2D Computer Graphics

Diego Nehab

Summer 2020

IMPA

Digital images and anti-aliasing

What is an image?

Let $S = [a, b] \times [c, d] \subset \mathbb{R}^2$ be a spatial domain

What is an image?

Let $S = [a, b] \times [c, d] \subset \mathsf{R}^2$ be a spatial domain

Let $V = \mathsf{R}_{>0} \to \mathsf{R}_{\geq 0}$ be the space of spectral exitance distributions

What is an image?

Let $S = [a, b] \times [c, d] \subset \mathsf{R}^2$ be a spatial domain

Let $V = \mathsf{R}_{>0} \to \mathsf{R}_{\geq 0}$ be the space of spectral exitance distributions

An ideal image is a function

$$I : S \to V$$

that takes point p to its spectral exitance $M_{e,\lambda}$

What is an image?

Let $S = [a, b] \times [c, d] \subset \mathbb{R}^2$ be a spatial domain

Let $V = \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ be the space of spectral exitance distributions

An ideal image is a function

$$I : S \to V$$

that takes point p to its spectral exitance $M_{e,\lambda}$

But computers are finite, so we must discretize

Domain discretization

Common to discretize the domain into a uniform grid

$$D = \{1, \ldots, w\} \times \{1, \ldots, h\}$$

Common to discretize the domain into a uniform grid

$$D = \{1, \dots, w\} \times \{1, \dots, h\}$$

Two popular ways of mapping between $(i, j) \in D$ and $(x, y) \in S$

$$(x, y) = \left(a + \tfrac{i-1}{w}(b - a),\ c + \tfrac{j-1}{h}(d - c)\right) \qquad \text{(primal)}$$

$$(x, y) = \left(a + \tfrac{i-0.5}{w}(b - a),\ c + \tfrac{j-0.5}{h}(d - c)\right) \qquad \text{(dual)}$$

Domain discretization

Common to discretize the domain into a uniform grid

$$D = \{1, \dots, w\} \times \{1, \dots, h\}$$

Two popular ways of mapping between $(i, j) \in D$ and $(x, y) \in S$

$$(x, y) = \left(a + \tfrac{i-1}{w}(b-a),\ c + \tfrac{j-1}{h}(d-c)\right) \qquad \text{(primal)}$$

$$(x, y) = \left(a + \tfrac{i-0.5}{w}(b-a),\ c + \tfrac{j-0.5}{h}(d-c)\right) \qquad \text{(dual)}$$

Could use other grids (e.g. hexagonal)

Domain discretization

Common to discretize the domain into a uniform grid

$$D = \{1, \ldots, w\} \times \{1, \ldots, h\}$$

Two popular ways of mapping between $(i, j) \in D$ and $(x, y) \in S$

$$(x, y) = \left(a + \tfrac{i-1}{w}(b - a),\ c + \tfrac{j-1}{h}(d - c)\right) \qquad \text{(primal)}$$

$$(x, y) = \left(a + \tfrac{i-0.5}{w}(b - a),\ c + \tfrac{j-0.5}{h}(d - c)\right) \qquad \text{(dual)}$$

Could use other grids (e.g. hexagonal)

"Resolution" is an ambiguous term

- In printers and scanners, refers to "dots per inch" (DPI)
- In images and cameras, typically refers to $w \times h$

Codomain discretization

Common to discretize the codomain as into a set C

- Luminous exitance levels 8-bits $\{0, \ldots, 255\}$, gamma encoded

Codomain discretization

Common to discretize the codomain as into a set C

- Luminous exitance levels 8-bits $\{0, \ldots, 255\}$, gamma encoded
- R, G, and B in 8-bits each, gamma encoded (*sRGB*)

Codomain discretization

Common to discretize the codomain as into a set C

- Luminous exitance levels 8-bits $\{0, \ldots, 255\}$, gamma encoded
- R, G, and B in 8-bits each, gamma encoded (*sRGB*)
- Include an alpha channel?

Codomain discretization

Common to discretize the codomain as into a set C

- Luminous exitance levels 8-bits $\{0, \ldots, 255\}$, gamma encoded
- R, G, and B in 8-bits each, gamma encoded (*sRGB*)
- Include an alpha channel?
- Use indices into a color palette?

