GLOBAL THEORY OF ONE-FREQUENCY SCHRÖDINGER OPERATORS I: STRATIFIED ANALYTICITY OF THE LYAPUNOV EXPONENT AND THE BOUNDARY OF NONUNIFORM HYPERBOLICITY

ARTUR AVILA

ABSTRACT. We study Schrödinger operators with a one-frequency analytic potential, focusing on the transition between the two distinct local regimes characteristic respectively of large and small potentials. From the dynamical point of view, the transition signals the emergence of nonuniform hyperbolicity, so the dependence of the Lyapunov exponent with respect to parameters plays a central role in the analysis. Though often ill-behaved by conventional measures, we show that the Lyapunov exponent is in fact remarkably regular in a "stratified sense" which we define: the irregularity comes from the matching of nice (analytic or smooth) functions along sets with complicated geometry. This result allows us to establish that the "critical set" for the transition lies within a codimension one subset in the parameter space. As a consequence, for a typical potential the set of critical energies is at most countable, hence typically not seen by spectral measures. Key to our approach are two results about the dependence of the Lyapunov exponent of one-frequency $SL(2, \mathbb{C})$ cocycles with respect to perturbations in the imaginary direction: on one hand there is a severe "quantization" restriction, and on the other hand "regularity" of the dependence characterizes uniform hyperbolicity when the Lyapunov exponent is positive. Our method is independent of arithmetic conditions on the frequency.

1. INTRODUCTION

This work is concerned with the dynamics of one-frequency $SL(2)$ cocycles, and has two distinct aspects: the analysis, from a new point of view, of the dependence of the Lyapunov exponent with respect to parameters, and the study of the "boundary" of nonuniform hyperbolicity. But our underlying motivation is to build a global theory of one-frequency Schrödinger operators with general analytic potentials, so we will start from there.

1.1. **One-frequency Schrödinger operators.** A one-dimensional quasiperiodic Schrödinger operator with one-frequency analytic potential $H = H_{\alpha,v} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is given by

$$(1) \qquad (Hu)_n = u_{n+1} + u_{n-1} + v(n\alpha)u_n,$$

where $v : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is an analytic function (the potential), and $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$ is the frequency. We denote by $\Sigma = \Sigma_{\alpha,v}$ the spectrum of H. Despite many recent

advances ([BG], [GS1], [B], [BJ1], [BJ2], [AK1], [GS2], [GS3], [AJ], [AFK], [A2])
key aspects of an authentic "global theory" of such operators have been missing.
Namely, progress has been made mainly into the understanding of the behavior
in regions of the spectrum belonging to two regimes with (at least some of the)
behavior caracteristic, respectively, of "large" and "small" potentials. But the
transition between the two regimes has been considerably harder to understand.

Until now, there has been only one case where the analysis has genuinely been
carried out at a global level. The almost Mathieu operator, $v(x) = 2\lambda \cos 2\pi(\theta+x)$,
is a highly symmetric model for which coupling strengths λ and λ^{-1} can be related
through the Fourier transform (Aubry duality). Due to this unique feature, it has
been possible to establish that the transition happens precisely at the (self-dual)
critical coupling $|\lambda| = 1$: in the subcritical regime $|\lambda| < 1$ all energies in the spec-
trum behave as for small potentials, while in the supercritical regime $|\lambda| > 1$ all
energies in the spectrum behave as for large potentials. Hence typical almost Math-
ieu operators fall entirely in one regime or the other. Related to this simple phase
transition picture, is the fundamental spectral result of [J], which implies that the
spectral measure of a typical Almost Mathieu operator has no singular continu-
ous components (it is either typically atomic for $|\lambda| > 1$ or typically absolutely
continuous for $|\lambda| < 1$).

One precise way to distinguish the subcritical and the supercritical regime for
the almost Mathieu operator is by means of the Lyapunov exponent. Recall that
for $E \in \mathbb{R}$, a formal solution $u \in \mathbb{C}^{\mathbb{Z}}$ of $Hu = Eu$ can be reconstructed from its
values at two consecutive points by application of n-step transfer matrices:

$$(2) \qquad\qquad A_n(k\alpha) \cdot \begin{pmatrix} u_k \\ u_{k-1} \end{pmatrix} = \begin{pmatrix} u_{k+n} \\ u_{k+n-1} \end{pmatrix},$$

where $A_n : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$, $n \in \mathbb{Z}$, are analytic functions defined on the same
band of analyticity of v, given in terms of $A = \begin{pmatrix} E - v & -1 \\ 1 & 0 \end{pmatrix}$ by

$$(3) \quad A_n(\cdot) = A(\cdot + (n-1)\alpha) \cdots A(\cdot), \ A_{-n}(\cdot) = A_n(\cdot - n\alpha)^{-1}, \ n \geq 1, \ A_0(\cdot) = id,$$

The Lyapunov exponent at energy E is denoted by $L(E)$ and given by

$$(4) \qquad\qquad \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{R}/\mathbb{Z}} \ln \|A_n(x)\| dx \geq 0.$$

It follows from the Aubry-André formula (proved by Bourgain-Jitomirskaya [BJ1])
that $L(E) = \max\{0, \ln |\lambda|\}$ for $E \in \Sigma_{\alpha,v}$. Thus the supercritical regime can be
distinguished by the positivity of the Lyapunov exponent: supercritical just means
nonuniformly hyperbolic in dynamical systems terminology.

How to distinguish subcritical energies from critical ones (since both have zero
Lyapunov exponent)? One way could be in terms of their stability: critical energies
are in the boundary of the supercritical regime, while subcritical ones are far away.
Another, more intrinsic way, consists of looking at the complex extensions of the
A_n: it can be shown (by a combination of [J] and [JKS]) that for subcritical energies
we have a uniform subexponential bound $\ln \|A_n(z)\| = o(n)$ through a band $|\Im z| < \delta(\lambda)$, while for critical energies this is not the case (it follows from [H]). (See also
the Appendix for a rederivation of both facts in the spirit of this paper.)

This work is not concerned with the almost Mathieu operator, whose global
theory was constructed around duality and many remarkable exact computations.

Still, what we know about it provides a powerful hint about what one can expect from the general theory. By analogy, we can always classify energies in the spectrum of an operator $H_{\alpha,v}$ as supercritical, subcritical, or critical in terms of the growth behavior of (complex extensions of) transfer matrices,[1] though differently from the almost Mathieu case the coexistence of regimes is possible [Bj2]. Beyond the "local" problems of describing precisely the behavior at the supercritical and subcritical regimes, a proper global theory should certainly explain how the "phase transition" between them occurs, and how this critical set of energies affects the spectral analysis of H.

In this direction, our main result in this paper can be stated as follows. Let $C_\delta^\omega(\mathbb{R}/\mathbb{Z},\mathbb{R})$ be the real Banach space of analytic functions $\mathbb{R}/\mathbb{Z} \to \mathbb{R}$ admitting a holomorphic extension to $|\Im z| < \delta$ which are continuous up to the boundary.

Theorem 1. *For any $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$, the set of potentials and energies (v, E) such that E is a critical energy for $H_{\alpha,v}$ is contained in a countable union of codimension-one analytic submanifolds of $C_\delta^\omega(\mathbb{R}/\mathbb{Z},\mathbb{R}) \times \mathbb{R}$.*[2]

In particular, a typical operator H will have at most countably many critical energies. In the continuation of this series [A3], this will be the starting point of the proof that the critical set is typically empty.

It was deliberately implied in the discussion above that the non-critical regimes were stable (with respect to perturbations of the energy, potential or frequency), but the critical one was not. Stability of nonuniform hyperbolicity was known (continuity of the Lyapunov exponent [BJ1]), while the stability of the subcritical regime is obtained here. The instability of the critical regime of course follows from Theorem 1. The stability of the subcritical regime implies that the critical set contains the boundary of nonuniform hyperbolicity.[3]

In the next section we will decribe our results about the dependence of the Lyapunov exponent with respect to parameters which play a key role in the proof of Theorem 1 and have otherwise independent interest. In Section 1.5, we will further comment on how our work on criticality relates to the spectral analysis of the operators, and in particular how it gives a framework to address the following:

Conjecture 1. *For a (measure-theoretically) typical operator H, the spectral measures have no singular continuous component.*

1.2. Stratified analyticity of the Lyapunov exponent.
As discussed above, the Lyapunov exponent L is fundamental in the understanding of the spectral properties of H. It is also closely connected with another important quantity, the *integrated density of states* (i.d.s.) N. As the Lyapunov exponent, the (i.d.s.) is a function of the energy: while the Lyapunov exponent measures the asymptotic average growth/decay of solutions (not necessary in ℓ^2) of the equation $Hu = Eu$, the integrated density of states gives the asymptotic distribution of eigenvalues of

---

[1]That large potentials fall into the supercritical regime then follows from [SS] and that small potentials fall into the subcritical one is a consequence of [BJ1] and [BJ2].

[2]A codimension-one analytic submanifold is a (not-necessarily closed) set X given locally (near any point of X) as the zero set of an analytic submersion $C_\delta^\omega(\mathbb{R}/\mathbb{Z},\mathbb{R}) \to \mathbb{R}$.

[3]It is actually true that any critical energy can be made supercritical under an arbitrarily small perturbation of the potential, see [A3].

restrictions to large boxes. Both are related by the Thouless formula:

$$(5) \qquad L(E) = \int \ln |E' - E| dN(E').$$

Much work has been dedicated to the regularity properties of L and N. For quite general reasons, the integrated density of states is a continuous non-decreasing function onto $[0, 1]$, and it is constant outside the spectrum. Notice that this is not enough to conclude continuity of the Lyapunov exponent from the Thouless formula. Other regularity properties (such as Hölder), do pass from N to L and vice-versa. This being said, our focus here is primarily on the Lyapunov exponent on its own.

