots, t_{\pi_n})$
- Multi-affinity: $P\big((1-\alpha)u + \alpha v, \ldots\big) = (1-\alpha)P(u, \ldots) + \alpha P(v, \ldots)$

Blossoms

Let $p : \mathbb{R} \to \mathbb{R}$ be a degree-n polynomial, and let $P : \mathbb{R}^n \to \mathbb{R}$

Every polynomial p can be associated to a *blossom* P as follows

- Symmetry: $P(t_1, t_2, \ldots, t_n) = P(t_{\pi_1}, t_{\pi_2}, \ldots, t_{\pi_n})$
- Multi-affinity: $P\big((1-\alpha)u + \alpha v, \ldots\big) = (1-\alpha)P(u, \ldots) + \alpha P(v, \ldots)$
- Diagonal property: $P(t, t, \ldots, t) = p(t)$

Let $p : \mathbb{R} \to \mathbb{R}$ be a degree-n polynomial, and let $P : \mathbb{R}^n \to \mathbb{R}$

Every polynomial p can be associated to a *blossom* P as follows

- Symmetry: $P(t_1, t_2, \ldots, t_n) = P(t_{\pi_1}, t_{\pi_2}, \ldots, t_{\pi_n})$
- Multi-affinity: $P\big((1-\alpha)u + \alpha v, \ldots\big) = (1-\alpha)P(u, \ldots) + \alpha P(v, \ldots)$
- Diagonal property: $P(t, t, \ldots, t) = p(t)$

Correspondence is unique

Let $p : \mathbb{R} \to \mathbb{R}$ be a degree-n polynomial, and let $P : \mathbb{R}^n \to \mathbb{R}$

Every polynomial p can be associated to a *blossom* P as follows

- Symmetry: $P(t_1, t_2, \ldots, t_n) = P(t_{\pi_1}, t_{\pi_2}, \ldots, t_{\pi_n})$
- Multi-affinity: $P\big((1-\alpha)u + \alpha v, \ldots\big) = (1-\alpha)P(u, \ldots) + \alpha P(v, \ldots)$
- Diagonal property: $P(t, t, \ldots, t) = p(t)$

Correspondence is unique

We want to reparameterize $p(t)$ for $t \mapsto r(1-u) + su$

Let $p : \mathbb{R} \to \mathbb{R}$ be a degree-n polynomial, and let $P : \mathbb{R}^n \to \mathbb{R}$

Every polynomial p can be associated to a *blossom* P as follows

- Symmetry: $P(t_1, t_2, \ldots, t_n) = P(t_{\pi_1}, t_{\pi_2}, \ldots, t_{\pi_n})$
- Multi-affinity: $P\big((1-\alpha)u + \alpha v, \ldots\big) = (1-\alpha)P(u, \ldots) + \alpha P(v, \ldots)$
- Diagonal property: $P(t, t, \ldots, t) = p(t)$

Correspondence is unique

We want to reparameterize $p(t)$ for $t \mapsto r(1-u) + su$

Evaluate $p\big(r(1-u) + su\big)$ using the blossom instead

Let $p : \mathbb{R} \to \mathbb{R}$ be a degree-n polynomial, and let $P : \mathbb{R}^n \to \mathbb{R}$

Every polynomial p can be associated to a *blossom* P as follows

- Symmetry: $P(t_1, t_2, \dots, t_n) = P(t_{\pi_1}, t_{\pi_2}, \dots, t_{\pi_n})$
- Multi-affinity: $P\big((1-\alpha)u + \alpha v, \dots\big) = (1-\alpha)P(u, \dots) + \alpha P(v, \dots)$
- Diagonal property: $P(t, t, \dots, t) = p(t)$

Correspondence is unique

We want to reparameterize $p(t)$ for $t \mapsto r(1-u) + su$

Evaluate $p\big(r(1-u) + su\big)$ using the blossom instead

$p\big(r(1-u) + su\big)$

Blossoms

Let $p : \mathbb{R} \to \mathbb{R}$ be a degree-n polynomial, and let $P : \mathbb{R}^n \to \mathbb{R}$

Every polynomial p can be associated to a *blossom* P as follows

- Symmetry: $P(t_1, t_2, \ldots, t_n) = P(t_{\pi_1}, t_{\pi_2}, \ldots, t_{\pi_n})$
- Multi-affinity: $P\big((1-\alpha)u + \alpha v, \ldots\big) = (1-\alpha)P(u, \ldots) + \alpha P(v, \ldots)$
- Diagonal property: $P(t, t, \ldots, t) = p(t)$

Correspondence is unique

We want to reparameterize $p(t)$ for $t \mapsto r(1-u) + su$

Evaluate $p\big(r(1-u) + su\big)$ using the blossom instead

$p\big(r(1-u) + su\big) = P\big(r(1-u) + su, r(1-u) + su, \ldots\big)$

Blossoms

Let $p : \mathbf{R} \to \mathbf{R}$ be a degree-n polynomial, and let $P : \mathbf{R}^n \to \mathbf{R}$

Every polynomial p can be associated to a *blossom* P as follows

- Symmetry: $P(t_1, t_2, \dots, t_n) = P(t_{\pi_1}, t_{\pi_2}, \dots, t_{\pi_n})$
- Multi-affinity: $P\big((1-\alpha)u + \alpha v, \dots\big) = (1-\alpha)P(u, \dots) + \alpha P(v, \dots)$
- Diagonal property: $P(t, t, \dots, t) = p(t)$

