ON THE KOTANI-LAST AND SCHRÖDINGER CONJECTURES

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ABSTRACT. In the theory of ergodic one-dimensional Schrödinger operators, ac spectrum has been traditionally expected to be very rigid. Two key conjectures in this direction state, on one hand, that ac spectrum demands almost periodicity of the potential, and, on the other hand, that the eigenfunctions are almost surely bounded in the essential suport of the ac spectrum. We show how the repeated slow deformation of periodic potentials can be used to break rigidity, and disprove both conjectures.

1. INTRODUCTION

In this paper we consider one-dimensional Schrödinger operators, both on the real line $\mathbb{R}$ and on the lattice $\mathbb{Z}$. In the first case, they act on $L^2(\mathbb{R})$ and have the form

$$(1) \qquad (H\mathfrak{u})(t) = -\frac{d^2}{dt^2}\mathfrak{u}(t) + V(t)\mathfrak{u}(t),$$

while in the second case they act on $\ell^2(\mathbb{Z})$ and have the form

$$(2) \qquad (H\mathfrak{u})_n = \mathfrak{u}_{n+1} + \mathfrak{u}_{n-1} + V(n)\mathfrak{u}_n.$$

We are interested in the so-called ergodic case, where one considers a measured family of potentials defined by the evaluation of a sampling function along the orbits of a dynamical system. Thus, in the first (continuum) case, we have $V(t) = v(F_t(x))$, where F_t is an ergodic flow and in the second (discrete) case $V(n) = v(f^n(x))$, where f is an ergodic invertible map. We denote the implied fixed probability measure by σ. We will also assume below that flows, maps, and sampling functions are continuous in some compact phase space X and that $\operatorname{supp}\sigma = X$.

By general reasoning, the spectrum of ergodic operators is almost surely constant. In general, the spectral measure is not almost surely independent of $x \in X$, but the ac part of the spectral measure is. There is much work dedicated to the understanding of the ac part of the spectral measure, with most results so far pointing to very rigid behavior [K], [DeS], [CJ] (see also [R] for recent developments regarding non-ergodic potentials). Two natural problems in this direction are:

Problem 1.1. *Does the existence of an absolutely continuous component of the spectrum (for almost every $x \in X$) imply that the potential is almost periodic?*

We recall that an almost periodic potential is one that can be obtained by evaluating a continuous sampling function along an ergodic translation of a compact Abelian group. Another way of formulating this is that the dynamics has a topological almost periodic factor, through which the sampling function factorizes.

Problem 1.2. *Are all eigenfunctions bounded, for almost every energy, with respect to the ac part of the spectral measure (for almost every $x \in X$)?*

Here, by an eigenfunction associated to energy E we mean a generalized solution of $H\mathfrak{u} = E\mathfrak{u}$ (i.e., without the requirement of belonging to $L^2(\mathbb{R})$ or $\ell^2(\mathbb{Z})$).

For the first problem, an affirmative answer has been explicitly conjectured in the discrete case, in what is now known as the *Kotani-Last Conjecture* (recently popularized in [D], [J], and [S], see also the earlier [KK])

For the second problem, and also in the discrete case, the affirmative answer would be a particular case of the so-called *Schrödinger Conjecture* (see [V], §1.7), according to which eigenfunctions should be bounded for almost every energy with respect to the ac part of the spectral measure, for any (possibly non-ergodic) potential. We note that in the continuum case, the corresponding general statement was known to be false if one does not assume the potential to be bounded: The famous counterexample in [MMG] is indeed unbounded both from above and from below (it is also sparse, hence non-ergodic). As it turns out, in such setup the absolutely continuous spectrum is not constrained by any strict "Parseval-like" bound on the average size of eigenfunctions.[1] Such a bound (10) is a key difficulty in our setup, and it is what makes it more similar to another situation of interest (square-integrable potentials), as we will discuss.

Remark 1.3. Here is one example of how those conjectures could be used to deduce further regularity properties. It is known that, almost surely in the essential support of ac spectrum, there is a pair of linearly independent complex-conjugate eigenfunctions $\mathfrak{u}$, $\bar{\mathfrak{u}}$, satisfying $|\mathfrak{u}(t)| = U(F_t(x))$ or $|\mathfrak{u}_n| = U(f^n(x))$, according to the setting, with $U : X \to (0, \infty)$ some L^2-function (depending on E but independent of x), see [DeS]. If the dynamics is minimal then boundedness of the eigenfunctions implies that U is in fact *continuous* [Y], so if the dynamics is almost periodic then the absolute value of these eigenfunctions is itself almost periodic.

