2D Computer Graphics

Diego Nehab

Summer 2020

IMPA

Resultants and implicitization

Why we need resultants

Two types of renderers and their applications

- Traditional vs. vector textures

Why we need resultants

Two types of renderers and their applications

- Traditional vs. vector textures
- Amortized vs. random access

Why we need resultants

Two types of renderers and their applications

- Traditional vs. vector textures
- Amortized vs. random access
- In both cases, parallelism is key

Why we need resultants

Two types of renderers and their applications

- Traditional vs. vector textures
- Amortized vs. random access
- In both cases, parallelism is key

Amortized renderers need actual point of intersection

Why we need resultants

Two types of renderers and their applications

- Traditional vs. vector textures
- Amortized vs. random access
- In both cases, parallelism is key

Amortized renderers need actual point of intersection

Random access only need to count them

Why we need resultants

Two types of renderers and their applications

- Traditional vs. vector textures
- Amortized vs. random access
- In both cases, parallelism is key

Amortized renderers need actual point of intersection

Random access only need to count them

Can count using implicit tests

Why we need resultants

Two types of renderers and their applications

- Traditional vs. vector textures
- Amortized vs. random access
- In both cases, parallelism is key

Amortized renderers need actual point of intersection

Random access only need to count them

Can count using implicit tests

For that, we will use resultants

Implicit vs. explicit

We say that a bivariate polynomial $\Gamma(u, v)$ is the implicit form of a parametric polynomial curve $\gamma(t) = \big(x(t), y(t)\big)$ if

$$\Gamma(p) = 0 \quad \Longleftrightarrow \quad \exists t \ \mid \ p = \gamma(t)$$

Implicit vs. explicit

We say that a bivariate polynomial $\Gamma(u, v)$ is the implicit form of a parametric polynomial curve $\gamma(t) = \big(x(t), y(t)\big)$ if

$$\Gamma(p) = 0 \quad \Longleftrightarrow \quad \exists t \ \mid \ p = \gamma(t)$$

Given expressions for x and y, how do we obtain an expression for Γ?

We say that a bivariate polynomial $\Gamma(u, v)$ is the implicit form of a parametric polynomial curve $\gamma(t) = \big(x(t), y(t)\big)$ if

$$\Gamma(p) = 0 \quad \iff \quad \exists t \;\mid\; p = \gamma(t)$$

Given expressions for x and y, how do we obtain an expression for Γ?

The condition $p = (x_p, y_p) = \big(x(t), y(t)\big) = \gamma(t)$ can be rewritten as

$$\begin{cases} f_p(t) = x(t) - x_p = 0 \\ g_p(t) = y(t) - y_p = 0 \end{cases}$$

Implicit vs. explicit

We say that a bivariate polynomial $\Gamma(u, v)$ is the implicit form of a parametric polynomial curve $\gamma(t) = \big(x(t), y(t)\big)$ if

$$\Gamma(p) = 0 \quad \Longleftrightarrow \quad \exists t \;|\; p = \gamma(t)$$

Given expressions for x and y, how do we obtain an expression for Γ?

The condition $p = (x_p, y_p) = \big(x(t), y(t)\big) = \gamma(t)$ can be rewritten as

$$\begin{cases} f_p(t) = x(t) - x_p = 0 \\ g_p(t) = y(t) - y_p = 0 \end{cases}$$

Polynomials f_p and g_p have a *common* root at t.

We say that a bivariate polynomial $\Gamma(u, v)$ is the implicit form of a parametric polynomial curve $\gamma(t) = \big(x(t), y(t)\big)$ if

$$\Gamma(p) = 0 \quad \iff \quad \exists t \ | \ p = \gamma(t)$$

Given expressions for x and y, how do we obtain an expression for Γ?

The condition $p = (x_p, y_p) = \big(x(t), y(t)\big) = \gamma(t)$ can be rewritten as

$$\begin{cases} f_p(t) = x(t) - x_p = 0 \\ g_p(t) = y(t) - y_p = 0 \end{cases}$$

Polynomials f_p and g_p have a *common* root at t.

We need a bivariate polynomial $\Gamma(p)$ that vanishes if and only if two one-variable polynomials f_p and y_p have a common root.