It is easy to see that the Lyapunov exponent is real analytic outside the spectrum. Beyond that, however, there are obvious limitations to its regularity. For a constant potential, say $v = 0$, the Lyapunov exponent is $L(E) = \max\{0, \ln \frac{1}{2}(E + \sqrt{E^2 - 4})\}$, so it is only $1/2$-Hölder continuous. With Diophantine frequencies and small potentials, the generic situation is to have Cantor spectrum with countably many square root singularities at the endpoints of gaps [E]. For small potential and generic frequencies, it is possible to show that the Lyapunov exponent escapes any fixed continuity modulus (such as Hölder), and it is also not of bounded variation. More delicately, Bourgain [B] has observed that in the case of the critical almost Mathieu operator the Lyapunov exponent needs not be Hölder even for Diophantine frequencies (another instance of complications arising at the boundary of non-uniform hyperbolicity). Though a surprising result, analytic regularity, was obtained in a related, but non-Schrödinger, context [AK2], the negative results described above seemed to impose serious limitations on the amount of regularity one should even try to look for in the Schrödinger case.

As for positive results, a key development was the proof by Goldstein-Schlag [GS1] that the Lyapunov exponent is Hölder continuous for Diophantine frequencies in the regime of positive Lyapunov exponent. Later Bourgain-Jitomirskaya [BJ1] proved that the Lyapunov exponent is continuous for all irrational frequencies, and this result played a fundamental role in the recent theory of the almost Mathieu operator. More delicate estimates on the Hölder regularity for Diophantine frequencies remained an important topic of the local theories [GS2], [AJ].

There is however one important case where, in a different sense, much stronger regularity holds. For small analytic potentials, it follows from the work of Bourgain-Jitomirskaya ([BJ1] and [BJ2], see [AJ]) that the Lyapunov exponent is zero (and hence constant) in the spectrum! In general, however, the Lyapunov exponent need not be constant in the spectrum. In fact, there are examples where the Lyapunov exponent vanishes in part of the spectrum and is positive in some other part [Bj1]. Particularly in this positive Lyapunov exponent regime, it would seem unreasonable, given the negative results outlined above, to expect much more regularity. In fact, from a dynamical systems perspective, it would be natural to expect bad behavior in such setting, since when the Lyapunov exponent is positive, the associated dynamical system in the two torus presents "strange attractors" with very complicated dependence of the parameters [Bj2].

In this respect, the almost Mathieu operator would seem to behave quite oddly. As we have seen, by the Aubry formula the Lyapunov exponent is always constant in the spectrum, and moreover, this constant is just a simple expression of the coupling $\max\{0, \ln \lambda\}$ (and in particular, it does become positive in the *supercritical regime*

$\lambda > 1$). It remains true that the Lyapunov exponent displays wild oscillations "just outside" the spectrum, so this is not inconsistent with the negative results discussed above.

However, for a long time, the general feeling has been that this just reinforces the special status of the almost Mathieu operator (which admits a remarkable symmetry, Aubry duality, relating the supercritical and the subcritical regimes), and such a phenomenon would seem to have little to do with the case of general potentials. This general feeling is wrong, as the following sample result shows.

Example Theorem. *Let $\lambda > 1$ and let w be any real analytic function. For $\epsilon \in \mathbb{R}$, let $v(x) = 2\lambda \cos 2\pi x + \epsilon w(x)$. Then for ϵ small enough, for every $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$, the Lyapunov exponent restricted to the spectrum is a positive real analytic function.*

Of course by a real analytic function on a set we just mean the restriction of some real analytic function defined on an open neighborhood.

For an arbitrary real analytic potential, the situation is just slightly lengthier to describe. Let X be a topological space. A stratification of X is a strictly decreasing finite or countable sequence of closed sets $X = X_0 \supset X_1 \supset \cdots$ such that $\cap X_i = \varnothing$. We call $X_i \smallsetminus X_{i+1}$ the i-th *stratum* of the stratification.

Let now X be a subset of a real analytic manifold, and let $f : X \to \mathbb{R}$ be a continuous function. We say that f is C^r-stratified if there exists a stratification such that the restriction of f to each stratum is C^r.

Theorem 2 (Stratified analyticity in the energy). *Let $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$ and v be any real analytic function. Then the Lyapunov exponent is a C^ω-stratified function of the energy.*

As we will see, in this theorem the stratification starts with $X_1 = \Sigma_{\alpha,v}$, which is compact, so the stratification is finite.

Nothing restricts us to look only at the energy as a parameter. For instance, in the case of the almost Mathieu operator, the Lyapunov exponent (restricted to the spectrum) is real analytic also in the coupling constant, except at $\lambda = 1$.

Theorem 3 (Stratified analyticity in the potential). *Let $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$, let X be a real analytic manifold, and let v_λ, $\lambda \in X$, be a real analytic family of real analytic potentials. Then the Lyapunov exponent is a C^ω-stratified function of both λ and E.*

It is quite clear how this result opens the doors for the analysis of the boundary of non-uniform hyperbolicity, since parameters corresponding to the vanishing of the Lyapunov exponent are contained in the set of solutions of equations (in infinitely many variables) with analytic coefficients. Of course, one still has to analyze the nature of the equations one gets, guaranteeing the non-vanishing of the coefficients. Indeed, in the subcritical regime, the coefficients do vanish. We will work out suitable expressions for the Lyapunov exponent restricted to strata which will allow us to show non-vanishing outside the subcritical regime.

In the case of the almost Mathieu operator, there is no dependence of the Lyapunov exponent on the frequency parameter. In general, Bourgain-Jitomirskaya [BJ1] proved that the Lyapunov exponent is a continuous function of $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$. This is a very subtle result, as the continuity is not in general uniform in α. We will show that the Lyapunov exponent is in fact C^∞-stratified as a function of $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$.

Theorem 4. *Let X be a real analytic manifold, and let v_λ, $\lambda \in X$, be a real analytic family of real analytic potentials. Then the Lyapunov exponent is a C^∞-stratified function of $(\alpha, \lambda, E) \in (\mathbb{R} \smallsetminus \mathbb{Q}) \times X \times \mathbb{R}$.*

With v as in the Example Theorem, the Lyapunov exponent is actually C^∞ as a function of α and E in the spectrum.

1.3. **Lyapunov exponents of** $\mathrm{SL}(2, \mathbb{C})$ **cocycles.** In the dynamical systems approach, which we follow here, the understanding of the Schrödinger operator is obtained through the detailed description of a certain family of dynamical systems.

A (one-frequency, analytic) quasiperiodic $\mathrm{SL}(2, \mathbb{C})$ cocycle is a pair (α, A), where $\alpha \in \mathbb{R}$ and $A : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2, \mathbb{C})$ is analytic, understood as defining a linear skew-product acting on $\mathbb{R}/\mathbb{Z} \times \mathbb{C}^2$ by $(x, w) \mapsto (x + \alpha, A(x) \cdot w)$. The iterates of the cocycle have the form $(n\alpha, A_n)$ where A_n is given by (3). The Lyapunov exponent $L(\alpha, A)$ of the cocycle (α, A) is given by the left hand side of (4). We say that (α, A) is *uniformly hyperbolic* if there exist analytic functions $u, s : \mathbb{R}/\mathbb{Z} \to \mathbb{P}\mathbb{C}^2$, called the unstable and stable directions, and $n \geq 1$ such that for every $x \in \mathbb{R}/\mathbb{Z}$, $A(x) \cdot u(x) = u(x + \alpha)$ and $A(x) \cdot s(x) = s(x + \alpha)$, for every unit vector $w \in s(x)$ we have $\|A_n(x) \cdot w\| < 1$ and for every unit vector $w \in u(x)$ we have $\|A_n(x) \cdot w\| > 1$ (clearly $u(x) \neq s(x)$ for every $x \in \mathbb{R}/\mathbb{Z}$). The unstable and stable directions are uniquely caracterized by those properties, and clearly $u(x) \neq s(x)$ for every $x \in \mathbb{R}/\mathbb{Z}$. It is clear that if (α, A) is uniformly hyperbolic then $L(\alpha, A) > 0$.

If $L(\alpha, A) > 0$ but (α, A) is not uniformly hyperbolic, we will say that (α, A) is *nonuniformly hyperbolic.*

Uniform hyperbolicity is a stable property: the set $\mathcal{UH} \subset \mathbb{R} \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{C}))$ of uniformly hyperbolic cocycles is open. Moreover, it implies good behavior of the Lyapunov exponent: the restriction of $(\alpha, A) \mapsto L(\alpha, A)$ to $\mathcal{UH}$ is a C^∞ function of both variables,[4] and it is a pluriharmonic function of the second variable.[5] In fact regularity properties of the Lyapunov exponent are consequence of the regularity of the unstable and stable directions, which depend smoothly on both variables (by normally hyperbolic theory [HPS]) and holomorphically on the second variable (by a simple normality argument).

On the other hand, a variation [JKS] of [BJ1] gives that $(\alpha, A) \mapsto L(\alpha, A)$ is continuous as a function on $(\mathbb{R} \smallsetminus \mathbb{Q}) \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{C}))$. It is important to notice (and in fact, fundamental in what follows) that the Lyapunov exponent is not continuous on $\mathbb{R} \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{C}))$. In the remaining of this section, we will restrict our attention (except otherwise noted) to cocycles with irrational frequencies.

Most important examples are Schrödinger cocycles $A^{(v)}$, determined by a real analytic function v by $A^{(v)} = \begin{pmatrix} v & -1 \\ 1 & 0 \end{pmatrix}$. In this notation, the Lyapunov exponent at energy E for the operator $H_{\alpha, v}$ becomes $L(E) = L(\alpha, A^{(E-v)})$. One of the most basic aspects of the connection between spectral and dynamical properties is that

---

[4]Since $\mathcal{UH}$ is not a Banach manifold, it might seem important to be precise about what notion of smoothness is used here. This issue can be avoided by enlarging the setting to include C^∞ non-analytic cocycles (say by considering a Gevrey condition), so that we end up with a Banach manifold.

[5]This means that, in addition to being continuous, given any family $\lambda \mapsto A^{(\lambda)} \in \mathcal{UH}$, $\lambda \in \mathbb{D}$, which holomorphic (in the sense that it is continuous and for every $x \in \mathbb{R}/\mathbb{Z}$ the map $\lambda \mapsto A^{(\lambda)}(x)$ is holomorphic), the map $\lambda \mapsto L(\alpha, A^{(\lambda)})$ is harmonic.

$E \notin \Sigma_{\alpha,v}$ if and only if $(\alpha, A^{(E-v)})$ is uniformly hyperbolic. Thus the analyticity of $E \mapsto L(E)$ outside of the spectrum just translates a general property of uniformly hyperbolic cocycles.