Correspondence is unique

We want to reparameterize $p(t)$ for $t \mapsto r(1-u) + su$

Evaluate $p\big(r(1-u) + su\big)$ using the blossom instead

$$\begin{aligned} p\big(r(1-u)+su\big) &= P\big(r(1-u)+su, r(1-u)+su, \dots\big) \\ &= \tbinom{n}{0}(1-u)^n\, P(r,r,r,\dots) + \tbinom{n}{1}(1-u)^{n-1}u\, P(s,r,r,\dots) \\ &+ \tbinom{n}{2}(1-u)^{n-2}u^2\, P(s,s,r,\dots) + \dots + \tbinom{n}{n}u^n\, P(s,s,s,\dots) \end{aligned}$$

Blossoms

Rewriting,

$$p\big(r(1-u)+su\big) = \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i \, P(\underbrace{s,\ldots,s}_{i},\overbrace{r\ldots,r}^{n-i})$$

Rewriting,

$$p\big(r(1-u)+su\big) = \sum_{i=0}^{n} \tbinom{n}{i}(1-t)^{n-i}t^i\, P(\underbrace{s,\ldots,s}_{i},\overbrace{r\ldots,r}^{n-i})$$

But this is simply $p\big(r(1-u)+su\big)$ written in Bernstein basis!

Rewriting,

$$p\big(r(1-u)+su\big) = \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i \, P(\underbrace{s,\ldots,s}_{i},\overbrace{r\ldots,r}^{n-i})$$

But this is simply $p\big(r(1-u)+su\big)$ written in Bernstein basis!

$P(\underbrace{s,\ldots,s}_{i},\overbrace{r\ldots,r}^{n-i})$ *must be* ith control point for $p_{[r,s]}(u)$

Rewriting,

$$p\big(r(1-u)+su\big) = \sum_{i=0}^{n} \tbinom{n}{i}(1-t)^{n-i}t^i\, P(\underbrace{s,\ldots,s}_{i},\overbrace{r\ldots,r}^{n-i})$$

But this is simply $p\big(r(1-u)+su\big)$ written in Bernstein basis!

$P(\underbrace{s,\ldots,s}_{i},\overbrace{r\ldots,r}^{n-i})$ *must be* ith control point for $p_{[r,s]}(u)$

Setting $r=0$ and $s=1$, $P(\underbrace{1,\ldots,1}_{i},\overbrace{0,\ldots,0}^{n-i})$ is ith control point for $p(t)$

Rewriting,

$$p\big(r(1-u)+su\big) = \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i \, P(\underbrace{s,\ldots,s}_{i},\overbrace{r\ldots,r}^{n-i})$$

But this is simply $p\big(r(1-u)+su\big)$ written in Bernstein basis!

$P(\underbrace{s,\ldots,s}_{i},\overbrace{r\ldots,r}^{n-i})$ *must be* ith control point for $p_{[r,s]}(u)$

Setting $r=0$ and $s=1$, $P(\underbrace{1,\ldots,1}_{i},\overbrace{0,\ldots,0}^{n-i})$ is ith control point for $p(t)$

From them, we can evaluate the blossom $P(t_1, t_2, \ldots, t_n)$

De Casteljou using blossoms

$P(\underbrace{1,\ldots,1}_{i},\overbrace{0,\ldots,0}^{n-i})$ and $P(\underbrace{1,\ldots,1}_{i+1},\overbrace{0,\ldots,0}^{n-(i+1)})$ differ only in entry i

De Casteljou using blossoms

$P(\underbrace{1,\ldots,1}_{i},\overbrace{0,\ldots,0}^{n-i})$ and $P(\underbrace{1,\ldots,1}_{i+1},\overbrace{0,\ldots,0}^{n-(i+1)})$ differ only in entry i

$$(1-t_1)P(\underbrace{1,\ldots,1}_{i},\overbrace{0,\ldots,0}^{n-i}) + t_1 P(\underbrace{1,\ldots,1}_{i+1},\overbrace{0,\ldots,0}^{n-(i+1)})$$

De Casteljou using blossoms

$P(\underbrace{1,\dots,1}_{i},\overbrace{0,\dots,0}^{n-i})$ and $P(\underbrace{1,\dots,1}_{i+1},\overbrace{0,\dots,0}^{n-(i+1)})$ differ only in entry i

$$(1-t_1)P(\underbrace{1,\dots,1}_{i},\overbrace{0,\dots,0}^{n-i}) + t_1P(\underbrace{1,\dots,1}_{i+1},\overbrace{0,\dots,0}^{n-(i+1)}) =$$

$$= P(\underbrace{1,\dots,1}_{i-1},t_1,\overbrace{0,\dots,0}^{n-i}) \qquad \text{(multi-affinity)}$$

De Casteljou using blossoms

$$P(\underbrace{1,\ldots,1}_{i},\overbrace{0,\ldots,0}^{n-i}) \text{ and } P(\underbrace{1,\ldots,1}_{i+1},\overbrace{0,\ldots,0}^{n-(i+1)}) \text{ differ only in entry } i$$

$$\begin{aligned}
(1-t_1)P(\underbrace{1,\ldots,1}_{i},\overbrace{0,\ldots,0}^{n-i}) + t_1P(\underbrace{1,\ldots,1}_{i+1},\overbrace{0,\ldots,0}^{n-(i+1)}) &= \\
&= P(\underbrace{1,\ldots,1}_{i-1},t_1,\overbrace{0,\ldots,0}^{n-i}) \qquad \text{(multi-affinity)} \\
&= P(t_1,\underbrace{1,\ldots,1}_{i-1},\overbrace{0,\ldots,0}^{(n-1)-(i-1)}) \qquad \text{(symmetry)}
\end{aligned}$$

