ON POINT SPECTRUM AT CRITICAL COUPLING

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Abstract. We give a short proof of absence of point spectrum at critical coupling for the almost Mathieu operator, for any irrational frequency, except (possibly) for countably many values of the phase.

1. Introduction

Here we consider the almost Mathieu operator with critical coupling $H = H_{\alpha,\theta} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$,

$$(Hu)_n = u_{n+1} + u_{n-1} + 2\cos(2\pi(\theta + n\alpha))u_n, \tag{1.1}$$

where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ (the frequency) and $\theta \in \mathbb{R}$ (the phase) are parameters. The spectrum of H is a θ independent set $\Sigma = \Sigma_\alpha$.

It is expected that $H_{\alpha,\theta}$ has purely singular continuous spectrum for every $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and every $\theta \in \mathbb{R}$. Since the Lebesgue measure of Σ_α is zero [AK1], there can not be absolutely continuous spectrum anyway, so singular continuous spectrum follows from absence of point spectrum in this context. It is known that eigenfunctions, if they exist, can not be in $l^1(\mathbb{Z})$ [D].

The first results on absence of point spectrum were obtained under certain topologically generic conditions on α [AS] or θ [JS]. In [GJLS] it was proved that for every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that $|\Sigma_\alpha| = 0$ (at the time this was only known for almost every α [L]), and for almost every $\theta \in \mathbb{R}$, $H_{\alpha,\theta}$ has no point spectrum.

Together with Raphaël Krikorian, using convergence of renormalization [AK2] and some deep non-perturbative results [BJ], we have shown that the possible exceptional set of θ is contained in an explicit countable set (this result is so far unpublished). Our goal in this note is to give a short direct proof of this last result. The proof is self-contained and not based on $|\Sigma_\alpha| = 0$.

We say that $\theta \in \mathbb{R}$ is rational with respect to α if there exists $k \in \mathbb{Z}$ such that $2\theta + k\alpha \in \mathbb{Z}$.

Theorem 1.1. *For every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and for every $\theta \in \mathbb{R}$ which is not rational with respect to α, the operator $H = H_{\alpha,\theta}$ has no eigenvalues.*

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2. Proof of Theorem 1.1

Let $H = H_{\alpha,\theta}$, and assume that $H\hat{u} = E\hat{u}$ for some $E \in \mathbb{R}$, $\hat{u} \in l^2(\mathbb{Z})$, $\hat{u} \neq 0$. Let $u(x) = \sum e^{2\pi i n x}\hat{u}_n$. Let $A(x) = \begin{pmatrix} E - 2\cos 2\pi x & -1 \\ 1 & 0 \end{pmatrix}$ be the transfer matrix,

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$B(x)=\begin{pmatrix} u(x) & \overline{u(x)} \\ e^{-2\pi i\theta}u(x-\alpha) & e^{2\pi i\theta}\overline{u(x-\alpha)}\end{pmatrix}$, $D=\begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta}\end{pmatrix}$, so that

$$A(x)\cdot B(x)=B(x+\alpha)D \tag{2.1}$$

(direct computation). This expresses the classical Aubry duality between eigenfunctions $\hat{u}$ and Bloch waves u at the critical point. In particular $\det B(x)=\det B(x+\alpha)$. By ergodicity of $x\mapsto x+\alpha$ on $\mathbb{R}/\mathbb{Z}$, $\det B(x)$ is constant. Ergodicity and (2.1) also imply that $\|B(x)\|>0$ for almost every x (otherwise $B(x)=0$ almost everywhere, contradicting $\hat{u}\neq 0$).

Let us show that $\det B\neq 0$ if θ is not rational with respect to α. Indeed, if $\det B=0$ then there exists a function $\phi:\mathbb{R}/\mathbb{Z}\to\mathbb{C}$, with $|\phi|=1$, such that $\begin{pmatrix} u(x) \\ e^{-2\pi i\theta}u(x-\alpha)\end{pmatrix}=\phi(x)\begin{pmatrix} \overline{u(x)} \\ e^{-2\pi i\theta}\overline{u(x-\alpha)}\end{pmatrix}$. Using (2.1), we get $\phi(x+\alpha)=e^{-4\pi i\theta}\phi(x)$. Expanding in Fourier series $\phi(x)=\sum\hat{\phi}_k e^{2\pi ikx}$ we get $e^{2\pi ik\alpha}\hat{\phi}_k=e^{-4\pi i\theta}\hat{\phi}_k$. Since θ is not rational with respect to α, we have that $\hat{\phi}_k=0$ for all k, contradicting $|\phi|=1$.

Let $A_k(x)=A(x+(k-1)\alpha)\cdots A(x)$, $k\geq 1$ be the k-step transfer matrix. Then $A_k(x)=B(x+k\alpha)D^kB(x)^{-1}$. Let $\Psi^{(k)}(x)=\mathrm{tr}A_k(x)-2\cos 2\pi k\theta=\mathrm{tr}A_k(x)-\mathrm{tr}D^k$ (here $\mathrm{tr}\begin{pmatrix} a & b \\ c & d\end{pmatrix}=a+d$). Then $\Psi^{(k)}(x)=\mathrm{tr}B(x+k\alpha)D^kB(x)^{-1}-\mathrm{tr}B(x)D^kB(x)^{-1}$, so that

$$|\Psi^{(k)}(x)|\leq\frac{2}{|\det B|}\|B(x+k\alpha)-B(x)\|\|B(x)\| \tag{2.2}$$

(notice that $\|B^{-1}\|=\frac{\|B\|}{|\det B|}$). By Cauchy-Schwartz,

$$\|\Psi^{(k)}(x)\|_{L^1}\leq\frac{2}{|\det B|}\|B(x+k\alpha)-B(x)\|_{L^2}\|B(x)\|_{L^2}. \tag{2.3}$$

We now notice that $\|B(x)\|_{L^2}<\infty$ by construction. This also implies that $\|B(x+k\alpha)-B(x)\|_{L^2}$ gets arbitrarily small when $k\alpha$ gets close to an integer.[1] Thus

$$\liminf_{k\to\infty}\|\Psi^{(k)}(x)\|_{L^1}=0. \tag{2.4}$$

On the other hand, it is readily seen that $\Psi^{(k)}(x)$ is a trigonometric polynomial, $\Psi^{(k)}(x)=\sum_{n=-k}^{k}\hat{\Psi}^{(k)}_n e^{2\pi inx}$. The extremal coefficients are easy to compute: $\hat{\Psi}^{(k)}_{\pm k}(x)=(-1)^k e^{\pm\pi ik(k-1)\alpha}$. Since $\hat{\Psi}^{(k)}_k=\int_{\mathbb{R}/\mathbb{Z}}\Psi^{(k)}(x)e^{-2\pi ikx}dx$, we have $\|\Psi^{(k)}(x)\|_{L^1}\geq|\hat{\Psi}^{(k)}_k|=1$ contradicting (2.4).

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[1] For any function $w:\mathbb{R}/\mathbb{Z}\to\mathbb{C}$ with $\|w(x)\|_{L^2}<\infty$, $\lim_{\epsilon\to 0}\|w(x+\epsilon)-w(x)\|_{L^2}=0$ (as can be seen using Fourier series).

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