The resultant

If we knew the roots of $a_1, a_2, \ldots, a_r$ of f_p and $b_1, b_2, \ldots, b_s$ of g_p, which depend on p, of course, we could write

$$\mathrm{R}(f_p, g_p) = \prod_{i=1}^{r} \prod_{j=i}^{s} (a_i - b_j)$$

The resultant

If we knew the roots of $a_1, a_2, \ldots, a_r$ of f_p and $b_1, b_2, \ldots, b_s$ of g_p, which depend on p, of course, we could write

$$\mathrm{R}(f_p, g_p) = \prod_{i=1}^{r} \prod_{j=i}^{s} (a_i - b_j)$$

We call $\mathrm{R}(f_p, g_p)$ the *resultant* of f_p, g_p

The resultant

If we knew the roots of $a_1, a_2, \ldots, a_r$ of f_p and $b_1, b_2, \ldots, b_s$ of g_p, which depend on p, of course, we could write

$$\mathrm{R}(f_p, g_p) = \prod_{i=1}^{r} \prod_{j=i}^{s} (a_i - b_j)$$

We call $\mathrm{R}(f_p, g_p)$ the *resultant* of f_p, g_p

Is there an expression for the resultant that does *not* require knowledge of the roots of f_p and g_p?

If we knew the roots of $a_1, a_2, \dots, a_r$ of f_p and $b_1, b_2, \dots, b_s$ of g_p, which depend on p, of course, we could write

$$\mathsf{R}(f_p, g_p) = \prod_{i=1}^{r} \prod_{j=i}^{s} (a_i - b_j)$$

We call $\mathsf{R}(f_p, g_p)$ the *resultant* of f_p, g_p

Is there an expression for the resultant that does *not* require knowledge of the roots of f_p and g_p?

It makes sense that there should be! Think about the Vieta formulas for sums of products of roots!

The Sylvester form for the resultant

Let f and g have a common root and let

$$\deg(f) = m \quad \text{and} \quad \deg(g) = n$$

Let f and g have a common root and let

$$\deg(f) = m \quad \text{and} \quad \deg(g) = n$$

There is h with $\deg(h) = 1$ such that

$$f(t) = h(t)r(t) \quad \text{and} \quad g(t) = h(t)s(t)$$

The Sylvester form for the resultant

Let f and g have a common root and let

$$\deg(f) = m \quad \text{and} \quad \deg(g) = n$$

There is h with $\deg(h) = 1$ such that

$$f(t) = h(t)r(t) \quad \text{and} \quad g(t) = h(t)s(t)$$

We can eliminate h from the equations by noticing that

$$f(t)s(t) = h(t)r(t)s(t) = g(t)r(t)$$

Let f and g have a common root and let

$$\deg(f) = m \quad \text{and} \quad \deg(g) = n$$

There is h with $\deg(h) = 1$ such that

$$f(t) = h(t)r(t) \quad \text{and} \quad g(t) = h(t)s(t)$$

We can eliminate h from the equations by noticing that

$$f(t)s(t) = h(t)r(t)s(t) = g(t)r(t)$$

These polynomials are *identical*, so all coefficients must be the same

$$f(t)s(t) = g(t)r(t)$$

The Sylvester form for the resultant

These polynomials are *identical*, so all coefficients must be the same

$$f(t)s(t) = g(t)r(t)$$

The Sylvester form for the resultant

These polynomials are *identical*, so all coefficients must be the same

$$f(t)s(t) = g(t)r(t)$$

We do not know the value of t, so we don't know the coefficients of $(r_0, r_1, \ldots, r_{m-1})$ of r and $(s_0, s_1, \ldots, s_{n-1})$ of s, but we know the coefficients f_i, g_j of f and g

These polynomials are *identical*, so all coefficients must be the same

$$f(t)s(t) = g(t)r(t)$$

We do not know the value of t, so we don't know the coefficients of $(r_0, r_1, \ldots, r_{m-1})$ of r and $(s_0, s_1, \ldots, s_{n-1})$ of s, but we know the coefficients f_i, g_j of f and g

The coefficient equations are

$$f_0 s_0 = g_0 r_0$$

$$f_1 s_0 + f_0 s_1 = g_1 r_0 + g_0 r_1$$

$$f_2 s_0 + f_1 s_1 + f_0 s_2 = g_2 r_0 + g_1 r_1 + g_0 r_2$$

$$\vdots$$

$$f_m s_{n-1} = g_n r_{m-1}$$

In matrix form

$$\begin{bmatrix} f_0 & & & & g_0 & & \\ \vdots & f_0 & & & g_1 & \ddots & \\ f_m & \vdots & \ddots & & \vdots & \ddots & g_0 \\ & f_m & & f_0 & g_n & & g_1 \\ & & \ddots & \vdots & & \ddots & \vdots \\ & & & f_m & & & g_n \end{bmatrix} \begin{bmatrix} s_0 \\ \vdots \\ s_{n-1} \\ -r_0 \\ \vdots \\ -r_{m-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