Codomain discretization

Common to discretize the codomain as into a set C

- Luminous exitance levels 8-bits $\{0, \ldots, 255\}$, gamma encoded
- R, G, and B in 8-bits each, gamma encoded (*sRGB*)
- Include an alpha channel?
- Use indices into a color palette?
- Use more than 3 color channels (multispectral images)?

Codomain discretization

Common to discretize the codomain as into a set C

- Luminous exitance levels 8-bits $\{0, \ldots, 255\}$, gamma encoded
- R, G, and B in 8-bits each, gamma encoded ($sRGB$)
- Include an alpha channel?
- Use indices into a color palette?
- Use more than 3 color channels (multispectral images)?
- Represent each channel in 16-bits or even 32-bit floats (HDR)?

Codomain discretization

Common to discretize the codomain as into a set C

- Luminous exitance levels 8-bits $\{0, \ldots, 255\}$, gamma encoded
- R, G, and B in 8-bits each, gamma encoded ($sRGB$)
- Include an alpha channel?
- Use indices into a color palette?
- Use more than 3 color channels (multispectral images)?
- Represent each channel in 16-bits or even 32-bit floats (HDR)?

A *pixel* is simply the value stored at coordinates (i, j)

Codomain discretization

Common to discretize the codomain as into a set C

- Luminous exitance levels 8-bits $\{0, \ldots, 255\}$, gamma encoded
- R, G, and B in 8-bits each, gamma encoded (*sRGB*)
- Include an alpha channel?
- Use indices into a color palette?
- Use more than 3 color channels (multispectral images)?
- Represent each channel in 16-bits or even 32-bit floats (HDR)?

A *pixel* is simply the value stored at coordinates (i, j)

It is *not* a little square [Smith, 1995]

Codomain discretization

Common to discretize the codomain as into a set C

- Luminous exitance levels 8-bits $\{0, \ldots, 255\}$, gamma encoded
- R, G, and B in 8-bits each, gamma encoded (*sRGB*)
- Include an alpha channel?
- Use indices into a color palette?
- Use more than 3 color channels (multispectral images)?
- Represent each channel in 16-bits or even 32-bit floats (HDR)?

A *pixel* is simply the value stored at coordinates (i, j)

It is *not* a little square [Smith, 1995]

How to select the values to store?

Rendering

How do we obtain an image from a vector graphics illustration?

Rendering

How do we obtain an image from a vector graphics illustration?

Represent where? Monitor, printer?

Rendering

How do we obtain an image from a vector graphics illustration?

Represent where? Monitor, printer?

Best according to what? Perceived difference, L_2 metric?

Rendering

How do we obtain an image from a vector graphics illustration?

Represent where? Monitor, printer?

Best according to what? Perceived difference, L_2 metric?

Typical situation is quite complex

How do we obtain an image from a vector graphics illustration?

Represent where? Monitor, printer?

Best according to what? Perceived difference, L_2 metric?

Typical situation is quite complex

Perceptual metrics are a work in progress

- s-CIELAB metric [Zhang and Wandell, 1996]
- SSIM [Wang et al., 2004]

How do we obtain an image from a vector graphics illustration?

Represent where? Monitor, printer?

Best according to what? Perceived difference, L_2 metric?

Typical situation is quite complex

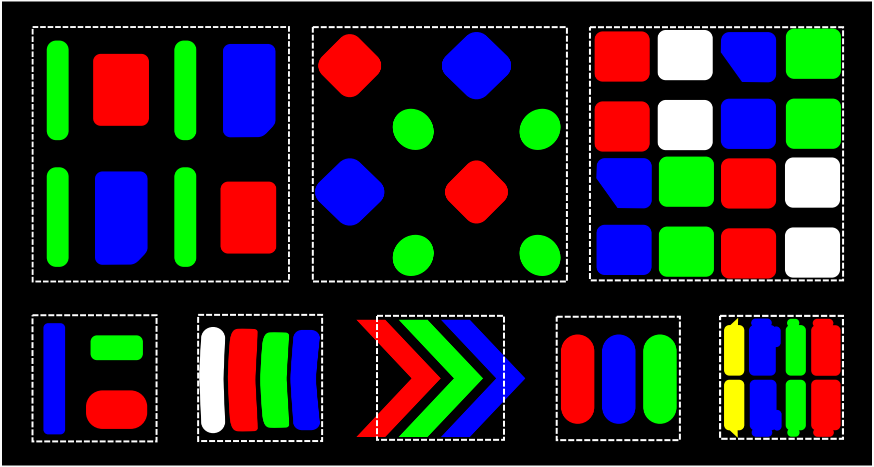
Perceptual metrics are a work in progress