$P(\underbrace{1,\ldots,1}_{i},\overbrace{0,\ldots,0}^{n-i})$ and $P(\underbrace{1,\ldots,1}_{i+1},\overbrace{0,\ldots,0}^{n-(i+1)})$ differ only in entry i

$$
\begin{aligned}
(1-t_1)P(\underbrace{1,\ldots,1}_{i},\overbrace{0,\ldots,0}^{n-i}) + t_1P(\underbrace{1,\ldots,1}_{i+1},\overbrace{0,\ldots,0}^{n-(i+1)}) &= \\
&= P(\underbrace{1,\ldots,1}_{i-1},t_1,\overbrace{0,\ldots,0}^{n-i}) \qquad \text{(multi-affinity)} \\
&= P(t_1,\underbrace{1,\ldots,1}_{i-1},\overbrace{0,\ldots,0}^{(n-1)-(i-1)}) \qquad \text{(symmetry)}
\end{aligned}
$$

Repeat for $t_2,\ldots,t_n$ until we reach $P(t_1,t_2,\ldots,t_n)$

$P(\underbrace{1,\ldots,1}_{i},\overbrace{0,\ldots,0}^{n-i})$ and $P(\underbrace{1,\ldots,1}_{i+1},\overbrace{0,\ldots,0}^{n-(i+1)})$ differ only in entry i

$$\begin{aligned}(1-t_1)P(\underbrace{1,\ldots,1}_{i},\overbrace{0,\ldots,0}^{n-i}) + t_1 P(\underbrace{1,\ldots,1}_{i+1},\overbrace{0,\ldots,0}^{n-(i+1)}) = \\ = P(\underbrace{1,\ldots,1}_{i-1},t_1,\overbrace{0,\ldots,0}^{n-i}) \qquad &\text{(multi-affinity)} \\ = P(t_1,\underbrace{1,\ldots,1}_{i-1},\overbrace{0,\ldots,0}^{(n-1)-(i-1)}) \qquad &\text{(symmetry)}\end{aligned}$$

Repeat for $t_2,\ldots,t_n$ until we reach $P(t_1,t_2,\ldots,t_n)$

Easier way to perform affine reparameterization!

Using affine reparameterization or blossoms

Subdivision of Bézier segments

Using affine reparameterization or blossoms

Sometimes needed

- To make sure all segments are monotonic

Subdivision of Bézier segments

Using affine reparameterization or blossoms

Sometimes needed

- To make sure all segments are monotonic
- To make sure no segment has a double point or an inflection point

Subdivision of Bézier segments

Using affine reparameterization or blossoms

Sometimes needed

- To make sure all segments are monotonic
- To make sure no segment has a double point or an inflection point
- To divide an integral into two parts

Subdivision of Bézier segments

Using affine reparameterization or blossoms

Sometimes needed

- To make sure all segments are monotonic
- To make sure no segment has a double point or an inflection point
- To divide an integral into two parts
- To *flatten* a segment

Curve is $\gamma(t) = \big(x(t), y(t)\big)$

Intersection between Bézier segments and rays

Curve is $\gamma(t) = \big(x(t), y(t)\big)$

Horizontal ray is $r(u) = (x_r + u, y_r)$, for $u > 0$

Intersection between Bézier segments and rays

Curve is $\gamma(t) = \big(x(t), y(t)\big)$

Horizontal ray is $r(u) = (x_r + u, y_r)$, for $u > 0$

Solve $y(t) = y_r$ for t

Intersection between Bézier segments and rays

Curve is $\gamma(t) = \big(x(t), y(t)\big)$

Horizontal ray is $r(u) = (x_r + u, y_r)$, for $u > 0$

Solve $y(t) = y_r$ for t

For each solution t_i, check that $0 \le t_i \le 1$ and that $x_r \le x(t_i)$

Curve is $\gamma(t) = \big(x(t), y(t)\big)$

Horizontal ray is $r(u) = (x_r + u, y_r)$, for $u > 0$

Solve $y(t) = y_r$ for t

For each solution t_i, check that $0 \leq t_i \leq 1$ and that $x_r \leq x(t_i)$

What could go wrong?