The Sylvester form for the resultant

In matrix form

$$\begin{bmatrix} f_0 & & & & g_0 & & \\ \vdots & f_0 & & & g_1 & \ddots & \\ f_m & \vdots & \ddots & & \vdots & \ddots & g_0 \\ & f_m & & f_0 & g_n & & g_1 \\ & & \ddots & \vdots & & \ddots & \vdots \\ & & & f_m & & & g_n \end{bmatrix} \begin{bmatrix} s_0 \\ \vdots \\ s_{n-1} \\ -r_0 \\ \vdots \\ -r_{m-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

The polynomials have a common root *iff* the linear system has a non-trivial solution

In matrix form

$$\begin{bmatrix} f_0 & & & & g_0 & & \\ \vdots & f_0 & & & g_1 & \ddots & \\ f_m & \vdots & \ddots & & \vdots & \ddots & g_0 \\ & f_m & & f_0 & g_n & & g_1 \\ & & \ddots & \vdots & & \ddots & \vdots \\ & & & f_m & & & g_n \end{bmatrix} \begin{bmatrix} s_0 \\ \vdots \\ s_{n-1} \\ -r_0 \\ \vdots \\ -r_{m-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

The polynomials have a common root *iff* the linear system has a non-trivial solution

The resultant is the determinant of this $(m+n) \times (m+n)$ matrix

The Cayley-Bezout form for the resultant

Consider the bivariate polynomial

$$p(s,t) = f(s)g(t) - f(t)g(s)$$

The Cayley-Bezout form for the resultant

Consider the bivariate polynomial

$$p(s,t) = f(s)g(t) - f(t)g(s)$$

It clearly has a root at $t = s$

The Cayley-Bezout form for the resultant

Consider the bivariate polynomial

$$p(s,t) = f(s)g(t) - f(t)g(s)$$

It clearly has a root at $t = s$

So we can factor it out and think about $r(s,t)$, where

$$r(s,t)(s-t) = p(s,t)$$

The Cayley-Bezout form for the resultant

Consider the bivariate polynomial

$$p(s,t) = f(s)g(t) - f(t)g(s)$$

It clearly has a root at $t = s$

So we can factor it out and think about $r(s,t)$, where

$$r(s,t)(s-t) = p(s,t)$$

If f, g have a common root at t, then $p(s,t)$ vanishes identically

The Cayley-Bezout form for the resultant

Consider the bivariate polynomial

$$p(s,t) = f(s)g(t) - f(t)g(s)$$

It clearly has a root at $t = s$

So we can factor it out and think about $r(s,t)$, where

$$r(s,t)(s-t) = p(s,t)$$

If f, g have a common root at t, then $p(s,t)$ vanishes identically

Therefore, so does $r(s,t)$

$p(s,t)$ is anti-symmetrical in s,t, but $r(s,t)$ is symmetrical

The Cayley-Bezout form for the resultant

$p(s,t)$ is anti-symmetrical in s,t, but $r(s,t)$ is symmetrical

Let $k = \max\left(\deg(f), \deg(g)\right)$, then

$$r(s,t) = \begin{bmatrix} 1 & \cdots & s^k \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ t^k \end{bmatrix}$$

The Cayley-Bezout form for the resultant

$p(s,t)$ is anti-symmetrical in s,t, but $r(s,t)$ is symmetrical

Let $k = \max\big(\deg(f), \deg(g)\big)$, then

$$r(s,t) = \begin{bmatrix} 1 & \cdots & s^k \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ t^k \end{bmatrix}$$

The resultant is the determinant of this $k \times k$ matrix

The Cayley-Bezout form for the resultant

$p(s,t)$ is anti-symmetrical in s,t, but $r(s,t)$ is symmetrical

Let $k = \max\left(\deg(f), \deg(g)\right)$, then

$$r(s,t) = \begin{bmatrix} 1 & \cdots & s^k \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ t^k \end{bmatrix}$$

The resultant is the determinant of this $k \times k$ matrix

Its rank-deficiency is the number of common roots in f, g

The Cayley-Bezout form for the resultant

$p(s,t)$ is anti-symmetrical in s, t, but $r(s,t)$ is symmetrical

Let $k = \max\big(\deg(f), \deg(g)\big)$, then

$$r(s,t) = \begin{bmatrix} 1 & \cdots & s^k \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ t^k \end{bmatrix}$$