If $A \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{C}))$ admits a holomorphic extension to $|\Im z| < \delta$, then for $|\epsilon| < \delta$ we can define $A_\epsilon \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{C}))$ by $A_\epsilon(x) = A(x + i\epsilon)$. The Lyapunov exponent $L(\alpha, A_\epsilon)$ is easily seen to be a convex function of ϵ. Thus we can define a function

$$(6) \qquad \omega(\alpha, A) = \lim_{\epsilon \to 0+} \frac{1}{2\pi\epsilon}(L(\alpha, A_\epsilon) - L(\alpha, A)),$$

called the *acceleration*. It follows from convexity and continuity of the Lyapunov exponent that the acceleration is an upper semi-continuous function in $(\mathbb{R} \smallsetminus \mathbb{Q}) \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{C}))$.

Our starting point is the following result.

Theorem 5 (Acceleration is quantizatized). *The acceleration of an* $\mathrm{SL}(2, \mathbb{C})$ *cocycle with irrational frequency is always an integer.*

Remark 1. It is easy to see that quantization does not extend to rational frequencies, see Remark 5.

This result allows us to break parameter spaces into suitable pieces restricted to which we can study the dependence of the Lyapunov exponent.

Quantization implies that $\epsilon \mapsto L(\alpha, A_\epsilon)$ is a piecewise affine function of ϵ. Knowing this, it makes sense to introduce the following:

Definition 2. We say that $(\alpha, A) \in (\mathbb{R} \smallsetminus \mathbb{Q}) \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{C}))$ is *regular* if $L(\alpha, A_\epsilon)$ is affine for ϵ in a neighborhood of 0.

Remark 3. If A takes values in $\mathrm{SL}(2, \mathbb{R})$ then $\epsilon \mapsto L(\alpha, A_\epsilon)$ is an even function. By convexity, $\omega(\alpha, A) \geq 0$, and if $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$ then (α, A) is regular if and only if $\omega(\alpha, A) = 0$.

Clearly regularity is an open condition in $(\mathbb{R} \smallsetminus \mathbb{Q}) \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{C}))$.

It is natural to assume that regularity has important consequences for the dynamics. Indeed, we have been able to completely characterize the dynamics of regular cocycles with positive Lyapunov exponent, which is the other cornerstone of this paper.

Theorem 6. *Let* $(\alpha, A) \in (\mathbb{R} \smallsetminus \mathbb{Q}) \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{C}))$. *Assume that* $L(\alpha, A) > 0$. *Then* (α, A) *is regular if and only if* (α, A) *is uniformly hyperbolic.*

One striking consequence is the following:

Corollary 7. *For any* $(\alpha, A) \in (\mathbb{R} \smallsetminus \mathbb{Q}) \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{C}))$, *there exists* $\epsilon_0 > 0$ *such that*

1. $L(\alpha, A_\epsilon) = 0$ *(and* $\omega(\alpha, A) = 0$*) for every* $0 < \epsilon < \epsilon_0$, *or*
2. (α, A_ϵ) *is uniformly hyperbolic for every* $0 < \epsilon < \epsilon_0$.

Proof. Since $\epsilon \to L(\alpha, A_\epsilon)$ is piecewise affine, it must be affine on $(0, \epsilon_0)$ for $\epsilon_0 > 0$ sufficiently small, so that (α, A_ϵ) is regular for every $0 < \epsilon < \epsilon_0$.

Since the Lyapunov exponent is non-negative, if $L(\alpha, A_\epsilon) > 0$ for some $0 < \epsilon < \epsilon_0$, then $L(\alpha, A_\epsilon) > 0$ for every $0 < \epsilon < \epsilon_0$. The result follows from the previous theorem. $\square$

This result plays an important role in our work (especially in [A3]), since it allow us to consider the dynamics of non-regular $\mathrm{SL}(2,\mathbb{R})$-cocycles as a non-tangential limit of better behaved (uniformly hyperbolic) cocycles.

As for the case of regular cocycles with zero Lyapunov exponent, this is the topic of the Almost Reducibility Conjecture, which we will discuss in section 1.5. For now, we will focus on the deduction of regularity properties of the Lyapunov exponent from Theorems 5 and 6.

1.3.1. *Stratified regularity: proof of Theorems 2, 3 and 4.* For $\delta > 0$, denote by $C_\delta^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{C})) \subset C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{C}))$ the set of all A which admit a bounded holomorphic extension to $|\Im z| < \delta$, continuous up to the boundary. It is naturally endowed with a complex Banach manifold structure.

For $j \neq 0$, let $\Omega_{\delta,j} \subset \mathbb{R} \times C_\delta^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{C}))$ be the set of all (α, A) such that there exists $0 < \delta' < \delta$ such that $(\alpha, A_{\delta'}) \in \mathcal{UH}$ and $\omega(\alpha, A_{\delta'}) = j$, and let $L_{\delta,j} : \Omega_{\delta,j} \to \mathbb{R}$ be given by $L_{\delta,j}(\alpha, A) = L(\alpha, A_{\delta'}) - 2\pi j \delta'$. Since if $0 < \delta' < \delta'' < \delta$, $\omega(\alpha, A_{\delta'}) = \omega(\alpha, A_{\delta''}) = j$ implies that $L(\alpha, A_{\delta'}) = L(\alpha, A_{\delta''}) - 2\pi j(\delta'' - \delta')$, we see that $L_{\delta,j}$ is well defined.

Proposition 4. $\Omega_{\delta,j}$ *is open and* $(\alpha, A) \mapsto L_{\delta,j}(\alpha, A)$ *is a C^∞ function, pluriharmonic in the second variable. Moreover, if $(\alpha, A) \in (\mathbb{R} \smallsetminus \mathbb{Q}) \times C_\delta^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{C}))$ has acceleration j, then $(\alpha, A) \in \Omega_{\delta,j}$ and $L(\alpha, A) = L_{\delta,j}(\alpha, A)$.*

Proof. The first part follows from openness of $\mathcal{UH}$ and the regularity of the Lyapunov exponent restricted to $\mathcal{UH}$. For the second part, we use Corollary 7 and upper semicontinuity of the Lyapunov exponent to conclude that $\omega(\alpha, A) = j$ implies that $(\alpha, A_{\delta'}) \in \mathcal{UH}$ and has acceleration j for every δ' sufficiently small, which gives also $L(\alpha, A) = L(\alpha, A_{\delta'}) - 2\pi j \delta'$.   □

We can now explain the proofs of Theorems 2, 3 and 4. For definiteness, we will consider Theorem 3, the argument is exactly the same for the other theorems. Define a stratification of the parameter space $\mathbb{X} = \mathbb{R} \times X$: $\mathbb{X}_0 = \mathbb{X}$, $\mathbb{X}_1 \subset \mathbb{X}_0$ is the set of (E, λ) such that $(\alpha, A^{(E-v_\lambda)})$ is not uniformly hyperbolic and for $j \geq 2$, $\mathbb{X}_j \subset \mathbb{X}_1$ is the set of (E, λ) such that $\omega(\alpha, A^{(E-v_\lambda)}) \geq j - 1$.

Since uniform hyperbolicity is open and the acceleration is upper semicontinuous, each $\mathbb{X}_j$ is closed, so this is indeed a stratification. Since the 0-th stratum $\mathbb{X}_0 \smallsetminus \mathbb{X}_1$ corresponds to uniformly hyperbolic cocycles, the Lyapunov exponent is analytic there.

By quantization, the j-th stratum, $j \geq 2$, corresponds to cocycles which are not uniformly hyperbolic and have acceleration $j - 1$. For each (E, λ_0) in such a stratum, choose $\delta > 0$ such that $\lambda \mapsto A^{(v_\lambda)}$ is an analytic function with values in $C_\delta^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ for λ in a neighborhood of λ_0. The analyticity of the Lyapunov exponent restricted to the stratum is then a consequence of Proposition 4.

As for a parameter (E, λ) in the first stratum $\mathbb{X}_1 \smallsetminus \mathbb{X}_2$, quantization implies that $(\alpha, A^{(E-v_\lambda)})$ has non-positive acceleration, so by Remark 3 $(\alpha, A^{(E-v_\lambda)})$ must be regular with zero acceleration. Since it is not uniformly hyperbolic, Theorem 6 implies that $L(\alpha, A^{(E-v_\lambda)}) = 0$. Thus the Lyapunov exponent is in fact identically 0 in the first stratum.

1.4. **Codimensionality of critical cocycles.** Non-regular cocycles split into two groups, the ones with positive Lyapunov exponent (non-uniformly hyperbolic cocycles), and the ones with zero Lyapunov exponent, which we call *critical cocycles*.[6]

As discussed before, the first group has been extensively studied recently ([BG], [GS1], [GS2], [GS3]). But very little is known about the second one.

Though our methods do not provide new information on the dynamics of critical cocycles, they are perfectly adapted to show that critical cocycles are rare. This is somewhat surprising, since in dynamical systems, it is rarely the case that the success of parameter exclusion precedes a detailed control of the dynamics!

Of course, for $\mathrm{SL}(2,\mathbb{C})$ cocycles, our previous results already show that critical cocycles are rare in certain one-parameter families, since for every (α, A), for every $\delta \neq 0$ small, (α, A_δ) is regular, and hence not critical. But for our applications we are mostly concerned with $\mathrm{SL}(2,\mathbb{R})$-valued cocycles, and even more specifically, with Schrödinger cocycles.

If $(\alpha, A) \in (\mathbb{R} \smallsetminus \mathbb{Q}) \times C_\delta^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{C}))$ is critical with acceleration j, then $(\alpha, A) \in \Omega_{\delta,j}$ and $L_{\delta,j} = 0$. Moreover, if A is $\mathrm{SL}(2,\mathbb{R})$-valued, criticality implies that the acceleration is positive (see Remark 3). So the locus of critical $\mathrm{SL}(2,\mathbb{R})$-valued cocycles is covered by countably many analytic sets $L_{\delta,j}^{-1}(0)$. Thus the main remaining issue is to show that the functions $L_{\delta,j}$ are non-degenerate.

Theorem 8. *For every $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$, $\delta > 0$ and $j > 0$, if $v_* \in C_\delta^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ and $\omega(\alpha, A^{(v_*)}) = j$ then $v \mapsto L_{\delta,j}(\alpha, A^{(v)})$ is a submersion in a neighborhood of v_*.*

This theorem immediately implies Theorem 1.

