2D Computer Graphics

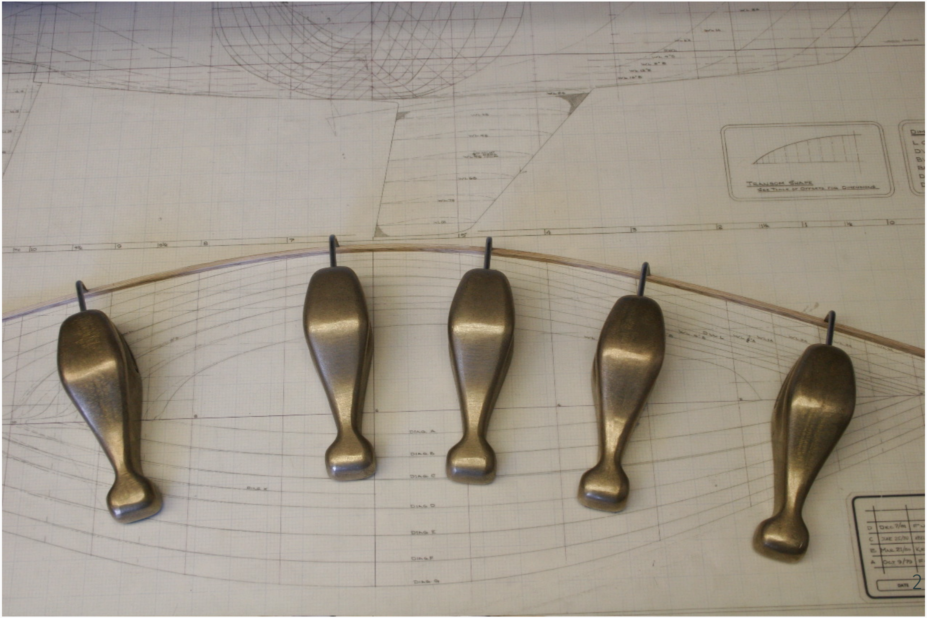
Diego Nehab

Summer 2020

IMPA

Bézier curves

Curve modeling by splines



Thin strip of wood used in building construction

Curve modeling by *splines*

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Anchored in place by lead weights called *ducks*

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Physical process

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Interpolating, smooth, energy minimizing ✓

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Thin strip of wood used in building construction

Anchored in place by lead weights called *ducks*

Physical process

Interpolating, smooth, energy minimizing ✓

No local control ✗


FABER-CASTELL
BURMESTER 2

Computational process

$k+1$ vertices $\{p_0, \ldots, p_k\}$ define a curve

$$\gamma(t) = \sum_{i=0}^{k} p_i \frac{\prod_{j \neq i}(t-j)}{\prod_{j \neq i}(i-j)}, \quad t \in [0,k]$$

Similar issues

Define a family of generating functions β^n recursively

$$\beta^0(t) = \begin{cases} 1, & -\frac{1}{2} \leq t < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\beta^n = \beta^{n-1} * \beta^0, \quad n \in \mathbb{N}$$

Notation for *convolution*

$$h = f * g \;\Leftrightarrow\; h(t) = \int_{-\infty}^{\infty} f(u)\, g(u - t) dt$$

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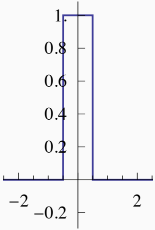

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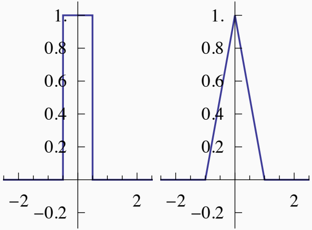

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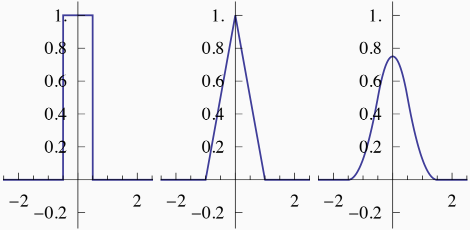

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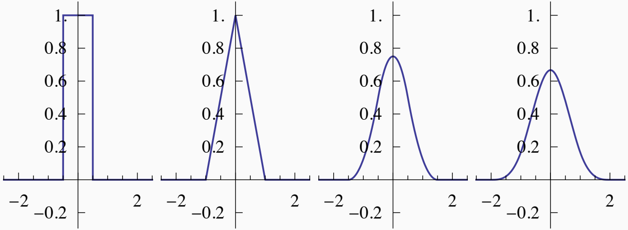

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$$\beta^2(t) = \begin{cases} \frac{1}{8}(3 + 2t)^2, & -\frac{3}{2} \leq t < -\frac{1}{2} \\ \frac{1}{4}(3 - 4t^2), & -\frac{1}{2} \leq t < \frac{1}{2} \\ \frac{1}{8}(9 - 12t + 4t^2), & \frac{1}{2} \leq t < \frac{2}{2} \\ 0, & \text{otherwise} \end{cases}$$

$k+1$ vertices $\{p_0, \ldots, p_k\}$ and generating function β^n define a curve

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B-splines

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Many, many interesting properties

Polylines

$k+1$ vertices $\{p_0, \ldots, p_k\}$ define k segments

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Bézier curves

Generalization of linear interpolation

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$kn+1$ vertices $\{p_0, \dots, p_{kn}\}$ define k segments of degree n

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De Casteljau algorithm

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De Casteljau algorithm

Geometric interpretation

Algebraic interpretation

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Expanding and collecting the p_i terms,

$$\gamma_i^n(t) = \sum_{j=0}^{n} \binom{n}{j} (1-t)^{n-j}\, t^j\, p_{i+j}$$

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$$\gamma_i^n(t) = \sum_{j=0}^{n} b_{j,n}(t)\, p_{i+j} \quad \text{with} \quad b_{j,n}(t) = \binom{n}{j} (1-t)^{n-j}\, t^j$$

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Basis for the space of polynomials P_n with degree n or less (Why?)