The resultant is the determinant of this $k \times k$ matrix

Its rank-deficiency is the number of common roots in f, g

The smaller matrices lead to smaller expressions for the resultant

$p(s,t)$ is anti-symmetrical in s,t, but $r(s,t)$ is symmetrical

Let $k = \max\left(\deg(f), \deg(g)\right)$, then

$$r(s,t) = \begin{bmatrix} 1 & \cdots & s^k \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ t^k \end{bmatrix}$$

The resultant is the determinant of this $k \times k$ matrix

Its rank-deficiency is the number of common roots in f, g

The smaller matrices lead to smaller expressions for the resultant

A good discussion of resultants, as applied to computer graphics, can be found in [de Montaudoin and Tiller, 1984, Goldman et al., 1984]. There are even formulas for polynomials in the Bernstein basis

Implicitization results

For a quadratic parametric curve

- We already knew how to solve this problem…

For a quadratic parametric curve

- We already knew how to solve this problem...

For a rational quadratic parametric curve

- The equation changes to

$$p = (x_p, y_p) = \big(x(t)/w(t), y(t)/w(t)\big) = \gamma(t)$$

For a quadratic parametric curve

- We already knew how to solve this problem...

For a rational quadratic parametric curve

- The equation changes to

$$p = (x_p, y_p) = \big(x(t)/w(t), y(t)/w(t)\big) = \gamma(t)$$

- It can be rewritten as

$$\begin{cases} f_p(t) = x(t) - x_p w(t) = 0 \\ g_p(t) = y(t) - y_p w(t) = 0 \end{cases}$$

For a quadratic parametric curve

- We already knew how to solve this problem...

For a rational quadratic parametric curve

- The equation changes to

$$p = (x_p, y_p) = \big(x(t)/w(t), y(t)/w(t)\big) = \gamma(t)$$

- It can be rewritten as

$$\begin{cases} f_p(t) = x(t) - x_p w(t) = 0 \\ g_p(t) = y(t) - y_p w(t) = 0 \end{cases}$$

- So it reduces to the integral case

For a cubic...

A “better” way to implicitize

Idea is to adapt the coordinate system to the curve $\gamma(t)$

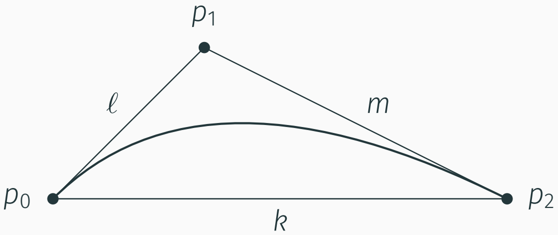
A "better" way to implicitize

Idea is to adapt the coordinate system to the curve $\gamma(t)$

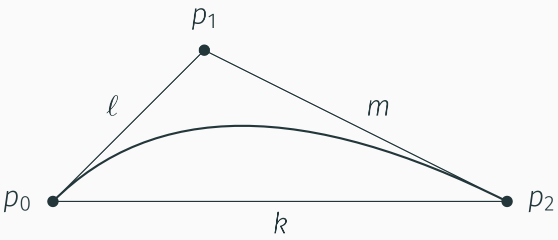
For the quadratic, consider the 3 linear functionals

$$k(x, y, w), \quad \ell(x, y, w) \quad \text{and} \quad m(x, y, w)$$

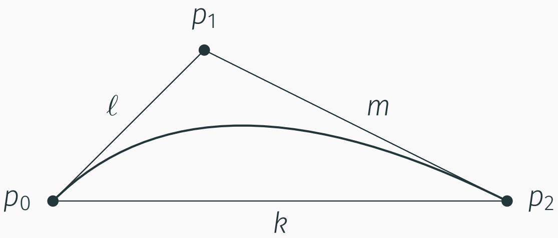
associated, respectively, to the line connecting the endpoints and the two tangents at the endpoints



Which points satisfy the quadratic equation $k^2 - \ell m = 0$?



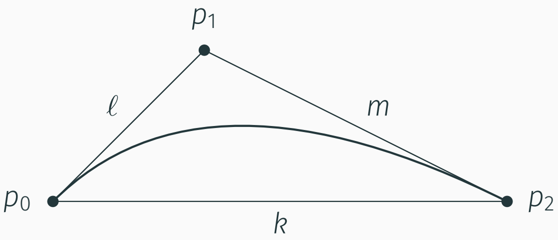
Which points satisfy the quadratic equation $k^2 - \ell m = 0$?