We are also able to show non-degeneracy in the case of non-Schrödinger cocycles, see Remark 11: though the derivative of $L_{\delta,j}$ may vanish, this forces the dynamics to be particularly nice, and it can be shown that the second derivative is non-vanishing.

1.5. **Almost reducibility.** The results of this paper give further motivation to the research on the set of regular cocycles with zero Lyapunov exponent. The central problem here is addressing the following:

Conjecture 2 (Almost Reducibility Conjecture). *Let $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$ and $A \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$. If (α, A) is regular and $L(\alpha, A) = 0$ then (α, A) is almost reducible: There exist $\delta > 0$, a constant $A_* \in \mathrm{SL}(2,\mathbb{R})$, and a sequence of analytic maps $B_n \in C_\delta^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{PSL}(2,\mathbb{R}))$ such that $\sup_{|\Im x| < \delta} \| B_n(x+\alpha) A(x) B_n(x)^{-1} - A_* \| \to 0$.*

This conjecture was first made in [AJ], and it can be generalized to $\mathrm{SL}(2,\mathbb{C})$-valued cocycles in the obvious way. What makes it so central is that almost reducibility was analyzed in much detail in recent works, see [AJ], [A1], [AFK] and [A2], so a proof would immediately give a very fine picture of the subcritical regime. In particular, coupled with the results of this paper about the critical regime, and the results of Bourgain, Goldstein and Schlag about the supercritical regime, a proof of the Almost Reducibility Conjecture would give a proof of Conjecture 1:

1. The Almost Reducibility Conjecture implies that the subcritical regime can only support absolutely continuous spectrum [A2],

---

[6]As explained before, this terminology is consistent with the almost Mathieu operator terminology: it turns out that if $v(x) = 2\lambda \cos 2\pi(\theta + x)$, $\lambda \in \mathbb{R}$, then $(\alpha, A^{(E-v)})$ is critical if and only if $\lambda = 1$ and $E \in \Sigma_{\alpha,v}$.

2. [BG] implies that pure point spectrum is typical throughout the supercritical regime,[7]
3. Theorem 1 implies that typically the critical regime is invisible to the spectral measures.[8]

We have previously proved the almost reducibility conjecture when α is exponentially well approximated by rational numbers p_n/q_n: $\limsup \frac{\ln q_{n+1}}{q_n} > 0$, [A2]. Coupled with [AJ], [A1], this established the almost reducibility conjecture in the case of the almost Mathieu operator. A complete proof of the almost reducibility conjecture was finally obtained recently [A4].

1.6. **Further comments.** As mentioned before, it follows from the combination of [BJ1] and [BJ2], that the Lyapunov exponent is zero in the spectrum, provided the potential is sufficiently small, irrespective of the frequency. This is a very surprising result from the dynamical point of view.

For instance, fix some non-constant small v, and consider α close to 0. Then the spectrum is close, in the Hausdorff topology, to the interval $[\inf v - 2, \sup v + 2]$. However, if $E \notin [\inf v + 2, \sup v - 2]$ we have

$$(7) \qquad \lim_{n \to \infty} \lim_{\alpha \to 0} \frac{1}{n} \int_{\mathbb{R}/\mathbb{Z}} \ln \|A^{(E-v)}(x + (n-1)\alpha) \cdots A(x)\| dx > 0.$$

At first it might seem that as $\alpha \to 0$ the dynamics of $(\alpha, A^{(E-v)})$ becomes increasingly complicated and we should expect the behavior of large potentials (with positive Lyapunov exponents by [SS]).[9] Somehow, delicate cancellation between expansion and contraction takes place precisely at the spectrum and kills the Lyapunov exponent.

Bourgain-Jitomirskya's result that the Lyapunov exponent must be zero on the spectrum in this situation involves duality and localization arguments which are far from the dynamical point of view. Our work provides a different explanation for it, and extends it from $\mathrm{SL}(2, \mathbb{R})$-cocycles to $\mathrm{SL}(2, \mathbb{C})$-cocycles. Indeed, quantization implies that all cocycles near constant have zero acceleration. Thus they are all regular. Thus if A is close to constant and (α, A) has a positive Lyapunov exponent then it must be uniformly hyperbolic.

We stress that while this argument explains why constant cocycles are far from non-uniform hyperbolicity, localization methods remain crucial to the understanding of several aspects of the dynamics of cocycles close to a constant one, at least in the Diophantine regime.

---

[7]More precisely, for every fixed potential, and for almost every frequency, the spectrum is pure point with exponentially decaying eigenfunctions throughout the region of the spectrum where the Lyapunov exponent is positive.

[8]Since in the continuation of this series, [A3], we will show a stronger fact (for fixed frequency, a typical potential has no critical energies), we just sketch the argument. For fixed frequency, Theorem 1 implies that a typical potential admits at most countably many critical energies. Considering phase changes $v_\theta(x) = v(x + \theta)$, which do not change the critical set, we see that for almost every θ the critical set, being a fixed countable set, can not carry any spectral weight (otherwise the average over θ of the spectral measures would have atoms, but this average has a continuous distribution, the integrated density of states [AS]).

[9]In fact, the Lyapunov exponent function converges in the L^1-sense, as $\alpha \to 0$, to a continuous function, positive outside $[\sup v - 2, \inf v + 2]$ (see the argument of [AD]). This is compact with the fact that the edges of the spectrum (located in two small intervals of size $\sup v - \inf v$) become increasingly thinner (in measure) as $\alpha \to 0$.

Let us finally make a few remarks and pose questions about the actual values taken by the acceleration.

1. If the coefficients of A are trigonometric polynomials of degree at most n, then $|\omega(\alpha, A)| \leq n$ by convexity (since $L(\alpha, A_\epsilon) \leq \sup_{x \in \mathbb{R}/\mathbb{Z}} \ln \|A(x + \epsilon i)\| \leq 2\pi n \epsilon + O(1)$).

2. On the other hand, if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $|\lambda| \geq 1$ and $n \in \mathbb{N}$, then for $v(x) = 2\lambda \cos 2\pi n x$ we have $\omega(\alpha, A^{(E-v)}) = n$ for every $E \in \Sigma_{\alpha,v}$. In the case $n = 1$ (the almost Mathieu operator), this is shown in the Appendix. The general case reduces to this one since for any $A \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{C}))$ and $n \in \mathbb{N}$, $L(n\alpha, A(x)) = L(\alpha, A(nx))$, which implies $n\omega(n\alpha, A(x)) = \omega(\alpha, A(nx))$.

3. If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and A takes values in $\mathrm{SO}(2, \mathbb{R})$, the acceleration is easily seen to be the norm of the topological degree of A. The results of [AK2] imply that this also holds for "premonotonic cocycles" which include small $\mathrm{SL}(2, \mathbb{R})$ perturbations of $\mathrm{SO}(2, \mathbb{R})$ valued cocycles with non-zero topological degree.

4. It seems plausible that the norm of the topological degree is always a lower bound for the acceleration of $\mathrm{SL}(2, \mathbb{R})$ cocycles. In case of non-zero degree, is this bound achieved precisely by premonotonic cocycles?

5. Consider a typical perturbation of the potential $2\lambda \cos 2\pi n x$, $\lambda > 1$. Do energies with any fixed acceleration $1 \leq k \leq n$ form a set of positive measure? It seems promising to use the "Benedicks-Carleson" method of Lai-Sang Young [Y] to address aspects of this question ($k = n$, large λ, allowing exclusion of small set of frequencies). One is also tempted to relate the acceleration to the number of "critical points" for the dynamics (which can be identified when her method works). Colisions between a few critical points might provide a mechanism for the appearance of energies with intermediate acceleration.

1.7. **Outline of the remaining of the paper.** The outstanding issues (not covered in the introduction) are the proofs of Theorems 5, 6 and 8.

We first address quantization (Theorem 5) in section 2. The proof uses periodic approximation. A Fourier series estimate shows that as the denominators of the approximations grow, quantization becomes more and more pronounced. The result then follows by continuity of the Lyapunov exponent [JKS].

Next we show, in section 3, that regularity with positive Lyapunov exponent implies uniform hyperbolicity (the hard part of Theorem 6). The proof again proceeds by periodic approximation. We first notice that the Fourier series estimate implies that periodic approximants are uniformly hyperbolic, and hence have unstable and stable directions. If we can show that we can take an analytic limit of those directions, then the uniform hyperbolicity of (α, A) will follow. A simple normality argument shows that we only need to prove that the invariant directions do not get too close as the denominators grow. We show (by direct computation) that if they would get too close, then the derivative of the Lyapunov exponent would be relatively large with respect to perturbations of some Fourier modes of the potential. This contradicts a "macroscopic" bound on the derivative which comes from pluriharmonicity.

We then show, in section 4, the non-vanishing of the derivative of the canonical analytic extension of the Lyapunov exponent, $L_{\delta,j}$ (Theorem 8). Under the hypothesis that $\omega(\alpha, A^{(v_*)}) = j > 0$, $(\alpha, A^{(v_*)}_{\delta'}) \in \mathcal{UH}$ for $0 < \delta' < \delta_0$ ($0 < \delta_0 < \delta$ small),

so we can define holomorphic invariant directions u and s, over $0 < \Im z < \delta_0$. Using the explicit expressions for the derivative of the Lyapunov exponent in terms of the unstable and stable directions u and s, derived in section 3, we conclude that the vanishing of the derivative would imply a symmetry of Fourier coefficients (of a suitable expression involving u and s), which is enough to conclude that u and s analytically continuate through $\Im z = 0$. This implies that $(\alpha, A^{(v_*)})$ is "conjugate to a cocycle of rotations", which implies that its acceleration is zero, contradicting the hypothesis.

We also include an appendix showing how to use quantization to compute the Lyapunov exponent and acceleration in the case of the almost Mathieu operator, which is used in deriving the Example Theorem.

Acknowledgements: I am grateful to Svetlana Jitomirskaya and David Damanik for several detailed comments which greatly improved the exposition.