- s-CIELAB metric [Zhang and Wandell, 1996]
- SSIM [Wang et al., 2004]

Monitors can be very different from one another

- Different subpixel layouts
- Different subpixel spectral properties

Different monitors



The general case

Define an approximation space

- Functions you can represent

The general case

Define an approximation space

- Functions you can represent

Define a metric

- A way to compare original function with approximation

The general case

Define an approximation space

- Functions you can represent

Define a metric

- A way to compare original function with approximation

Discretization becomes an optimization problem

The general case

Define an approximation space

- Functions you can represent

Define a metric

- A way to compare original function with approximation

Discretization becomes an optimization problem

Approximation errors can be coherent in a disturbing way

- Jagged edges (your renderer)

The general case

Define an approximation space

- Functions you can represent

Define a metric

- A way to compare original function with approximation

Discretization becomes an optimization problem

Approximation errors can be coherent in a disturbing way

- Jagged edges (your renderer)
- Weird artefacts

The general case

Define an approximation space

- Functions you can represent

Define a metric

- A way to compare original function with approximation

Discretization becomes an optimization problem

Approximation errors can be coherent in a disturbing way

- Jagged edges (your renderer)
- Weird artefacts
- Show resolution chart, zonal plate, infinite checkerboard

The general case

Define an approximation space

- Functions you can represent

Define a metric

- A way to compare original function with approximation

Discretization becomes an optimization problem

Approximation errors can be coherent in a disturbing way

- Jagged edges (your renderer)
- Weird artefacts
- Show resolution chart, zonal plate, infinite checkerboard
- This is what we call *aliasing*

The general case

Define an approximation space

- Functions you can represent

Define a metric

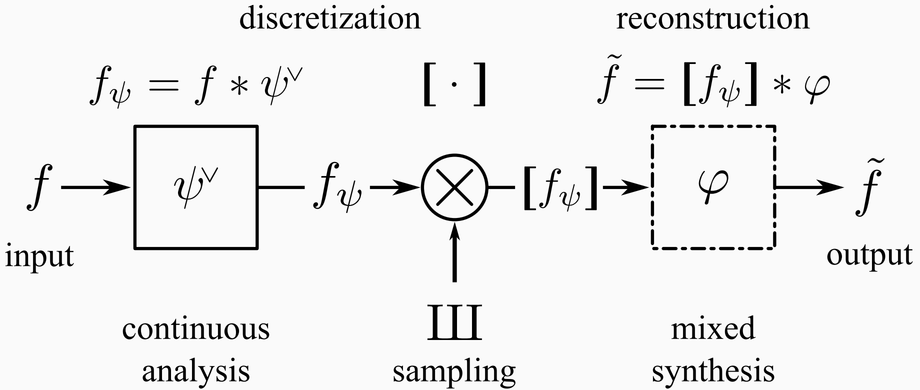
- A way to compare original function with approximation

Discretization becomes an optimization problem

Approximation errors can be coherent in a disturbing way

- Jagged edges (your renderer)
- Weird artefacts
- Show resolution chart, zonal plate, infinite checkerboard
- This is what we call *aliasing*

General case is too difficult to analyse. So we simplify


discretization
reconstruction
$f_\psi = f * \psi^\vee$
$[\,\cdot\,]$
$\tilde{f} = [f_\psi] * \varphi$
f
$\psi^\vee$
f_ψ
$[f_\psi]$
φ
$\tilde{f}$
input
output
Ш
continuous
analysis
sampling
mixed
synthesis