Solve for $y'(t) = 0$ and $x'(t) = 0$ for t

Monotonization of Bézier segments

Solve for $y'(t) = 0$ and $x'(t) = 0$ for t

Sort solutions $t_1 < t_2 < \cdots < t_k$

Monotonization of Bézier segments

Solve for $y'(t) = 0$ and $x'(t) = 0$ for t

Sort solutions $t_1 < t_2 < \cdots < t_k$

Split original at corresponding parameters

Monotonization of Bézier segments

Solve for $y'(t) = 0$ and $x'(t) = 0$ for t

Sort solutions $t_1 < t_2 < \cdots < t_k$

Split original at corresponding parameters

Endpoints form tight bounding boxes

Monotonization of Bézier segments

Solve for $y'(t) = 0$ and $x'(t) = 0$ for t

Sort solutions $t_1 < t_2 < \cdots < t_k$

Split original at corresponding parameters

Endpoints form tight bounding boxes

Axis aligned rays can intersect only once

Intersection between monotonic Bézier segments and rays

Use bounding box $(x_{min}, y_{min}, x_{max}, y_{max})$

Use bounding box $(x_{min}, y_{min}, x_{max}, y_{max})$

- If $y_r > y_{max}$ or $y_r \leq y_{min} \quad \rightarrow \quad$ no intersection

Use bounding box $(x_{min}, y_{min}, x_{max}, y_{max})$

- If $y_r > y_{max}$ or $y_r \leq y_{min} \quad \rightarrow \quad$ no intersection
- Otherwise, if $x_r > x_{max} \quad \rightarrow \quad$ no intersection

Intersection between monotonic Bézier segments and rays

Use bounding box $(x_{min}, y_{min}, x_{max}, y_{max})$

- If $y_r > y_{max}$ or $y_r \leq y_{min}$ $\quad \rightarrow \quad$ no intersection
- Otherwise, if $x_r > x_{max}$ $\quad \rightarrow \quad$ no intersection
- Otherwise, if $x_r \leq x_{min}$ $\quad \rightarrow \quad$ intersection

Intersection between monotonic Bézier segments and rays

Use bounding box $(x_{min}, y_{min}, x_{max}, y_{max})$

- If $y_r > y_{max}$ or $y_r \leq y_{min} \quad \rightarrow \quad$ no intersection
- Otherwise, if $x_r > x_{max} \quad \rightarrow \quad$ no intersection
- Otherwise, if $x_r \leq x_{min} \quad \rightarrow \quad$ intersection
- Otherwise, must test!

Intersection between monotonic Bézier segments and rays

Use bounding box $(x_{min}, y_{min}, x_{max}, y_{max})$

- If $y_r > y_{max}$ or $y_r \leq y_{min}$ $\quad\rightarrow\quad$ no intersection
- Otherwise, if $x_r > x_{max}$ $\quad\rightarrow\quad$ no intersection
- Otherwise, if $x_r \leq x_{min}$ $\quad\rightarrow\quad$ intersection
- Otherwise, must test!

Inside the box, use bisection to solve $y(t_i) = y_r$ for t_i

- There *is* a solution $t_i \in [0, 1]$

Intersection between monotonic Bézier segments and rays

Use bounding box $(x_{min}, y_{min}, x_{max}, y_{max})$

- If $y_r > y_{max}$ or $y_r \leq y_{min} \quad \rightarrow \quad$ no intersection
- Otherwise, if $x_r > x_{max} \quad \rightarrow \quad$ no intersection
- Otherwise, if $x_r \leq x_{min} \quad \rightarrow \quad$ intersection
- Otherwise, must test!

Inside the box, use bisection to solve $y(t_i) = y_r$ for t_i

- There *is* a solution $t_i \in [0, 1]$

Check that $x_r \leq x(t_i)$

Flattening of Bézier segments

Recursively subdivide segments until the polyline connecting the interpolation points forms a good approximation for the curve

Flattening of Bézier segments

Recursively subdivide segments until the polyline connecting the interpolation points forms a good approximation for the curve

Sometimes needed

- If output driver does not support curves

Flattening of Bézier segments

Recursively subdivide segments until the polyline connecting the interpolation points forms a good approximation for the curve

Sometimes needed

- If output driver does not support curves
- Simple way to approximate length of a segment

Flattening of Bézier segments

Recursively subdivide segments until the polyline connecting the interpolation points forms a good approximation for the curve

Sometimes needed

- If output driver does not support curves
- Simple way to approximate length of a segment

Show in Mathematica

Rational Bézier curves

Every integral quadratic Bézier segment is a parabola

There is T affine that maps any quadratic Bézier to $y = x^2$

$$\begin{bmatrix} x(t) \\ y(t) \\ 1 \end{bmatrix} = \begin{bmatrix} t \\ t^2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} (1-t)^2 \\ 2(1-t)t \\ t^2 \end{bmatrix} = \mathsf{C}\, B_2(t)$$

There is T affine that maps any quadratic Bézier to $y = x^2$

$$\begin{bmatrix} x(t) \\ y(t) \\ 1 \end{bmatrix} = \begin{bmatrix} t \\ t^2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} (1-t)^2 \\ 2(1-t)t \\ t^2 \end{bmatrix} = \mathsf{C}\, B_2(t)$$

$$\mathsf{T} \begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \Rightarrow \quad \mathsf{T} = \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

Every integral quadratic Bézier segment is a parabola

There is T affine that maps any quadratic Bézier to $y = x^2$

$$\begin{bmatrix} x(t) \\ y(t) \\ 1 \end{bmatrix} = \begin{bmatrix} t \\ t^2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} (1-t)^2 \\ 2(1-t)t \\ t^2 \end{bmatrix} = \mathsf{C}\, B_2(t)$$

$$\mathsf{T} \begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \Rightarrow \quad \mathsf{T} = \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

How can we tell T is affine?