2. Quantization of acceleration: proof of Theorem 5

We will use the continuity in the frequency of the Lyapunov exponent [BJ1], [JKS].[10]

Theorem 9 ([JKS]). *If $A \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{C}))$, then the map $\alpha \mapsto L(\alpha, A)$, $\alpha \in \mathbb{R}$, is continuous at every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.*

This result is very delicate: the restriction of $\alpha \mapsto L(\alpha, A)$ to $\mathbb{R} \setminus \mathbb{Q}$ is not, in general, uniformly continuous.

Notice that if p/q is a rational number, then there exists a simple expression for the Lyapunov exponent $L(p/q, A)$

$$(8) \qquad L(p/q, A) = \frac{1}{q} \int_{\mathbb{R}/\mathbb{Z}} \ln \rho(A_{(p/q)}(x)) dx$$

where $A_{(p/q)}(x) = A(x + (q-1)p/q) \cdots A(x)$ and $\rho(B)$ is the spectral radius of an $\mathrm{SL}(2, \mathbb{C})$ matrix $\rho(B) = \lim_{n \to \infty} \|B^n\|^{1/n}$. A key observation is that if p and q are coprime then the trace $\mathrm{tr} A_{(p/q)}(x)$ is a $1/q$-periodic function of x. This follows from the relation

$$(9) \qquad A(x)A_{(p/q)}(x) = A_{(p/q)}(x + p/q)A(x),$$

expressing the fact that $A_{(p/q)}(x)$ and $A_{(p/q)}(x + p/q)$ are conjugate in $\mathrm{SL}(2, \mathbb{C})$, and hence $A_{(p/q)}(x)$ is conjugate to $A_{(p/q)}(x + kp/q)$ for any $k \in \mathbb{Z}$.

Fix $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $A \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{C}))$ and let p_n/q_n be a sequence of rational numbers (p_n and q_n coprime) approaching α (not necessarily continued fraction approximants).

Let $\epsilon > 0$ and $C > 0$ be such that A admits a bounded extension to $|\Im z| < \epsilon$ with $\sup_{|\Im z| < \epsilon} \|A(z)\| < C$. Since $\mathrm{tr} A_{(p_n/q_n)}$ is $1/q_n$-periodic,

$$(10) \qquad \mathrm{tr} A_{(p_n/q_n)}(x) = \sum_{k \in \mathbb{Z}} a_{k,n} e^{2\pi i k q_n x},$$

with $a_{k,n} \leq 2C^{q_n} e^{-2\pi k q_n \epsilon}$.

---

[10]Bourgain-Jitomirskaya actually restricted considerations to the case of Schrödinger (in particular, $\mathrm{SL}(2, \mathbb{R})$ valued) cocycles. Their result was generalized to the $\mathrm{SL}(2, \mathbb{C})$ case in the work of Jitomirskaya-Koslover-Schulteis [JKS].

Fix $0 < \epsilon' < \epsilon$. Fixing k_0 sufficiently large, we get

$$(11) \qquad \mathrm{tr}A_{(p_n/q_n)}(x) = \sum_{|k| \le k_0} a_{k,n} e^{2\pi i k q_n x} + O(e^{-q_n}), \quad |\Im x| < \epsilon',$$

for n large. Since $\max\{1, \frac{1}{2}|\mathrm{tr}|\} \le \rho \le \max\{1, |\mathrm{tr}|\}$, it follows that

$$(12) \qquad L(p_n/q_n, A_\delta) = \max_{k \le |k_0|} \max\{\frac{\ln|a_{k,n}|}{q_n} - 2\pi k\delta, 0\} + o(1), \quad \delta < \epsilon'.$$

Thus for large n, $\delta \mapsto L(p_n/q_n, A_\delta)$ is close, over $|\delta| < \epsilon'$, to a convex piecewise linear function with slopes in $\{-2\pi k_0, ..., 2\pi k_0\}$. By Theorem 9, these functions converge uniformly on compacts of $|\delta| < \epsilon$ to $\delta \mapsto L(\alpha, A_\delta)$. It follows that $\delta \mapsto L(\alpha, A_\delta)$ is a convex piecewise linear function of $|\delta| < \epsilon'$, with slopes in $\{-2\pi k_0, ..., 2\pi k_0\}$, so $\omega(\alpha, A) \in \mathbb{Z}$. This completes the proof of Theorem 5. $\qquad\square$

Remark 5. Consider say $A(x) = \begin{pmatrix} e^{\lambda(x)} & 0 \\ 0 & e^{-\lambda(x)} \end{pmatrix}$ with $\lambda(x) = e^{2\pi i q_0 x}$ for some $q_0 > 0$. Then $L(\alpha, A_\epsilon) = \frac{2}{\pi} e^{-2\pi q_0 \epsilon}$ if $\alpha = p/q$ for some q dividing q_0, and $L(\alpha, A_\epsilon) = 0$ otherwise. This gives an example both of discontinuity of the Lyapunov exponent and of lack of quantization of acceleration at rationals.

If we had chosen λ as a more typical function of zero average, we would get discontinuity of the Lyapunov exponent and lack of quantization at all rationals, both becoming increasingly less pronounced as the denominators grow.

3. CHARACTERIZATION OF UNIFORM HYPERBOLICITY: PROOF OF THEOREM 6

Since the Lyapunov exponent is a C^∞ function in $\mathcal{UH}$, the "if" part is obvious from quantization. In order to prove the "only if" direction, we will first show the uniform hyperbolicity of periodic approximants and then show that uniform hyperbolicity persists in the limit. To do this last part, we will use an explicit formula for the derivative of the Lyapunov exponent (fixed frequency) in $\mathcal{UH}$.

3.1. **Uniform hyperbolicity of approximants.**

Lemma 6. *Let* $(\alpha, A) \in (\mathbb{R} \setminus \mathbb{Q}) \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{R}))$ *and assume that* (α, A_δ) *is regular with positive Lyapunov exponent. If* p/q *is close to* α *and* $\tilde{A}$ *is close to* A *then* $(p/q, \tilde{A})$ *is uniformly hyperbolic.*

Proof. Let us show that if $p_n/q_n \to \alpha$ and $A^{(n)} \to A$ then there exists $\epsilon'' > 0$ such that

$$(13) \qquad \frac{1}{q_n} \ln \rho(A^{(n)}_{(p_n/q_n)}(x)) = L(\alpha, A_{\Im x}) + o(1), \quad |\Im x| < \epsilon'',$$

which implies the result. In fact this estimate is just a slight adaptation of what we did in section 2.

Since $A^{(n)} \to A$ and A is regular, we may choose $\epsilon > 0$ such that (α, A_δ) is regular for $|\delta| < \epsilon$, $A_n \in C^\omega_\epsilon(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{C}))$ for every n and $A_n \to A$ uniformly in $|\Im z| < \epsilon$.

Choose $\epsilon'' < \epsilon' < \epsilon$. We have seen in section 2 that there exists k_0 such that

$$(14) \qquad \mathrm{tr}A^{(n)}_{(p_n/q_n)}(x) = \sum_{|k| \le k_0} a_{k,n} e^{2\pi i k q_n x} + O(e^{-q_n}), \quad |\Im x| < \epsilon',$$

$$(15) \qquad L(p_n/q_n, A_\delta^{(n)}) = \max_{k \leq |k_0|} \max\{\frac{\ln|a_{k,n}|}{q_n} - 2\pi k\delta, 0\} + o(1), \quad |\delta| < \epsilon'.$$

By Theorem 9, $L(p_n/q_n, A_\delta^{(n)}) \to L(\alpha, A_\delta)$ uniformly on compacts of $|\delta| < \epsilon$, so we may rewrite (15) as

$$(16) \qquad L(\alpha, A_\delta^{(n)}) = \max_{k \leq |k_0|} \max\{\frac{\ln|a_{k,n}|}{q_n} - 2\pi k\delta, 0\} + o(1), \quad |\delta| < \epsilon'.$$

Since the left hand side in (16) is an affine positive function of δ, with slope $2\pi\omega(\alpha, A)$, over $|\delta| < \epsilon$, it follows that $|\omega(\alpha, A)| \leq k_0$,

$$(17) \qquad L(\alpha, A_\delta) = \frac{\ln|a_{-\omega(\alpha,A),n}|}{q_n} + 2\pi\omega(\alpha, A)\delta + o(1), \quad |\delta| < \epsilon'',$$

and morever, if $|j| \leq k_0$ is such that $j \neq -\omega(\alpha, A)$ we have

$$(18) \qquad \frac{\ln|a_{j,n}|}{q_n} - 2\pi j\delta + 2\pi(\epsilon' - \epsilon'') \leq L(\alpha, A_\delta) + o(1), \quad |\delta| < \epsilon''.$$

Together, (14), (17) and (18) imply (13), as desired. $\qquad\qquad\square$

3.2. Derivative of the Lyapunov exponent at uniformly hyperbolic cocycles.
Fix $(\alpha, A) \in \mathcal{UH}$. Let $u, s : \mathbb{R}/\mathbb{Z} \to \mathbb{PC}^2$ be the unstable and stable directions.

Let $B : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2, \mathbb{C})$ be analytic with column vectors in the directions of $u(x)$ and $s(x)$. Then

$$(19) \qquad B(x+\alpha)^{-1}A(x)B(x) = \begin{pmatrix} \lambda(x) & 0 \\ 0 & \lambda(x)^{-1} \end{pmatrix} = D(x).$$

Obviously $L(\alpha, A) = L(\alpha, D) = \int \Re \ln \lambda(x)dx.$[11]

Write $B(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$. We note that though the definition of B involves arbitrary choices, it is clear that $q_1(x) = a(x)d(x) + b(x)c(x)$, $q_2(x) = c(x)d(x)$ and $q_3(x) = -b(x)a(x)$ depend only on (α, A). We will call q_i, $i = 1, 2, 3$, the *coefficients of the derivative of the Lyapunov exponent*, for reasons that will be clear in a moment.