Let $U = \mathbf{R} \to \mathbf{R}$ and let $f, g \in U$

Let $U = \mathbf{R} \to \mathbf{R}$ and let $f, g \in U$

$L : U \to U$ is *linear* if

$$L\{\alpha f + g\} = \alpha L\{f\} + L\{g\}$$

Let $U = \mathbf{R} \to \mathbf{R}$ and let $f, g \in U$

$L : U \to U$ is *linear* if

$$L\{\alpha f + g\} = \alpha L\{f\} + L\{g\}$$

$S_\alpha : U \to U$ is a *shift* if

$$S_\alpha\{f\}(t) = f(t - \alpha)$$

Let $U = \mathbf{R} \to \mathbf{R}$ and let $f, g \in U$

$L : U \to U$ is *linear* if

$$L\{\alpha f + g\} = \alpha L\{f\} + L\{g\}$$

$S_\alpha : U \to U$ is a *shift* if

$$S_\alpha\{f\}(t) = f(t - \alpha)$$

$L : U \to U$ is *shift-invariant* if

$$L\big\{S_\alpha\{f\}\big\} = S_\alpha\big\{L\{f\}\big\}$$

The unit impulse

Also known as Dirac's delta "function"

Also known as Dirac's delta "function"

Defined by the *sifting property* or the *sampling property*

$$f(t) = \int_{-\infty}^{\infty} f(u)\,\delta(t-u)\,du$$

The unit impulse

Also known as Dirac's delta "function"

Defined by the *sifting property* or the *sampling property*

$$f(t) = \int_{-\infty}^{\infty} f(u)\,\delta(t-u)\,du$$

The set of shifted deltas looks like some kind of "basis"

Also known as Dirac's delta "function"

Defined by the *sifting property* or the *sampling property*

$$f(t) = \int_{-\infty}^{\infty} f(u)\,\delta(t-u)\,du$$

The set of shifted deltas looks like some kind of "basis"

Each element is perfectly located in space (or time)

Any linear, time-invariant operator L is a convolution

$$(f * g)(t) = \int_{-\infty}^{\infty} f(u)\, g(t-u)\, du$$

Any linear, time-invariant operator L is a convolution

$$(f * g)(t) = \int_{-\infty}^{\infty} f(u)\, g(t-u)\, du$$

Proof

$$L\{f\} = L\{f * \delta\}$$

Any linear, time-invariant operator L is a convolution

$$(f * g)(t) = \int_{-\infty}^{\infty} f(u)\, g(t-u)\, du$$

Proof

$$\begin{aligned} L\{f\} &= L\{f * \delta\} \\ &= L\left\{\int_{-\infty}^{\infty} f(u)\, \delta(\cdot - u)\, du\right\} \end{aligned}$$

Any linear, time-invariant operator L is a convolution

$$(f * g)(t) = \int_{-\infty}^{\infty} f(u)\, g(t-u)\, du$$

Proof

$$\begin{aligned} L\{f\} &= L\{f * \delta\} \\ &= L\left\{ \int_{-\infty}^{\infty} f(u)\, \delta(\cdot - u)\, du \right\} \\ &= \int_{-\infty}^{\infty} f(u)\, L\big\{\delta(\cdot - u)\big\}\, du \end{aligned}$$

Any linear, time-invariant operator L is a convolution

$$(f * g)(t) = \int_{-\infty}^{\infty} f(u)\, g(t-u)\, du$$

Proof

$$\begin{aligned} L\{f\} &= L\{f * \delta\} \\ &= L\left\{\int_{-\infty}^{\infty} f(u)\, \delta(\cdot - u)\, du\right\} \\ &= \int_{-\infty}^{\infty} f(u)\, L\big\{\delta(\cdot - u)\big\}\, du \\ &= \int_{-\infty}^{\infty} f(u)\, L\{\delta\}(\cdot - u)\, du \end{aligned}$$

Any linear, time-invariant operator L is a convolution

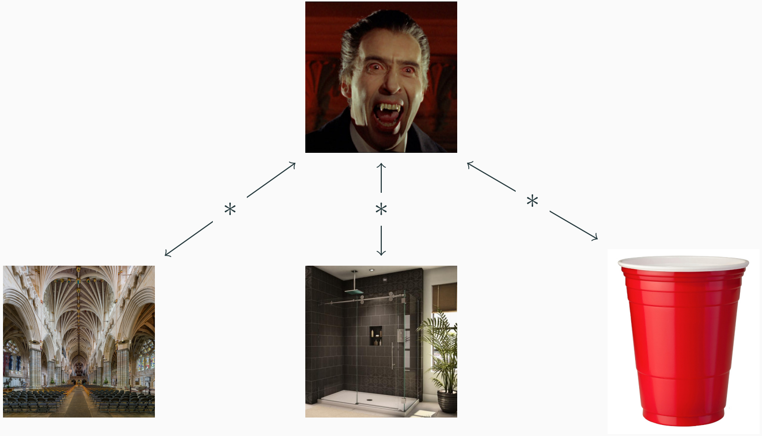
$$(f * g)(t) = \int_{-\infty}^{\infty} f(u)\, g(t-u)\, du$$