Every integral quadratic Bézier segment is a parabola

There is T affine that maps any quadratic Bézier to $y = x^2$

$$\begin{bmatrix} x(t) \\ y(t) \\ 1 \end{bmatrix} = \begin{bmatrix} t \\ t^2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} (1-t)^2 \\ 2(1-t)t \\ t^2 \end{bmatrix} = \mathsf{C}\, B_2(t)$$

$$\mathsf{T} \begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \Rightarrow \quad \mathsf{T} = \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

How can we tell T is affine?

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathsf{A} = v \quad \Rightarrow \quad v\mathsf{A}^{-1} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathsf{B} = v \quad \Rightarrow \quad \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathsf{B}\,\mathsf{A}^{-1} = v\mathsf{A}^{-1} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Rational polynomial parameterization of unit circle

Start with the unit circle in first quadrant

Rational polynomial parameterization of unit circle

Start with the unit circle in first quadrant

Obvious parameterization is

$$\gamma(t) = \begin{bmatrix} \cos t & \sin t \end{bmatrix}^T, \quad \text{for} \quad t \in [0, \tfrac{\pi}{2}]$$

Rational polynomial parameterization of unit circle

Start with the unit circle in first quadrant

Obvious parameterization is

$$\gamma(t) = \begin{bmatrix} \cos t & \sin t \end{bmatrix}^T, \quad \text{for} \quad t \in [0, \tfrac{\pi}{2}]$$

But

$$\cos t = \frac{\cos^2 \frac{t}{2} - \sin^2 \frac{t}{2}}{\sin^2 \frac{t}{2} + \cos^2 \frac{t}{2}}$$

$$\sin t = \frac{2 \sin \frac{t}{2} \cos \frac{t}{2}}{\sin^2 \frac{t}{2} + \cos^2 \frac{t}{2}}$$

Rational polynomial parameterization of unit circle

Start with the unit circle in first quadrant

Obvious parameterization is

$$\gamma(t) = \begin{bmatrix} \cos t & \sin t \end{bmatrix}^T, \quad \text{for} \quad t \in [0, \tfrac{\pi}{2}]$$

But

$$\cos t = \frac{\cos^2 \frac{t}{2} - \sin^2 \frac{t}{2}}{\sin^2 \frac{t}{2} + \cos^2 \frac{t}{2}} = \frac{1 - \tan^2 \frac{t}{2}}{1 + \tan^2 \frac{t}{2}}$$

$$\sin t = \frac{2 \sin \frac{t}{2} \cos \frac{t}{2}}{\sin^2 \frac{t}{2} + \cos^2 \frac{t}{2}} = \frac{2 \tan \frac{t}{2}}{1 + \tan^2 \frac{t}{2}}$$

Rational polynomial parameterization of unit circle

Start with the unit circle in first quadrant

Obvious parameterization is

$$\gamma(t) = \begin{bmatrix}\cos t & \sin t\end{bmatrix}^T, \quad \text{for} \quad t \in [0, \tfrac{\pi}{2}]$$

But

$$\cos t = \frac{\cos^2 \frac{t}{2} - \sin^2 \frac{t}{2}}{\sin^2 \frac{t}{2} + \cos^2 \frac{t}{2}} = \frac{1 - \tan^2 \frac{t}{2}}{1 + \tan^2 \frac{t}{2}} = \frac{1 - u^2}{1 + u^2}$$

$$\sin t = \frac{2 \sin \frac{t}{2} \cos \frac{t}{2}}{\sin^2 \frac{t}{2} + \cos^2 \frac{t}{2}} = \frac{2 \tan \frac{t}{2}}{1 + \tan^2 \frac{t}{2}} = \frac{2u}{1 + u^2}$$

Rational polynomial parameterization of unit circle

Start with the unit circle in first quadrant

Obvious parameterization is

$$\gamma(t) = \begin{bmatrix}\cos t & \sin t\end{bmatrix}^T, \quad \text{for} \quad t \in [0, \tfrac{\pi}{2}]$$

But

$$\cos t = \frac{\cos^2 \frac{t}{2} - \sin^2 \frac{t}{2}}{\sin^2 \frac{t}{2} + \cos^2 \frac{t}{2}} = \frac{1 - \tan^2 \frac{t}{2}}{1 + \tan^2 \frac{t}{2}} = \frac{1 - u^2}{1 + u^2}$$

$$\sin t = \frac{2 \sin \frac{t}{2} \cos \frac{t}{2}}{\sin^2 \frac{t}{2} + \cos^2 \frac{t}{2}} = \frac{2 \tan \frac{t}{2}}{1 + \tan^2 \frac{t}{2}} = \frac{2u}{1 + u^2}$$

So we also have

$$\gamma(u) = \begin{bmatrix}\frac{1-u^2}{1+u^2} & \frac{2u}{1+u^2}\end{bmatrix}^T, \quad \text{for} \quad u \in [0, 1]$$

Move to $\mathbb{RP}^2$

In the projective plane, using homogeneous coordinates, we have

$$\gamma(u) = \begin{bmatrix} x(u) & y(u) & w(u) \end{bmatrix}^T = \begin{bmatrix} 1-u^2 & 2u & 1+u^2 \end{bmatrix}^T, \quad \text{for } u \in [0,1]$$

Move to $\mathbb{RP}^2$

In the projective plane, using homogeneous coordinates, we have

$$\gamma(u) = \begin{bmatrix} x(u) & y(u) & w(u) \end{bmatrix}^T = \begin{bmatrix} 1-u^2 & 2u & 1+u^2 \end{bmatrix}^T, \quad \text{for } u \in [0,1]$$