Lemma 7. *Let $(\alpha, A) \in \mathcal{UH}$ and let $q_1, q_2, q_3 : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ be the coefficients of the derivative of the Lyapunov exponent. Let $w : \mathbb{R}/\mathbb{Z} \to \mathrm{sl}(2, \mathbb{C})$ be analytic, and write $w = \begin{pmatrix} w_1 & w_2 \\ w_3 & -w_1 \end{pmatrix}$. Then*

$$(20) \qquad \frac{d}{dt}L(\alpha, Ae^{tw}) = \Re \int_{\mathbb{R}/\mathbb{Z}} \sum_{i=1}^{3} q_i(x)w_i(x)dx, \quad at \quad t = 0.$$

Proof. Write $B(x+p/q)^{-1}A(x)e^{tw(x)}B(x) = D^t(x)$. We notice that

$$(21) \qquad D(x)^{-1}\frac{d}{dt}D^t(x) = B(x)^{-1}w(x)B(x), \quad at \quad t = 0,$$

and

$$(22) \qquad \sum_{i=1}^{3} q_i(x)w_i(x) = \text{u.l.c. of } B(x)^{-1}w(x)B(x),$$

---

[11]Notice that the quantization of the acceleration, in the uniformly hyperbolic case, follows immediately from this expression (the integer arising being the number of turns $\lambda(x)$ does around 0).

where u.l.c. stands for the upper left coefficient.

Suppose first that α is a rational number p/q. Then

$$(23) \qquad \frac{d}{dt}L(p/q, Ae^{tw}) = \frac{1}{q}\frac{d}{dt}\int_{\mathbb{R}/\mathbb{Z}}\ln\rho(D_{(p/q)}^t(x))dx,$$

so it is enough to show that

$$(24) \qquad \frac{d}{dt}\ln\rho(D_{(p/q)}^t(x)) = \Re\sum_{j=0}^{q-1}\sum_{i=1}^{3}q_i(x+jp/q)w_i(x+jp/q), \quad \text{at} \quad t=0.$$

Since $D_{(p/q)}(x)$ is diagonal and its u.l.c. has norm bigger than 1,

$$(25) \qquad \frac{d}{dt}\ln\rho(D_{(p/q)}^t(x)) = \Re \text{ u.l.c. of } D_{(p/q)}(x)^{-1}\frac{d}{dt}D_{(p/q)}^t(x), \quad \text{at} \quad t=0.$$

Writing $D_{[j]}(x) = D(x+(j-1)p/q)\cdots D(x)$, and using (21), we see that

$$(26) \quad D_{(p/q)}(x)^{-1}\frac{d}{dt}D_{(p/q)}^t(x)$$

$$= \sum_{j=0}^{q-1}D_{[j]}(x)^{-1}B(x+jp/q)^{-1}w(x+jp/q)B(x+jp/q)D_{[j]}(x), \quad \text{at} \quad t=0.$$

Since the $D_{[j]}$ are diagonal,

$$(27) \qquad \text{u.l.c. of } D_{[j]}(x)^{-1}B(x+jp/q)^{-1}w(x+jp/q)B(x+jp/q)D_{[j]}(x)$$

$$= \text{u.l.c. of } B(x+jp/q)^{-1}w(x+jp/q)B(x+jp/q).$$

Putting together (22), (25), (26) and (27), we get (24).

The validity of the formula in the rational case yields the irrational case by approximation (since the Lyapunov exponent is C^∞ in $\mathcal{UH}$). $\square$

3.3. **Proof of Theorem 6.** Let $(\alpha, A) \in (\mathbb{R}\smallsetminus\mathbb{Q})\times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{C}))$ be such that (α, A) is regular. Then there exists $\epsilon > 0$ such that $L(\alpha, A_\delta)$ is regular for $|\delta| < \epsilon$.

Fix $0 < \epsilon' < \epsilon$. Choose a sequence $p_n/q_n \to \alpha$. By Lemma 6, if n is large then $(p_n/q_n, A_\delta)$ is uniformly hyperbolic for $\delta < \epsilon'$. So one can define functions $u_n(x), s_n(x)$ with values in $\mathbb{PC}^2$, corresponding to the eigendirections of $A_{(p_n/q_n)}(x)$ with the largest and smallest eigenvalues. Our strategy will be to show that the sequences $u_n(x)$ and $s_n(x)$ converge uniformly (in a band) to functions $u(x)$ and $s(x)$.

The coefficients of the derivative of $L(p_n/q_n, A)$ will be denoted q_i^n, $i=1,2,3$. The basic idea now is that if $q_2^n(x)$ and $q_3^n(x)$ are bounded, then it follows directly from the definitions that the angle between $u_n(x)$ and $s_n(x)$ is not too small, and this is enough to guarantee convergence. On the other hand, the derivative of the Lyapunov exponent is under control by pluriharmonicity, which yields the desired bound on the coefficients.

There are various way to proceed here, and we will just do an estimate of the Fourier coefficients of the q_j^n, $j = 2, 3$. Write

$$(28) \qquad \zeta_{n,k,j} = \int_{\mathbb{R}/\mathbb{Z}}q_j^n(x)e^{2\pi ikx}dx.$$

Lemma 8. *There exist $C > 0$, $\gamma > 0$ such that for every n sufficiently large,*

$$(29) \qquad |\zeta_{n,k,j}| \le Ce^{-\gamma|k|}, \quad j=2,3, \quad k\in\mathbb{Z}.$$

Proof. Choose $0 < \gamma < 2\pi\epsilon'$. Then for each fixed n large we have $|\zeta_{n,k,j}| \leq C_n e^{-\gamma|k|}$ (since q_j^n extend to $|\Im z| < \epsilon'$). If the result did not hold, then there would exist $n_l \to \infty$, $k_l \in \mathbb{Z}$, $j_l = 2, 3$ such that $|\zeta_{n_l,k_l,j_l}| > l e^{-\gamma|k_l|}$. We may assume that j_l is a constant and either $k_l > 0$ for all l or $k_l \leq 0$ for all l.

For simplicity, we will assume that $j_l = 2$ and $k_l \leq 0$ for all l. Let

$$\text{(30)} \qquad w_{(l)}(x) = \frac{|\zeta_{n_l,k_l,2}|}{\zeta_{n_l,k_l,2}} e^{\gamma|k_l|} \begin{pmatrix} 0 & e^{2\pi i k_l x} \\ 0 & 0 \end{pmatrix}.$$

Choose $\gamma < \gamma' < 2\pi\epsilon'$. Setting $\tilde{A}(x) = A(x - i\gamma'/2\pi)$ and $\tilde{w}_{(l)}(x) = w_{(l)}(x - i\gamma'/2\pi)$, we get

$$\text{(31)} \qquad \frac{d}{dt} L(p_{n_l}/q_{n_l}, \tilde{A}e^{t\tilde{w}_{(l)}}) = e^{\gamma|k_l|}|\zeta_{n_l,k_l,2}| \geq l,$$

since the coefficients of the derivative at $(p_{n_l}/q_{n_l}, \tilde{A})$ are $\tilde{q}_j^{n_l}(x) = q_j^{n_l}(x - i\gamma'/2\pi)$.

Notice that $\tilde{w}_{(l)}$ admits a holomorphic extension bounded by 1 on $|\Im z| < (\gamma' - \gamma)/2\pi$. Since $(\alpha, \tilde{A})$ is regular with positive Lyapunov exponent, it follows from Lemma 6, that there exists $r > 0$ such that for every l large $(p_{n_l}/q_{n_l}, \tilde{A}e^{t\tilde{w}_{(l)}})$ is uniformly hyperbolic for complex t with $|t| < r$. In particular, the functions $t \mapsto L(p_{n_l}/q_{n_l}, \tilde{A}e^{t\tilde{w}_{(l)}})$ are harmonic on $|t| < r$ for l large. Those functions are also clearly uniformly bounded. Harmonicity gives then that the derivative at $t = 0$ is uniformly bounded as well. This contradicts (31). □

Lemma 9. *If $a, b, c, d \in \mathbb{C}$ are such that $ad - bc = 1$, and the angle between the complex lines through $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$ is small, then $\max\{|ab|, |cd|\}$ is large.*

Proof. Straightforward computation. □

Lemma 8 implies that there exists $\gamma > 0$ such that q_2^n and q_3^n are uniformly bounded, as $n \to \infty$, on $|\Im x| < \gamma$. By Lemma 9, this implies that there exists $\eta > 0$ such that the angle between $u_n(x)$ and $s_n(x)$ is at least η, for every n large and $|\Im x| < \gamma$. We are in position to apply a normality argument.

Lemma 10. *Let $u_n(x)$ and $s_n(x)$ be holomorphic functions defined in some complex manifold, with values in $\mathbb{PC}^2$. If the angle between $u_n(x)$ and $s_n(x)$ is bounded away from 0 uniformly in x and n, then $u_n(x)$ and $s_n(x)$ form normal families, and limits of u_n and of s_n (taken along the same subsequence) are holomorphic functions such that $u(x) \neq s(x)$ for every x.*

Proof. We may identify $\mathbb{PC}^2$ with the Riemann Sphere. Write $\phi_n(x) = u_n(x)/s_n(x)$. Then $\phi_n(x)$ avoids a neighborhood of 1, hence it forms a normal family. Let us now take a sequence along which ϕ_n converges, and let us show $u_n(x)$ and $s_n(x)$ form normal families. This is a local problem, so we may work in a neighborhood of a point z. If $\lim \phi_n(z) \neq \infty$, then for every n large ϕ_n must be bounded (uniformly in a neighborhood of z), so u_n and $1/s_n$ must also be bounded. If $\lim \phi_n(z) = \infty$, then for every n large $1/\phi_n$ must be bounded (uniformly in a neighborhood of z), so s_n and $1/u_n$ must be bounded. In either case we conclude that s_n and u_n are normal in a neighborhood of z.

The last statement is obvious by pointwise convergence. □

Let $u(x)$ and $s(x)$ be limits of $u_n(x)$ and $s_n(x)$ over $|\Im x| < \gamma$, taken along the same subsequence. Then $A(x) \cdot u(x) = u(x + \alpha)$, $A(x) \cdot s(x) = s(x + \alpha)$

and $u(x) \neq s(x)$. Since $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $L(\alpha, A) > 0$, this easily implies that $(\alpha, A) \in \mathcal{UH}$. $\qquad \square$

4. Local non-triviality of the Lyapunov function in strata: Proof of Theorem 8

Let δ, j, v_* be as in the statement of Theorem 8. Notice that $(\alpha, A^{(v_*)}) \notin \mathcal{UH}$, since otherwise we would have $j = \omega(\alpha, A^{(v_*)}) = 0$.