Proof

$$\begin{aligned} L\{f\} &= L\{f * \delta\} \\ &= L\left\{\int_{-\infty}^{\infty} f(u)\, \delta(\cdot - u)\, du\right\} \\ &= \int_{-\infty}^{\infty} f(u)\, L\big\{\delta(\cdot - u)\big\}\, du \\ &= \int_{-\infty}^{\infty} f(u)\, L\{\delta\}(\cdot - u)\, du \\ &= f * L\{\delta\} \end{aligned}$$

Examples

Linear shift-invariant systems model many physical phenomena



All linear shift-invariant systems can be simultaneously diagonalized

All linear shift-invariant systems can be simultaneously diagonalized

Complex exponentials are the common “basis”

$$f(t) = \int_{-\infty}^{\infty} F(\omega)\, e^{2\pi i \omega t}\, d\omega$$

Fourier transform

All linear shift-invariant systems can be simultaneously diagonalized

Complex exponentials are the common “basis”

$$f(t) = \int_{-\infty}^{\infty} F(\omega)\, e^{2\pi i\omega t}\, d\omega$$

Each element is perfectly located in frequency

All linear shift-invariant systems can be simultaneously diagonalized

Complex exponentials are the common "basis"

$$f(t) = \int_{-\infty}^{\infty} F(\omega)\, e^{2\pi i \omega t}\, d\omega$$

Each element is perfectly located in frequency

Fourier transform is just a "change of basis"

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i \omega t}\, dt$$

All linear shift-invariant systems can be simultaneously diagonalized

Complex exponentials are the common “basis”

$$f(t) = \int_{-\infty}^{\infty} F(\omega)\, e^{2\pi i\omega t}\, d\omega$$

Each element is perfectly located in frequency

Fourier transform is just a “change of basis”

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i\omega t}\, dt$$

Works because, in the sense of distributions,

$$\delta(t) = \int_{-\infty}^{\infty} e^{2\pi i\omega t}\, d\omega$$

Interesting pairs and properties

$$\delta \stackrel{\mathcal{F}}{\longleftrightarrow} 1$$

Interesting pairs and properties

$$\delta \xleftrightarrow{\mathcal{F}} 1$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}} \xleftrightarrow{\mathcal{F}} e^{-2\pi^2\sigma^2\omega^2}$$

Interesting pairs and properties

$$\delta \xleftrightarrow{\mathcal{F}} 1$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}} \xleftrightarrow{\mathcal{F}} e^{-2\pi^2\sigma^2\omega^2}$$

$$\text{box} \xleftrightarrow{\mathcal{F}} \text{sinc}$$

$$\delta \xleftrightarrow{\mathcal{F}} 1$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}} \xleftrightarrow{\mathcal{F}} e^{-2\pi^2\sigma^2\omega^2}$$

$$\text{box} \xleftrightarrow{\mathcal{F}} \text{sinc}$$

$$Ш \xleftrightarrow{\mathcal{F}} Ш$$

$$\delta \xleftrightarrow{\mathcal{F}} 1$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}} \xleftrightarrow{\mathcal{F}} e^{-2\pi^2\sigma^2\omega^2}$$

$$\text{box} \xleftrightarrow{\mathcal{F}} \text{sinc}$$

$$Ш \xleftrightarrow{\mathcal{F}} Ш$$

$$f(at) \xleftrightarrow{\mathcal{F}} \frac{1}{a}F(\frac{\omega}{a})$$

Interesting pairs and properties

$$\delta \xleftrightarrow{\mathcal{F}} 1$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}} \xleftrightarrow{\mathcal{F}} e^{-2\pi^2\sigma^2\omega^2}$$

$$\text{box} \xleftrightarrow{\mathcal{F}} \text{sinc}$$

$$\text{III} \xleftrightarrow{\mathcal{F}} \text{III}$$

$$f(at) \xleftrightarrow{\mathcal{F}} \frac{1}{a} F\left(\frac{\omega}{a}\right)$$