Reparameterize to make more symmetrical $u \mapsto (1-v)\tan\frac{\alpha}{2} + v\tan\frac{\alpha}{2}$

$$\gamma(v) = \begin{bmatrix} \cos\alpha & 2(1-\cos\alpha) & -2(1-\cos\alpha) \\ -\sin\alpha & 2\sin\alpha & 0 \\ 1 & -2(1-\cos\alpha) & 2(1-\cos\alpha) \end{bmatrix} P_2(v)$$

Move to $\mathbb{RP}^2$

In the projective plane, using homogeneous coordinates, we have

$$\gamma(u) = \begin{bmatrix} x(u) & y(u) & w(u) \end{bmatrix}^T = \begin{bmatrix} 1-u^2 & 2u & 1+u^2 \end{bmatrix}^T, \quad \text{for } u \in [0,1]$$

Reparameterize to make more symmetrical $u \mapsto (1-v)\tan\frac{\alpha}{2} + v\tan\frac{\alpha}{2}$

$$\gamma(v) = \begin{bmatrix} \cos\alpha & 2(1-\cos\alpha) & -2(1-\cos\alpha) \\ -\sin\alpha & 2\sin\alpha & 0 \\ 1 & -2(1-\cos\alpha) & 2(1-\cos\alpha) \end{bmatrix} P_2(v)$$

Covert to Bernstein basis

Move to $\mathbb{RP}^2$

In the projective plane, using homogeneous coordinates, we have

$$\gamma(u) = \begin{bmatrix} x(u) & y(u) & w(u) \end{bmatrix}^T = \begin{bmatrix} 1 - u^2 & 2u & 1 + u^2 \end{bmatrix}^T, \quad \text{for } u \in [0, 1]$$

Reparameterize to make more symmetrical $u \mapsto (1 - v) \tan \frac{\alpha}{2} + v \tan \frac{\alpha}{2}$

$$\gamma(v) = \begin{bmatrix} \cos\alpha & 2(1 - \cos\alpha) & -2(1 - \cos\alpha) \\ -\sin\alpha & 2\sin\alpha & 0 \\ 1 & -2(1 - \cos\alpha) & 2(1 - \cos\alpha) \end{bmatrix} P_2(v)$$

Covert to Bernstein basis

$$\gamma(v) = \begin{bmatrix} \cos\alpha & 1 & \cos\alpha \\ -\sin\alpha & 0 & \sin\alpha \\ 1 & \cos\alpha & 1 \end{bmatrix} B_2(v)$$

Move to RP^2

In the projective plane, using homogeneous coordinates, we have

$$\gamma(u) = \begin{bmatrix} x(u) & y(u) & w(u) \end{bmatrix}^T = \begin{bmatrix} 1-u^2 & 2u & 1+u^2 \end{bmatrix}^T, \quad \text{for } u \in [0,1]$$

Reparameterize to make more symmetrical $u \mapsto (1-v)\tan\frac{\alpha}{2} + v\tan\frac{\alpha}{2}$

$$\gamma(v) = \begin{bmatrix} \cos\alpha & 2(1-\cos\alpha) & -2(1-\cos\alpha) \\ -\sin\alpha & 2\sin\alpha & 0 \\ 1 & -2(1-\cos\alpha) & 2(1-\cos\alpha) \end{bmatrix} P_2(v)$$

Covert to Bernstein basis

$$\gamma(v) = \begin{bmatrix} \cos\alpha & 1 & \cos\alpha \\ -\sin\alpha & 0 & \sin\alpha \\ 1 & \cos\alpha & 1 \end{bmatrix} B_2(v)$$

We will call this the *canonical arc segment*

Projective reparameterization

Recall our affine reparameterization was of the form

$$t \mapsto a + bu$$

Recall our affine reparameterization was of the form

$$t \mapsto a + bu$$

Polynomials are closed under affine reparameterizations

Recall our affine reparameterization was of the form

$$t \mapsto a + bu$$

Polynomials are closed under affine reparameterizations

We can now apply a projective reparameterization

$$t \mapsto \frac{a + bu}{c + du}$$

Projective reparameterization

Recall our affine reparameterization was of the form

$$t \mapsto a + bu$$

Polynomials are closed under affine reparameterizations

We can now apply a projective reparameterization

$$t \mapsto \frac{a + bu}{c + du}$$

Also known as a *Möbius transformation*

Projective reparameterization

Recall our affine reparameterization was of the form

$$t \mapsto a + bu$$

Polynomials are closed under affine reparameterizations

We can now apply a projective reparameterization

$$t \mapsto \frac{a + bu}{c + du}$$

Also known as a *Möbius transformation*

Rational polynomials are closed under projective reparameterizations

In particular, those for which $[0, 1] \mapsto [0, 1]$

$$t \mapsto \frac{a + bu}{c + du} \qquad \text{(general case)}$$

In particular, those for which $[0,1] \mapsto [0,1]$

$$t \mapsto \frac{a+bu}{c+du} \qquad \text{(general case)}$$

$$t \mapsto \frac{bu}{c+du} \qquad (0 \mapsto 0 \;\Rightarrow\; a = 0)$$

In particular, those for which $[0,1] \mapsto [0,1]$

$$t \mapsto \frac{a + bu}{c + du} \qquad \text{(general case)}$$

$$t \mapsto \frac{bu}{c + du} \qquad (0 \mapsto 0 \;\Rightarrow\; a = 0)$$

$$t \mapsto \frac{(c + d)u}{c + du} \qquad (1 \mapsto 1 \;\Rightarrow\; b = c + d)$$