In matrix form

$$\gamma_i^n(t) = \begin{bmatrix} p_{ni} & p_{ni+1} & \cdots & p_{ni+n} \end{bmatrix} \begin{bmatrix} b_{0,n}(t) \\ b_{1,n}(t) \\ \vdots \\ b_{n,n}(t) \end{bmatrix}$$

In matrix form

$$\gamma_i^n(t) = \underbrace{\begin{bmatrix} p_{ni} & p_{ni+1} & \cdots & p_{ni+n} \end{bmatrix}}_{\text{Bézier control points } \mathbf{C}_\mathbf{i}^\mathbf{B}} \left.\begin{bmatrix} b_{0,n}(t) \\ b_{1,n}(t) \\ \vdots \\ b_{n,n}(t) \end{bmatrix}\right\} \text{blending weights } B_n(t)$$

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Linear invariance is quite obvious in this form

Can be converted back and forth to power basis

$$P_n(t) = \begin{bmatrix} 1 & t & \cdots & t^n \end{bmatrix}^T \qquad\qquad B_n(t) = \mathbf{B_n}\, P_n(t)$$

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Examples

$$\mathsf{B_1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad \mathsf{B_2} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathsf{B_3} = \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\gamma_i^n(t) = \mathsf{C_i^B}\, B_n(t) = \underbrace{\mathsf{C_i^B} \overbrace{\mathsf{B_n}}^{\text{Change of basis}}}_{\text{Power basis control points } \mathsf{C_i^P}} P_n(t) = \mathsf{C_i^P}\, P_n(t)$$

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PostScript, SVG (Inkscape), RVG

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Mathematica

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We want to show that

$$T\Big(\sum_{j=0}^{n} b_{j,n}(t)\, p_j\Big) = \sum_{j=0}^{n} b_{j,n}(t)\, T(p_j).$$

Affine invariance of Bézier segments

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This will be true if and only if all points in the Bézier curve are affine combinations of the control points.

Indeed,

$$\sum_{j=0}^{n} b_{j,n}(t) = \sum_{j=0}^{n} \binom{n}{j} (1-t)^{n-j}\, t^j$$

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The Bernstein polynomials therefore form a *partition of unity*

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To apply an affine transformation to a Bézier curve, simply transform the control points

Convex hull property for Bézier segments

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If γ is a Bézier curve, then $\{\gamma(t) \mid t \in [0, 1]\}$ is contained in the convex hull of its control points

- From partition of unity and positivity in $[0, 1]$
- Useful for curve intersection, quick bounding box, etc

Derivative of Bézier segment

Since derivative operator is linear and $\gamma^n(t) = \sum_{j=0}^{n} b_{j,n}(t)\, p_j$, all we have to do is differentiate the Bernstein polynomials

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\end{aligned}$$

Therefore,

$$(\gamma^n)'(t) = \sum_{j=0}^{n-1} b_{j,n-1}(t)\, q_j \quad \text{with} \quad q_j = n(p_{j+1} - p_j).$$

What is the derivative at the endpoints?

Derivative of Bézier segment

What is the derivative at the endpoints?

How do we connect segments so that they are C^1 continuous?

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What about G^1 continuity?

Derivative of Bézier segment

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How do we connect segments so that they are C^1 continuous?

What about G^1 continuity?

Show in Inkscape

Degree elevation of Bézier segment

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$$b_{j,n+1}(t) = \binom{n+1}{j} (1-t)^{n+1-j}\, t^j$$

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$$b_{j,n+1}(t) = \binom{n+1}{j} (1-t)^{n+1-j}\, t^j = \tfrac{n+1}{n+1-j}\, (1-t)\, b_{j,n}(t)$$

Degree elevation of Bézier segment

Express a segment $\gamma^n(t)$ as $\gamma^{n+1}(t)$? (write $b_{j,n}(t)$ in terms of $b_{k,n+1}(t)$?)

Easy to express both $b_{j,n+1}(t)$ and $b_{j+1,n+1}(t)$ in terms of $b_{j,n}(t)$

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From these,

$$b_{j,n}(t) = \tfrac{n+1-j}{n+1}\, b_{j,n+1}(t) + \tfrac{j+1}{n+1}\, b_{j+1,n+1}(t)$$

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Expanding and collecting terms,

$$\gamma^n(t) = \sum_{i=0}^{n} b_{i,n}(t)\, p_i = \sum_{j=0}^{n+1} b_{j,n+1}(t)\, q_j = \gamma^{n+1}(t)$$

with $q_0 = p_0$, $q_{n+1} = p_n$, and

$$q_i = \tfrac{j}{n+1}\, p_{i-1} + (1 - \tfrac{j}{n+1})\, p_i$$

Examples

$$\begin{bmatrix} p_0 & p_1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} p_0 & \frac{1}{2}(p_0 + p_1) & p_1 \end{bmatrix}$$

$$\begin{bmatrix} p_0 & p_1 & p_2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} p_0 & \frac{1}{3}(p_0 + 2p_1) & \frac{1}{3}(2p_1 + p_2) & p_2 \end{bmatrix}$$

References

G. Farin. *Curves and Surfaces for CAGD, A Practical Guide, 5th edition*. Morgan Kaufmann, 2002.