$$f(t-a) \xleftrightarrow{\mathcal{F}} e^{-2\pi i \omega a} F(\omega)$$

Interesting pairs and properties

$$\delta \overset{\mathcal{F}}{\longleftrightarrow} 1$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}} \overset{\mathcal{F}}{\longleftrightarrow} e^{-2\pi^2\sigma^2\omega^2}$$

$$\text{box} \overset{\mathcal{F}}{\longleftrightarrow} \text{sinc}$$

$$\text{III} \overset{\mathcal{F}}{\longleftrightarrow} \text{III}$$

$$f(at) \overset{\mathcal{F}}{\longleftrightarrow} \tfrac{1}{a} F(\tfrac{\omega}{a})$$

$$f(t-a) \overset{\mathcal{F}}{\longleftrightarrow} e^{-2\pi i \omega a} F(\omega)$$

$$f * g \overset{\mathcal{F}}{\longleftrightarrow} F\,G \qquad \text{(convolution theorem)}$$

Let $f_k = f(k), k \in \mathbb{Z}$ be a sampling of f

Let $f_k = f(k), k \in \mathbb{Z}$ be a sampling of f

Use f_k in the Fourier series expansion of some periodic function $\tilde{F}(\omega)$

$$\tilde{F}(\omega) = \sum_{k=-\infty}^{\infty} f_k e^{2\pi i \omega k}$$

Let $f_k = f(k), k \in \mathbb{Z}$ be a sampling of f

Use f_k in the Fourier series expansion of some periodic function $\tilde{F}(\omega)$

$$\begin{aligned}\tilde{F}(\omega) &= \sum_{k=-\infty}^{\infty} f_k \, e^{2\pi i \omega k} \\ &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \, e^{-2\pi i \omega t} \, \delta(t-k) \, dt\end{aligned}$$

Let $f_k = f(k), k \in \mathbb{Z}$ be a sampling of f

Use f_k in the Fourier series expansion of some periodic function $\tilde{F}(\omega)$

$$\begin{aligned}\tilde{F}(\omega) &= \sum_{k=-\infty}^{\infty} f_k\, e^{2\pi i\omega k} \\ &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i\omega t}\, \delta(t-k)\, dt \\ &= \int_{-\infty}^{\infty} e^{-2\pi i\omega t} f(t) \sum_{k=-\infty}^{\infty} \delta(t-k)\, dt\end{aligned}$$

Let $f_k = f(k), k \in \mathbb{Z}$ be a sampling of f

Use f_k in the Fourier series expansion of some periodic function $\tilde{F}(\omega)$

$$\begin{aligned}
\tilde{F}(\omega) &= \sum_{k=-\infty}^{\infty} f_k \, e^{2\pi i \omega k} \\
&= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \, e^{-2\pi i \omega t} \, \delta(t-k) \, dt \\
&= \int_{-\infty}^{\infty} e^{-2\pi i \omega t} f(t) \sum_{k=-\infty}^{\infty} \delta(t-k) \, dt \quad = \int_{-\infty}^{\infty} e^{-2\pi i \omega t} \, (f \cdot \text{Ш})(t) \, dt
\end{aligned}$$

Let $f_k = f(k), k \in \mathbb{Z}$ be a sampling of f

Use f_k in the Fourier series expansion of some periodic function $\tilde{F}(\omega)$

$$\begin{aligned}
\tilde{F}(\omega) &= \sum_{k=-\infty}^{\infty} f_k\, e^{2\pi i \omega k} \\
&= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i \omega t}\, \delta(t-k)\, dt \\
&= \int_{-\infty}^{\infty} e^{-2\pi i \omega t} f(t) \sum_{k=-\infty}^{\infty} \delta(t-k)\, dt \quad = \int_{-\infty}^{\infty} e^{-2\pi i \omega t}\, (f \cdot Ш)(t)\, dt
\end{aligned}$$

So we see that $\tilde{F}(\omega)$ is the Fourier transform of $f \cdot Ш$.