Intersections of $k^2 - \ell m = 0$ with $k = 0$ happen when $\ell = 0$ or $m = 0$

A "better" way to implicitize

Which points satisfy the quadratic equation $k^2 - \ell m = 0$?

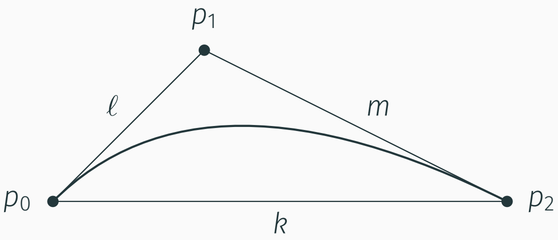


Intersections of $k^2 - \ell m = 0$ with $k = 0$ happen when $\ell = 0$ or $m = 0$

Intersection of $k^2 - \ell m = 0$ with line $\ell = 0$ happens when $k = 0$

A "better" way to implicitize

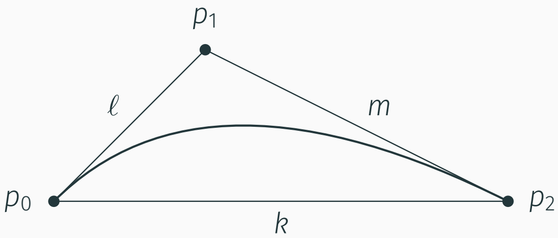
Which points satisfy the quadratic equation $k^2 - \ell m = 0$?



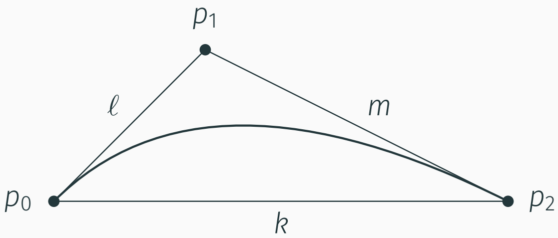
Intersections of $k^2 - \ell m = 0$ with $k = 0$ happen when $\ell = 0$ or $m = 0$

Intersection of $k^2 - \ell m = 0$ with line $\ell = 0$ happens when $k = 0$

- Furthermore, $\ell = 0$ is *tangent* to the curve at intersection



Are there any additional degrees of freedom?

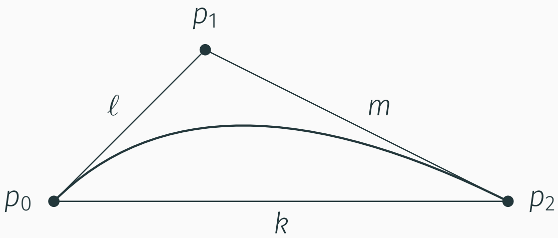


Are there any additional degrees of freedom?

To find the values of the linear functionals k, ℓ, m at control-points p_1, p_2, and p_3, consider their restriction to the curve γ

$$k\big(\gamma(t)\big) = k(p_0)\,(1-t)^2 + k(p_1)\,2t(1-t) + k(p_2)\,t^2$$
$$\ell\big(\gamma(t)\big) = \ell(p_0)\,(1-t)^2 + \ell(p_1)\,2t(1-t) + \ell(p_2)\,t^2$$
$$m\big(\gamma(t)\big) = m(p_0)\,(1-t)^2 + m(p_1)\,2t(1-t) + m(p_2)\,t^2$$

A "better" way to implicitize



Are there any additional degrees of freedom?

To find the values of the linear functionals k, ℓ, m at control-points p_1, p_2, and p_3, consider their restriction to the curve γ

$$\begin{aligned}
k\big(\gamma(t)\big) &= k(p_0)\,(1-t)^2 + k(p_1)\,2t(1-t) + k(p_2)\,t^2 &&= t\,(1-t) \\
\ell\big(\gamma(t)\big) &= \ell(p_0)\,(1-t)^2 + \ell(p_1)\,2t(1-t) + \ell(p_2)\,t^2 &&= t^2 \\
m\big(\gamma(t)\big) &= m(p_0)\,(1-t)^2 + m(p_1)\,2t(1-t) + m(p_2)\,t^2 &&= (1-t)^2
\end{aligned}$$