In particular, those for which $[0,1] \mapsto [0,1]$

$$t \mapsto \frac{a+bu}{c+du} \qquad \text{(general case)}$$

$$t \mapsto \frac{bu}{c+du} \qquad (0 \mapsto 0 \;\Rightarrow\; a = 0)$$

$$t \mapsto \frac{(c+d)u}{c+du} \qquad (1 \mapsto 1 \;\Rightarrow\; b = c+d)$$

$$t \mapsto \frac{u}{\lambda + (1-\lambda)u} \qquad (\lambda = \tfrac{c}{c+d})$$

Projective reparameterization

$$\gamma^n(t) = \sum_{i=0}^{n} p_i b_{n,i}(t) = \sum_{i=0}^{n} p_i \binom{n}{i} (1-t)^{n-i} t^i$$

Projective reparameterization

$$\gamma^n(t) = \sum_{i=0}^{n} p_i b_{n,i}(t) = \sum_{i=0}^{n} p_i \binom{n}{i} (1-t)^{n-i} t^i$$

$$t \mapsto \frac{u}{\lambda + (1-\lambda)u} \qquad 1 - t \mapsto \frac{\lambda(1-u)}{\lambda + (1-\lambda)u}$$

Projective reparameterization

$$\gamma^n(t) = \sum_{i=0}^{n} p_i b_{n,i}(t) = \sum_{i=0}^{n} p_i \binom{n}{i} (1-t)^{n-i} t^i$$

$$t \mapsto \frac{u}{\lambda + (1-\lambda)u} \qquad 1 - t \mapsto \frac{\lambda(1-u)}{\lambda + (1-\lambda)u}$$

$$\gamma^n\left(\frac{u}{\lambda + (1-\lambda)u}\right) = \sum_{i=0}^{n} p_i \binom{n}{i} \frac{\lambda^{n-i}(1-u)^{n-i}u^i}{\left(\lambda + (1-\lambda)u\right)^n}$$

Projective reparameterization

$$\gamma^n(t) = \sum_{i=0}^{n} p_i b_{n,i}(t) = \sum_{i=0}^{n} p_i \binom{n}{i} (1-t)^{n-i} t^i$$

$$t \mapsto \frac{u}{\lambda + (1-\lambda)u} \qquad 1 - t \mapsto \frac{\lambda(1-u)}{\lambda + (1-\lambda)u}$$

$$\begin{aligned}\gamma^n\left(\frac{u}{\lambda + (1-\lambda)u}\right) &= \sum_{i=0}^{n} p_i \binom{n}{i} \frac{\lambda^{n-i}(1-u)^{n-i}u^i}{\big(\lambda + (1-\lambda)u\big)^n} \\ &= \sum_{i=0}^{n} (\lambda^{n-i} p_i) \binom{n}{i} (1-u)^{n-i} u^i\end{aligned}$$

Projective reparameterization

$$\gamma^n(t) = \sum_{i=0}^{n} p_i b_{n,i}(t) = \sum_{i=0}^{n} p_i \tbinom{n}{i} (1-t)^{n-i} t^i$$

$$t \mapsto \frac{u}{\lambda + (1-\lambda)u} \qquad 1 - t \mapsto \frac{\lambda(1-u)}{\lambda + (1-\lambda)u}$$

$$\begin{aligned}
\gamma^n\left(\frac{u}{\lambda + (1-\lambda)u}\right) &= \sum_{i=0}^{n} p_i \tbinom{n}{i} \frac{\lambda^{n-i}(1-u)^{n-i}u^i}{\big(\lambda + (1-\lambda)u\big)^n} \\
&= \sum_{i=0}^{n} (\lambda^{n-i} p_i) \tbinom{n}{i} (1-u)^{n-i} u^i \\
&= \sum_{i=0}^{n} (\lambda^{n-i} p_i) b_{n,i}(t)
\end{aligned}$$

Canonic form for rational Bézier

Given control points $p_i = \begin{bmatrix} x_i & y_i & w_i \end{bmatrix}$, can we trace the same curve but use control points $p'_i = \begin{bmatrix} x'_i & y'_i & w'_i \end{bmatrix}$ such that $w'_0 = w'_n = 1$?

Canonic form for rational Bézier

Given control points $p_i = \begin{bmatrix} x_i & y_i & w_i \end{bmatrix}$, can we trace the same curve but use control points $p'_i = \begin{bmatrix} x'_i & y'_i & w'_i \end{bmatrix}$ such that $w'_0 = w'_n = 1$?

Use a projective reparameterization that maps $[0, 1]$ to $[0, 1]$

Canonic form for rational Bézier

Given control points $p_i = \begin{bmatrix} x_i & y_i & w_i \end{bmatrix}$, can we trace the same curve but use control points $p'_i = \begin{bmatrix} x'_i & y'_i & w'_i \end{bmatrix}$ such that $w'_0 = w'_n = 1$?

Use a projective reparameterization that maps $[0, 1]$ to $[0, 1]$

Result is $p'_i = \lambda^{n-i} p_i$

Canonic form for rational Bézier

Given control points $p_i = \begin{bmatrix} x_i & y_i & w_i \end{bmatrix}$, can we trace the same curve but use control points $p'_i = \begin{bmatrix} x'_i & y'_i & w'_i \end{bmatrix}$ such that $w'_0 = w'_n = 1$?