Let $f_k = f(k), k \in \mathbf{Z}$ be a sampling of f

Use f_k in the Fourier series expansion of some periodic function $\tilde{F}(\omega)$

$$\begin{aligned}\tilde{F}(\omega) &= \sum_{k=-\infty}^{\infty} f_k\, e^{2\pi i\omega k} \\ &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i\omega t}\, \delta(t-k)\, dt \\ &= \int_{-\infty}^{\infty} e^{-2\pi i\omega t} f(t) \sum_{k=-\infty}^{\infty} \delta(t-k)\, dt \quad = \int_{-\infty}^{\infty} e^{-2\pi i\omega t}\, (f \cdot \text{Ш})(t)\, dt\end{aligned}$$

So we see that $\tilde{F}(\omega)$ is the Fourier transform of $f \cdot \text{Ш}$.

This associates the sequence f_k with the function $f \cdot \text{Ш}$.

Using the convolution theorem, we get

$$f \cdot \text{Ш} \overset{\mathcal{F}}{\longleftrightarrow} F * \text{Ш}$$

Using the convolution theorem, we get

$$f \cdot Ш \overset{\mathcal{F}}{\longleftrightarrow} F * Ш$$

Now recall that

$$\text{box} \overset{\mathcal{F}}{\longleftrightarrow} \text{sinc}$$

Shannon-Whittaker-Nyquist-Kotelnikov theorem

Using the convolution theorem, we get

$$f \cdot \text{Ш} \xleftrightarrow{\mathcal{F}} F * \text{Ш}$$

Now recall that

$$\mathsf{box} \xleftrightarrow{\mathcal{F}} \mathsf{sinc}$$

If F is supported in $(-\frac{1}{2}, \frac{1}{2})$, then

$$(F * \text{Ш}) \cdot \mathsf{box} = F$$

Using the convolution theorem, we get

$$f \cdot Ш \overset{\mathcal{F}}{\longleftrightarrow} F * Ш$$

Now recall that

$$\mathsf{box} \overset{\mathcal{F}}{\longleftrightarrow} \mathsf{sinc}$$

If F is supported in $(-\frac{1}{2}, \frac{1}{2})$, then

$$(F * Ш) \cdot \mathsf{box} = F$$

And so,

$$f = (f \cdot Ш) * \mathsf{sinc}$$

Shannon-Whittaker-Nyquist-Kotelnikov theorem

Using the convolution theorem, we get

$$f \cdot Ш \stackrel{\mathcal{F}}{\longleftrightarrow} F * Ш$$

Now recall that

$$\mathsf{box} \stackrel{\mathcal{F}}{\longleftrightarrow} \mathsf{sinc}$$

If F is supported in $(-\frac{1}{2}, \frac{1}{2})$, then

$$(F * Ш) \cdot \mathsf{box} = F$$

And so,

$$f = (f \cdot Ш) * \mathsf{sinc}$$

Show theorem graphically

Let $\varphi : \mathrm{R} \to \mathrm{R}$ be a *generating function*

Let $\varphi : \mathrm{R} \to \mathrm{R}$ be a *generating function*

Let T be a *sampling period*

Let $\varphi : \mathsf{R} \to \mathsf{R}$ be a *generating function*

Let T be a *sampling period*

The shift-invariant approximation space $V_{\varphi,T}$ is

$$V_{\varphi,T} = \left\{ \tilde{f} : \mathsf{R} \to \mathsf{R} \,\middle|\, \tilde{f}(t) = \sum_{i=-\infty}^{\infty} c_i\, \varphi(t - iT),\ c_i \in \mathsf{R},\ i \in \mathsf{Z} \right\}$$

Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a *generating function*

Let T be a *sampling period*

The shift-invariant approximation space $V_{\varphi,T}$ is

$$V_{\varphi,T} = \left\{ \tilde{f} : \mathbb{R} \rightarrow \mathbb{R} \,\middle|\, \tilde{f}(t) = \sum_{i=-\infty}^{\infty} c_i \, \varphi(t - iT), \; c_i \in \mathbb{R}, \; i \in \mathbb{Z} \right\}$$

Reconstruction with φ transforms c_i into $\tilde{f}$

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a *generating function*

Let T be a *sampling period*

The shift-invariant approximation space $V_{\varphi,T}$ is

$$V_{\varphi,T} = \left\{ \tilde{f} : \mathbb{R} \to \mathbb{R} \,\middle|\, \tilde{f}(t) = \sum_{i=-\infty}^{\infty} c_i \, \varphi(t - iT),\ c_i \in \mathbb{R},\ i \in \mathbb{Z} \right\}$$