Let $0 < \epsilon_0 < \delta$ be such that $(\alpha, A_\epsilon^{(v_*)}) \in \mathcal{UH}$ and $\omega(\alpha, A_\epsilon^{(v_*)}) = j$ for $0 < \epsilon < \epsilon_0$. By definition, for every $0 < \epsilon < \epsilon_0$, we have $L_{\delta,j}(\alpha, A) = L(\alpha, A_\epsilon) - 2\pi j\epsilon$ for v in a neighborhood of v_*.

Let $u, s : \{0 < \Im x < \epsilon_0\} \to \mathbb{PC}^2$ be such that $x \mapsto u(x + i\epsilon)$ and $x \mapsto s(x + i\epsilon)$ are the unstable and stable directions of $(\alpha, A_\epsilon^{(v_*)})$, and let $q_1, q_2, q_3 : \{0 < \Im x < \epsilon_0\} \to \mathbb{C}$ be such that $x \mapsto q_j(x + i\epsilon)$ is the j-th coefficient of the derivative of $(\alpha, A_\epsilon^{(v_*)})$.

Notice that $A^{(v_*+w)} = A^{(v_*)} e^{\tilde{w}}$, where $\tilde{w}(x) = \begin{pmatrix} 0 & 0 \\ -w(x) & 0 \end{pmatrix}$. Thus the derivative of $w \mapsto L_{\delta,j}(\alpha, A^{(v_*+tw)})$ at $t = 0$ is

$$\tag{32} \Re \int_{\mathbb{R}/\mathbb{Z}} -w(x + i\epsilon)q_3(x + i\epsilon)dx.$$

If the result does not hold, then (32) must vanish for every $w \in C_\delta^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$. Testing this with w of the form $a \cos 2\pi k x + b \sin 2\pi k x$, $a, b \in \mathbb{R}$, $k \in \mathbb{Z}$, we see that the k-th Fourier coefficient of q_3 must be minus the complex conjugate of the $-k$-th Fourier coefficient of q_3 for every $k \in \mathbb{Z}$. Since the Fourier series converges for $0 < \Im x < \epsilon_0$, this implies that it actually converges for $|\Im x| < \epsilon_0$, and at $\mathbb{R}/\mathbb{Z}$ it defines a purely imaginary function. Thus $q_3(x)$ extends analytically through $|\Im x| = 0$, and hence $q_2(x) = c(x)d(x) = a(x - \alpha)b(x - \alpha) = -q_3(x - \alpha)$ (the middle equality holding due to the Schrödinger form) also does.

Identifying $\mathbb{PC}^2$ with the Riemann sphere in the usual way (the line through $\begin{pmatrix} z \\ w \end{pmatrix}$ corresponding to z/w), we get $q_2 = \frac{1}{u-s}$ and $-q_3 = \frac{us}{u-s}$. These formulas allow us to analytically continuate u and s through $\Im x = 0$.[12] Since q_2 and q_3 are purely imaginary when $\Im x = 0$, we conclude that u and s are complex conjugate directions in $\mathbb{PC}^2$ when $\Im x = 0$.[13] Note that if $u(x) = s(x)$ for some x then $u(x + n\alpha) = s(x + n\alpha)$ for every n and thus by analytic continuation $u = s$ everywhere, which is impossible. So when $\Im x = 0$, u and s are in fact distinct complex conjugate directions, and in particular they are also not real.

---

[12]Indeed, if q_2 is identically vanishing then we set either $u = \infty$, $s = -q_3$, or $s = \infty$, $u = q_3$ (so to match the previous definition when $\Im z > 0$). If q_2 is not identically vanishing but $1 - 4q_2 q_3$ vanishes identically, then we can take $u = -s = \frac{1}{2q_2}$. Assume now that there exists $x \in \mathbb{R}/\mathbb{Z}$ such that $q_2(x) \neq 0$ and $1 - 4q_2(x)q_3(x) \neq 0$. Then we can define $u(z)$ and $-s(z)$ in a small open square Q of side $2r$ centered on x (with sides parallel to the coordinate axis) as the distinct solutions of the equation $w^2 - \frac{1}{q_2}w + \frac{q_3}{q_2} = 0$ (so to match the previous definition when $\Im z > 0$). We then spread it using the dynamics to $|\Im z| < r$ in order to have $A_k(z) \cdot u(z) = u(z + k\alpha)$ and $A_k(z) \cdot s(z) = s(z + k\alpha)$ for every $z \in Q$ and $k \in \mathbb{Z}$. This gives well defined analytic functions because whenever z and $z + n\alpha \in Q$ we have $A_n(z) \cdot u(z) = u(z + n\alpha)$ and $A_n(z) \cdot s(z) = s(z + n\alpha)$ (this is clear when $\Im z > 0$ and holds in general by analytic continuation).

[13]In principle, $\Re q_2 = \Re q_3 = 0$ is also compatible with values of u and s such that $\Re u = \Re s = 0$ (where we set $\Re\infty = 0$ for simplicity). But if $\Re u = 0$ then $\Re(A \cdot u) = \Re v_*$, so if $u \neq \bar{s}$ somewhere then v_* must be identically vanishing. This contradicts $\omega(\alpha, A^{(v_*)} \neq 0$.

Let $B(x) \in \mathrm{SL}(2,\mathbb{R})$ be the unique upper triangular matrix with positive diagonal coefficients taking $u(x)$ to $\pm i$ (and hence $s(x)$ to $\mp i$). Then $B : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ is analytic. Define $A(x) = B(x+\alpha)A^{(v_*)}(x)B(x)^{-1}$. Since $A^{(v_*)}(x)$ takes $u(x)$ and $s(x)$ to $u(x+\alpha)$ and $s(x+\alpha)$, we conclude that $B(x+\alpha)A^{(v_*)}(x)B(x)^{-1} \in \mathrm{SO}(2,\mathbb{R})$. Since $x \mapsto A^{(v_*)}(x)$ is homotopic to a constant as a function $\mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$, $x \mapsto B(x+\alpha)A^{(v_*)}(x)B(x)^{-1} \in \mathrm{SL}(2,\mathbb{R})$ is homotopic to a constant as a function $\mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$, thus also as a function $\mathbb{R}/\mathbb{Z} \to \mathrm{SO}(2,\mathbb{R})$. It follows that there exists an analytic function $\phi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ such that $B(x+\alpha)A^{(v_*)}(x)B(x)^{-1} = A(x)$, where $A(x)$ is the rotation of angle $2\pi\phi(x)$.

Obviously this relation implies that $L(\alpha, A_\epsilon^{(v_*)}) = L(\alpha, A_\epsilon)$ for $\epsilon > 0$ small. If we show that $L(\alpha, A_\epsilon) = 0$ for $\epsilon > 0$ small, we will conclude that $\omega(\alpha, A^{(v_*)}) = 0$, contradicting the hypothesis.

To see that $L(\alpha, A_\epsilon) = 0$, notice that for $n \geq 1$, $A_n(x)$ is the rotation of angle $\sum_{k=0}^{n-1} \phi(x+k\alpha)$. Thus

$$(33) \qquad \frac{1}{n}\int_{\mathbb{R}/\mathbb{Z}} \ln \|A_n(x+i\epsilon)\| dx = 2\pi \int_{\mathbb{R}/\mathbb{Z}} \frac{1}{n}|\sum_{k=0}^{n-1}\Im\phi(x+k\alpha+i\epsilon)|dx.$$

Since $x \mapsto x+\alpha$ is ergodic with respect to Lebesgue measure, the integrand of the right hand side converges uniformly, as $n \to \infty$, to $|\int_{\mathbb{R}/\mathbb{Z}} \Im\phi(x+i\epsilon)dx| = |\int_{\mathbb{R}/\mathbb{Z}} \Im\phi(x)dx| = 0$. Thus the limit of the right hand side, which is $L(\alpha, A_\epsilon)$ by definition, is zero as well. $\qquad \square$

Remark 11. The analysis of the function $A \mapsto L_{\delta,j}(\alpha, A)$ from $C_\delta^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ to $\mathbb{R}$, with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and near some $\tilde{A}$ with $\omega(\alpha, \tilde{A}) > 0$ can be carried out as above with one important difference.

The argument above does allow one to establish that if $L_{\delta,j}$ is not a local submersion, then the coefficients of the derivative q_2 and q_3 extend from some half band $0 < \Im x < \epsilon_0$ to a full band $|\Im x| < \epsilon_0$.[14] This again leads to the conclusion that there exists $B : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ analytic such that $A(x) = B(x+\alpha)\tilde{A}(x)B(x)^{-1}$ takes values in $\mathrm{SO}(2,\mathbb{R})$. But now there are two cases.

1. $x \mapsto \tilde{A}(x)$ is homotopic to a constant. In this case, the above argument goes through and one concludes that $\omega(\alpha, \tilde{A}) = 0$, contradiction.
2. $x \mapsto \tilde{A}(x)$ is not homotopic to a constant. In this case, there is no contradiction, and the reader is invited to check that if $\tilde{A}(x)$ is the rotation of angle $2\pi x$ then indeed the derivative of $L_{\delta,j}$ vanishes, though $\omega(\alpha, \tilde{A}) = 1$.

The analysis of the second case has been carried out by different means in [AK2], where it is shown that the Lyapunov exponent is real analytic near cocycles with values in $\mathrm{SO}(2,\mathbb{R})$ provided they are not homotopic to a constant. We should emphasize that this result is obtained for any number of frequencies, which is certainly beyond the scope of the techniques we develop in this paper.

Interpreting their results (in the one-frequency case) from our new point of view, [AK2] shows that all real perturbations of $(\alpha, \tilde{A})$ have the same acceleration (the absolute value of the topological degree of $\tilde{A}$ as a map $\mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$). Real analyticity implies that the derivative of the Lyapunov exponent then is forced

---

[14]Though one lacks the symmetry between q_2 and q_3 exploited above, we just separately evaluate the extensions of q_2 and q_3, since we are not constrained to consider just perturbations of a specific form.

to vanish whenever the Lyapunov exponent is zero. In [AK2] it is shown that the second derivative is non-zero. The locus of zero exponents can be shown to intersect a neighborhood of $\tilde{A}$ in $C^\omega_\delta(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ in an analytic submanifold of codimension $4|\omega(\alpha, \tilde{A})|$.