Use a projective reparameterization that maps $[0, 1]$ to $[0, 1]$

Result is $p'_i = \lambda^{n-i} p_i$

In particular, $w'_0 = \lambda^n w_0$ and $w'_n = w_n$

Canonic form for rational Bézier

Given control points $p_i = \begin{bmatrix} x_i & y_i & w_i \end{bmatrix}$, can we trace the same curve but use control points $p_i' = \begin{bmatrix} x_i' & y_i' & w_i' \end{bmatrix}$ such that $w_0' = w_n' = 1$?

Use a projective reparameterization that maps $[0, 1]$ to $[0, 1]$

Result is $p_i' = \lambda^{n-i} p_i$

In particular, $w_0' = \lambda^n w_0$ and $w_n' = w_n$

Set $w_0' = w_n'$ and solve for λ: $\quad \lambda^n w_0 = w_n \;\Rightarrow\; \sqrt[n]{\frac{w_n}{w_0}}$

Canonic form for rational Bézier

Given control points $p_i = \begin{bmatrix} x_i & y_i & w_i \end{bmatrix}$, can we trace the same curve but use control points $p'_i = \begin{bmatrix} x'_i & y'_i & w'_i \end{bmatrix}$ such that $w'_0 = w'_n = 1$?

Use a projective reparameterization that maps $[0, 1]$ to $[0, 1]$

Result is $p'_i = \lambda^{n-i} p_i$

In particular, $w'_0 = \lambda^n w_0$ and $w'_n = w_n$

Set $w'_0 = w'_n$ and solve for λ: $\quad \lambda^n w_0 = w_n \ \Rightarrow \ \sqrt[n]{\frac{w_n}{w_0}}$

Requires $w_0 \neq 0$, $w_n \neq 0$ and either $w_n / w_0 \geq 0$ or n odd

Canonic form for rational Bézier

Given control points $p_i = \begin{bmatrix} x_i & y_i & w_i \end{bmatrix}$, can we trace the same curve but use control points $p_i' = \begin{bmatrix} x_i' & y_i' & w_i' \end{bmatrix}$ such that $w_0' = w_n' = 1$?

Use a projective reparameterization that maps $[0, 1]$ to $[0, 1]$

Result is $p_i' = \lambda^{n-i} p_i$

In particular, $w_0' = \lambda^n w_0$ and $w_n' = w_n$

Set $w_0' = w_n'$ and solve for λ: $\quad \lambda^n w_0 = w_n \;\Rightarrow\; \sqrt[n]{\frac{w_n}{w_0}}$

Requires $w_0 \neq 0$, $w_n \neq 0$ and either $w_n / w_0 \geq 0$ or n odd

If so, apply reparameterization and then divide by w_n

Given control points $p_i = \begin{bmatrix} x_i & y_i & w_i \end{bmatrix}$, can we trace the same curve but use control points $p'_i = \begin{bmatrix} x'_i & y'_i & w'_i \end{bmatrix}$ such that $w'_0 = w'_n = 1$?

Use a projective reparameterization that maps $[0, 1]$ to $[0, 1]$

Result is $p'_i = \lambda^{n-i} p_i$

In particular, $w'_0 = \lambda^n w_0$ and $w'_n = w_n$

Set $w'_0 = w'_n$ and solve for λ: $\quad \lambda^n w_0 = w_n \;\Rightarrow\; \sqrt[n]{\frac{w_n}{w_0}}$

Requires $w_0 \neq 0$, $w_n \neq 0$ and either $w_n / w_0 \geq 0$ or n odd

If so, apply reparameterization and then divide by w_n

$p'_i = p_i \sqrt[n]{w_n^{i-n} w_0^i}$

Parabola, ellipse, or hyperbola?

Parabola, ellipse, or hyperbola?

Points-with-weight interpretation

Miscellanea

Parabola, ellipse, or hyperbola?

Points-with-weight interpretation

Projective invariance of rational Bézier segments

Miscellanea

Parabola, ellipse, or hyperbola?

Points-with-weight interpretation

Projective invariance of rational Bézier segments

How to complement a rational Bézier segment?

Miscellanea

Parabola, ellipse, or hyperbola?

Points-with-weight interpretation

Projective invariance of rational Bézier segments

How to complement a rational Bézier segment?

- Easy for circular arcs

Parabola, ellipse, or hyperbola?

Points-with-weight interpretation

Projective invariance of rational Bézier segments

How to complement a rational Bézier segment?

- Easy for circular arcs
- By projective reparameterization

Miscellanea

Parabola, ellipse, or hyperbola?

Points-with-weight interpretation

Projective invariance of rational Bézier segments

How to complement a rational Bézier segment?

- Easy for circular arcs
- By projective reparameterization
- Parametrization issues

Parabola, ellipse, or hyperbola?

Points-with-weight interpretation

Projective invariance of rational Bézier segments

How to complement a rational Bézier segment?

- Easy for circular arcs
- By projective reparameterization
- Parametrization issues

How to find the affine transformation that maps the unit circle into a given rational quadratic Bézier segment?

References

G. Farin. *Curves and Surfaces for CAGD, A Practical Guide, 5th edition*. Morgan Kaufmann, 2002.