Values of functionals at l.h.s are coefficients in Bernstein basis

$$k(p_0)\,(1-t)^2 + k(p_1)\,2t(1-t) + k(p_2)\,t^2 = t\,(1-t)$$

$$\ell(p_0)\,(1-t)^2 + \ell(p_1)\,2t(1-t) + \ell(p_2)\,t^2 = t^2$$

$$m(p_0)\,(1-t)^2 + m(p_1)\,2t(1-t) + m(p_2)\,t^2 = (1-t)^2$$

Values of functionals at l.h.s are coefficients in Bernstein basis

$$k(p_0)\,(1-t)^2 + k(p_1)\,2t(1-t) + k(p_2)\,t^2 = t\,(1-t)$$
$$\ell(p_0)\,(1-t)^2 + \ell(p_1)\,2t(1-t) + \ell(p_2)\,t^2 = t^2$$
$$m(p_0)\,(1-t)^2 + m(p_1)\,2t(1-t) + m(p_2)\,t^2 = (1-t)^2$$

Convert polynomials on r.h.s to the Bernstein basis

$$\begin{bmatrix} k_a & k_b & k_c \\ \ell_a & \ell_b & \ell_c \\ m_a & m_b & m_c \end{bmatrix} \begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ w_0 & w_1 & w_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

A "better" way to implicitize

Values of functionals at l.h.s are coefficients in Bernstein basis

$$k(p_0)\,(1-t)^2 + k(p_1)\,2t(1-t) + k(p_2)\,t^2 = t\,(1-t)$$
$$\ell(p_0)\,(1-t)^2 + \ell(p_1)\,2t(1-t) + \ell(p_2)\,t^2 = t^2$$
$$m(p_0)\,(1-t)^2 + m(p_1)\,2t(1-t) + m(p_2)\,t^2 = (1-t)^2$$

Convert polynomials on r.h.s to the Bernstein basis

$$\begin{bmatrix} k_a & k_b & k_c \\ \ell_a & \ell_b & \ell_c \\ m_a & m_b & m_c \end{bmatrix} \begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ w_0 & w_1 & w_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Solve the linear system

Values of functionals at l.h.s are coefficients in Bernstein basis

$$k(p_0)\,(1-t)^2 + k(p_1)\,2t(1-t) + k(p_2)\,t^2 = t\,(1-t)$$
$$\ell(p_0)\,(1-t)^2 + \ell(p_1)\,2t(1-t) + \ell(p_2)\,t^2 = t^2$$
$$m(p_0)\,(1-t)^2 + m(p_1)\,2t(1-t) + m(p_2)\,t^2 = (1-t)^2$$

Convert polynomials on r.h.s to the Bernstein basis

$$\begin{bmatrix} k_a & k_b & k_c \\ \ell_a & \ell_b & \ell_c \\ m_a & m_b & m_c \end{bmatrix} \begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ w_0 & w_1 & w_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Solve the linear system

This representation is very useful in graphics hardware!

For the cubic, the same ideas apply.

For the cubic, the same ideas apply.

Unfortunately, now the linear functionals k, ℓ, m, n cannot be placed anywhere on the curve.

A "better" way to implicitize

For the cubic, the same ideas apply.

Unfortunately, now the linear functionals k, ℓ, m, n cannot be placed anywhere on the curve.

They have to be positioned at the inflection points and/or double-point.

A “better” way to implicitize

For the cubic, the same ideas apply.

Unfortunately, now the linear functionals k, ℓ, m, n cannot be placed anywhere on the curve.

They have to be positioned at the inflection points and/or double-point.

To read more about this, check [Loop and Blinn, 2005]

A "better" way to implicitize

For the cubic, the same ideas apply.

Unfortunately, now the linear functionals k, ℓ, m, n cannot be placed anywhere on the curve.

They have to be positioned at the inflection points and/or double-point.

To read more about this, check [Loop and Blinn, 2005]

Can we replace the root-finding with implicit tests? Not yet.

References

Y. de Montaudoin and W. Tiller. The Cayley method in computer aided geometric design. *Computer Aided Design*, 1(4):309–326, 1984.

R. N. Goldman, T. W. Sederberg, and D. C. Anderson. Vector elimination: A technique for the implicitazion, inversion, and intersection of planar parametric rational polynomial curves. *Computer Aided Design*, 1(4):327–356, 1984.

C. Loop and J. F. Blinn. Resolution independent curve rendering using programmable graphics hardware. *ACM Transactions on Graphics (Proceedings of ACM SIGGRAPH 2005)*, 24(3):1000–1009, 2005.