Reconstruction with φ transforms c_i into $\tilde{f}$

Typically, c_i are obtained from some f being approximated

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a *generating function*

Let T be a *sampling period*

The shift-invariant approximation space $V_{\varphi,T}$ is

$$V_{\varphi,T} = \left\{ \tilde{f} : \mathbb{R} \to \mathbb{R} \;\middle|\; \tilde{f}(t) = \sum_{i=-\infty}^{\infty} c_i\, \varphi(t - iT),\ c_i \in \mathbb{R},\ i \in \mathbb{Z} \right\}$$

Reconstruction with φ transforms c_i into $\tilde{f}$

Typically, c_i are obtained from some f being approximated

Process uses a *prefilter* ψ

$$c_i = \int_{-\infty}^{\infty} f(t)\, \psi(t - iT)\, dt$$

Let $\varphi : \mathrm{R} \to \mathrm{R}$ be a *generating function*

Let T be a *sampling period*

The shift-invariant approximation space $V_{\varphi,T}$ is

$$V_{\varphi,T} = \left\{ \tilde{f} : \mathrm{R} \to \mathrm{R} \,\middle|\, \tilde{f}(t) = \sum_{i=-\infty}^{\infty} c_i \, \varphi(t - iT),\ c_i \in \mathrm{R},\ i \in \mathrm{Z} \right\}$$

Reconstruction with φ transforms c_i into $\tilde{f}$

Typically, c_i are obtained from some f being approximated

Process uses a *prefilter* ψ

$$c_i = \int_{-\infty}^{\infty} f(t)\, \psi(t - iT)\, dt$$

Sampling is the special case where $\psi = \delta$

Examples

Another case study: $\varphi = \text{sinc}$, L_2, $T = 1$

- Prove that optimal prefilter is sinc
- This is the “ideal sampling”

Another case study: $\varphi = \textsf{sinc}$, L_2, $T = 1$

- Prove that optimal prefilter is sinc
- This is the "ideal sampling"

A simple case study: $\varphi = \textsf{box}$, L_2, $T = 1$

- Prove that optimal prefilter is box

Another case study: $\varphi = \textsf{sinc}$, L_2, $T = 1$

- Prove that optimal prefilter is sinc
- This is the "ideal sampling"

A simple case study: $\varphi = \textsf{box}$, L_2, $T = 1$

- Prove that optimal prefilter is box

Simple because shifted generating functions are *orthogonal*

- What happens with the non-orthogonal case?

Problems

Big problem: L_2 metric is not perceptual

Problems

Big problem: L_2 metric is not perceptual

There are no practical bandlimited functions

Problems

Big problem: L_2 metric is not perceptual

There are no practical bandlimited functions

Bandlimited approximations are not perceptual close to original

Big problem: L_2 metric is not perceptual

There are no practical bandlimited functions

Bandlimited approximations are not perceptual close to original

Show ringing

Big problem: L_2 metric is not perceptual

There are no practical bandlimited functions

Bandlimited approximations are not perceptual close to original

Show ringing

Common practice is to reconstruct with hat, prefilter with box

Problems

Big problem: L_2 metric is not perceptual

There are no practical bandlimited functions

Bandlimited approximations are not perceptual close to original

Show ringing

Common practice is to reconstruct with **hat**, prefilter with **box**

Common practice is not very good…

Big problem: L_2 metric is not perceptual

There are no practical bandlimited functions

Bandlimited approximations are not perceptual close to original

Show ringing

Common practice is to reconstruct with hat, prefilter with box

Common practice is not very good…

Show comparisons

References

A. R. Smith. A pixel is not a little square, a pixel is not a little square, a pixel is not a little square! (and a voxel is not a little cube). Technical Report 6, Microsoft Research, 1995.

Z. Wang, A. Bovik, H. Sheikh, and E. Simoncelli. Image quality assessment: From error visibility to structural similarity. *IEEE Transactions on Image Processing*, 13(4):600–612, 2004.

X. Zhang and B. A. Wandell. A spatial extension to CIELAB for digital color image reproduction. In *Society for Information Display Symposium Technical Digest*, volume 27, pages 731–734, 1996.