Thus our result implies that among cocycles non-homotopic to constants (and with a given irrational frequency), the locus of zero exponents is contained in a countable union of positive codimension submanifolds of $C^\omega_\delta(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$.

APPENDIX A. SOME ALMOST MATHIEU COMPUTATIONS

Through this section, we let $v(x) = 2\cos 2\pi x$.

Theorem 10. *If $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$, $\lambda > 0$, $E \in \mathbb{R}$ and $\epsilon \geq 0$ then $L(\alpha, A_\epsilon^{(E-\lambda v)}) = \max\{L(\alpha, A^{(E-\lambda v)}), (\ln\lambda) + 2\pi\epsilon\}$, $\epsilon \geq 0$.*

Proof. A direct computation shows that if E and λ are fixed then for every $\delta > 0$, there exists $0 < \xi < \pi/2$ such that if ϵ is large and $w \in \mathbb{C}^2$ makes angle at most ξ with the horizontal line then for every $x \in \mathbb{R}/\mathbb{Z}$, $w' = A^{(E-\lambda v)}(x + \epsilon i) \cdot w$ makes angle at most $\xi/2$ with the horizontal line and $|\ln\|w'\| - (\ln\lambda + 2\pi\epsilon)| < \delta$.

This implies that $L(\alpha, A_\epsilon^{(E-\lambda v)}) = 2\pi\epsilon + \ln\lambda + o(1)$ as $\epsilon \to \infty$. By quantization of acceleration, for every ϵ sufficiently large, $\omega(\alpha, A_\epsilon^{(E-\lambda v)}) = 1$ and $L(\alpha, A_\epsilon^{(E-\lambda v)}) = 2\pi\epsilon + \ln\lambda$. By real-symmetry, $\omega(\alpha, A_\epsilon^{(E-\lambda v)})$ is either 0 or 1 for $\epsilon \geq 0$. This implies the given formula for $L(\alpha, A_\epsilon^{(E-\lambda v)})$. $\square$

For completeness, let us give a contrived rederivation of the Aubry-André formula.

Corollary 11 ([BJ1])**.** *If $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$, $\lambda > 0$, $E \in \mathbb{R}$ then $L(\alpha, A^{(E-\lambda v)}) \geq \max\{0, \ln\lambda\}$ with equality if and only if $E \in \Sigma_{\alpha,v}$.*

Proof. The complement of the spectrum consists precisely of energies with positive Lyapunov exponent and zero acceleration (as those two properties characterize uniform hyperbolicity for $\mathrm{SL}(2,\mathbb{R})$-valued cocycles by Theorem 6).

The previous theorem gives the inequality, and shows that it is strict if and only if $L(\alpha, A^{(E-\lambda v)}) > 0$ and $\omega(\alpha, A^{(E-\lambda v)}) = 0$. $\square$

A.1. **Proof of the Example Theorem.** Fix $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$, $\lambda > 1$ and $w \in C^\omega_\delta(\mathbb{R}/\mathbb{Z}, \mathbb{R})$. Let $v_\epsilon = \lambda v + \epsilon w$.

Lemma 12. *If ϵ is sufficiently small, and $E \in \Sigma_{\alpha,v_\epsilon}$ then $\omega(\alpha, A^{(E-v_\epsilon)}) = 1$.*

Proof. By Theorem 10 and Corollary 11, $L(\alpha, A^{(E-\lambda v)}) \geq \ln\lambda$ and $\omega(\alpha, A^{(E-\lambda v)}) \leq 1$ for every $E \in \mathbb{R}$.

For ϵ small we have $\Sigma_{\alpha,v_\epsilon} \subset [-4\lambda, 4\lambda]$. By continuity of the Lyapunov exponent and upper semicontinuity of the acceleration, we get $\omega(\alpha, A^{(E-v_\epsilon)}) \leq 1$ and $L(\alpha, A^{(E-v_\epsilon)}) > 0$ for every $E \in \Sigma_{\alpha,v_\epsilon}$.

Since $A^{(E-v_\epsilon)}$ is real symmetric, $\omega(\alpha, A^{(E-v_\epsilon)}) \geq 0$ as well, and if $\omega(\alpha, A^{(E-v_\epsilon)}) = 0$ with $E \in \Sigma_{\alpha,v_\epsilon}$ then $(\alpha, A^{(E-v_\epsilon)})$ is regular. This last possibility can not happen: since the Lyapunov exponent is positive, it would imply uniform hyperbolicity, which can not happen in the spectrum. We conclude that $\omega(\alpha, A^{(E-v_\epsilon)}) > 0$ for $E \in \Sigma_{\alpha,v_\epsilon}$. By quantization, this forces $\omega(\alpha, A^{(E-v_\epsilon)}) = 1$. $\square$

By Proposition 4, $E \mapsto L(\alpha, A^{(E-v_\epsilon)})$ coincides in the spectrum with the restriction of an analytic function $(E \mapsto L_{\delta,1}(\alpha, A^{(E-v_\epsilon)}))$ defined in some neighborhood. This concludes the proof of the Example Theorem. $\qquad\square$

References

[A1] Avila, A. The absolutely continuous spectrum of the almost Mathieu operator. Preprint (www.arXiv.org).

[A2] Avila, A. Almost reducibility and absolute continuity I. Preprint (www.impa.br/∼avila/).

[A3] Avila, A. Global theory of one-frequency Schrödinger operators II: acriticality and the finiteness of phase transitions for typical potentials. Preprint (www.impa.br/∼avila/).

[A4] Avila, A. KAM, Lyapunov exponents and the spectral dichotomy for one-frequency Schrödinger operators. In preparation.

[AD] Avila, Artur; Damanik, David Generic singular spectrum for ergodic Schrödinger operators. Duke Math. J. 130 (2005), no. 2, 393–400.

[AFK] Avila, A.; Fayad, B.; Krikorian, R. A KAM scheme for SL(2, ℝ) cocycles with Liouvillean frequencies. Geometric and Functional Analysis 21 (2011), 1001-1019.

[AJ] Avila, A.; Jitomirskaya, S. Almost localization and almost reducibility. Journal of the European Mathematical Society 12 (2010), 93-131.

[AK1] Avila, A.; Krikorian, R. Reducibility or non-uniform hyperbolicity for quasiperiodic Schrödinger cocycles. Ann. of Math. 164 (2006), 911-940.

[AK2] Avila, A.; Krikorian, R. Monotonic cocycles. Preprint (www.impa.br/∼avila/).

[AS] Avron, Joseph; Simon, Barry Almost periodic Schrödinger operators. II. The integrated density of states. Duke Math. J. 50 (1983), no. 1, 369–391.

[Bj1] Bjerklöv, K. Explicit examples of arbitrarily large analytic ergodic potentials with zero Lyapunov exponent. Geom. Funct. Anal. 16 (2006), no. 6, 1183-1200.

[Bj2] Bjerklöv, Kristian Positive Lyapunov exponent and minimality for a class of one-dimensional quasi-periodic Schrödinger equations. Ergodic Theory Dynam. Systems 25 (2005), no. 4, 1015–1045.

[B] Bourgain, J. Green's function estimates for lattice Schrödinger operators and applications. Annals of Mathematics Studies, 158. Princeton University Press, Princeton, NJ, 2005. x+173 pp.

[BG] Bourgain, J.; Goldstein, M. On nonperturbative localization with quasi-periodic potential. Ann. of Math. (2) 152 (2000), no. 3, 835–879.

[BJ1] Bourgain, J.; Jitomirskaya, S. Continuity of the Lyapunov exponent for quasiperiodic operators with analytic potential. J. Statist. Phys. 108 (2002), no. 5-6, 1203–1218.

[BJ2] Bourgain, J.; Jitomirskaya, S. Absolutely continuous spectrum for 1D quasiperiodic operators. Invent. Math. 148 (2002), no. 3, 453–463.

[E] Eliasson, L. H. Floquet solutions for the 1-dimensional quasi-periodic Schrödinger equation. Comm. Math. Phys. 146 (1992), no. 3, 447–482.

[GS1] Goldstein, M.; Schlag, W. Hölder continuity of the integrated density of states for quasi-periodic Schrödinger equations and averages of shifts of subharmonic functions. Ann. of Math. (2) 154 (2001), no. 1, 155–203.

[GS2] Goldstein, M.; Schlag, W. Fine properties of the integrated density of states and a quantitative separation property of the Dirichlet eigenvalues. Geom. Funct. Anal. 18 (2008), no. 3, 755–869.

[GS3] Goldstein, M.; Schlag, W. On resonances and the formation of gaps in the spectrum of quasi-periodic Schrödinger equations. Ann. of Math. 173 (2011), 337-475.

[H] Herman, M. Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2. Comment. Math. Helv. 58 (1983), no. 3, 453–502.

[HPS] Hirsch, M. W.; Pugh, C. C.; Shub, M. Invariant manifolds. Lecture Notes in Mathematics, Vol. 583. Springer-Verlag, Berlin-New York, 1977. ii+149 pp.

[J] Jitomirskaya, S. Metal-insulator transition for the almost Mathieu operator. Ann. of Math. (2) 150 (1999), no. 3, 1159–1175.

[JKS] Jitomirskaya S.; Koslover D.; Schulteis M. Continuity of the Lyapunov exponent for quasiperiodic Jacobi matrices. Ergodic Theory and Dynamical Systems 29 (2009), 1881-1905.

[JL]   Jitomirskaya, S.; Last, Y. Power law subordinacy and singular spectra. II. Line operators. Comm. Math. Phys. 211 (2000), no. 3, 643–658.
[SS]   Sorets, E.; Spencer, T. Positive Lyapunov exponents for Schrödinger operators with quasi-periodic potentials. Comm. Math. Phys. 142 (1991), no. 3, 543–566.
[Y]   Young, L.-S. Lyapunov exponents for some quasi-periodic cocycles. Ergodic Theory Dynam. Systems 17 (1997), no. 2, 483–504.

CNRS UMR 7586, Institut de Mathématiques de Jussieu-Paris Rive Gauche, Bâtiment Sophie Germain, Case 7012, 75205 Paris Cedex 13, France

IMPA, Estrada Dona Castorina 110, 22460-320, Rio de Janeiro, Brazil
URL: www.impa.br/∼avila/
E-mail address: artur@math.jussieu.fr