Remark 1.4. There are of course many examples of almost periodic potentials with ac spectrum, dating from the KAM based work of Dinaburg-Sinai [DS]. KAM approaches do tend to produce bounded eigenfunctions. In [AFK], it has been proved that if f is an irrational rotation of the circle, and v is analytic, then (up to taking some sufficiently deep renormalization) a non-standard KAM scheme converges almost everywhere in the essential support of the ac part of the spectral measure, so that eigenfunctions are indeed bounded as predicted by the Schrödinger Conjecture.

In this paper we give negative answers to both problems, in both the discrete and the continuous setting.

Theorem 1. *There exists a uniquely ergodic map, a sampling function, and a positive measure set $\Lambda \subset \mathbb{R}$ such that for almost every x, Λ is contained in the essential support of the absolutely continuous spectrum, and for every $E \in \Lambda$ and almost every x, any non-trivial eigenfunction is unbounded.[2]*

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[1]Particularly, moments of growth do not have to be spread out according to the energy, and in fact in [MMG] many eigenfunctions become simultaneously large (in short bursts).

[2]Notice that, by general reasoning, for any E in the spectrum there exists always some x with a one-dimensional subspace of bounded eigenfunctions. On the other hand, if the dynamics is minimal, then the existence of an unbounded eigenfunction for some x implies that there are unbounded eigenfunctions for every x (with the same E).

Theorem 2. *There exists a weak mixing uniquely ergodic map and a non-constant sampling function such that the spectrum has an absolutely continuous component for every x.*

Theorem 3. *There exists a uniquely ergodic flow and a sampling function, such that the spectrum is purely absolutely continuous for almost every x, and for almost every energy in the spectrum, and almost every x, any non-trivial eigenfunction is unbounded.*

Theorem 4. *There exists a weak mixing uniquely ergodic flow and a non-constant sampling function, such that the spectrum is purely absolutely continuous for every x.*

We recall that weak mixing means the absence of a measurable almost periodic factor. In particular, potentials associated to non-constant sampling functions are never almost periodic.

Remark 1.5. One may wonder whether there is some natural condition (stronger than lack of almost periodicity) on the dynamics that would prevent the existence of ac spectrum. After we announced in 2009 the earliest result of this work (a less precise version of Theorem 2), Svetlana Jitomirskaya asked us whether weak mixing would be such a condition. Though our original (unpublished) construction did not yield weak mixing, the underlying mechanism could indeed be used to answer her question, as shown in the argument we present here.

Remark 1.6. Unbounded eigenfunctions can appear with or without almost periodicity: the example provided in the proof of Theorem 3 can be shown to be weak mixing (though it is not done here), while the example provided in the proof of Theorem 1 is almost periodic. In the other direction, the proofs of Theorems 2 and 4 (see Remarks 6.5 and 4.8) show that bounded eigenfunctions are also compatible with weak mixing.

Our methods do give considerable more control on the continuum case (in that we get control on the entire spectrum). The arguments are also much simpler. For this reason, we first develop all arguments in full detail for the continuum case. We then describe more leisurely the additional complications involved in the discrete case.

1.1. **Further perspective.** Besides its natural interest in the theory of orthogonal polynomials and one-dimensional Schrödinger operators, much of the motivation behind the Schrödinger Conjecture lies in its interpretation as a generalization of the sought after non-linear version of Carleson's Theorem about pointwise convergence of the Fourier series of an $L^2(\mathbb{R}/\mathbb{Z})$ function. Recall that this theorem (which solved Lusin's Conjecture) is equivalent to the statement that for any sequence of complex numbers $\{\lambda_n\}_{n\in\mathbb{N}}$ with $\sum |\lambda_n|^2 < \infty$, and for almost every $\theta \in \mathbb{R}$, the series $\sum \lambda_n e^{2\pi i n\theta}$ is bounded.

One simple formulation of a (conjectural) non-linear version of Carleson's Theorem goes as follows: for any sequence of $\mathrm{SL}(2,\mathbb{R})$ matrices $\{A_j\}_{j\in\mathbb{N}}$ such that

$$(3) \qquad\qquad \sum (\ln \|A_j\|)^2 < \infty$$

and for almost every $\theta \in \mathbb{R}$, the sequence $A^{(n)}(\theta) = A_n R_\theta \cdots A_1 R_\theta$ is bounded (here R_θ is the rotation of angle $2\pi\theta$). To see the connection, notice that it is

enough (by polar decomposition) to consider the case where $A_j = D_{e^{\lambda_j}} R_{\beta_j}$, where $\beta_j \in \mathbb{R}$ is arbitrary and $\lambda_j \geq 0$ satisfy $\sum |\lambda_j|^2 < \infty$. Expand

$$(4) \qquad \begin{pmatrix} e^{\lambda_j} & 0 \\ 0 & e^{-\lambda_j} \end{pmatrix} = \sum_{k \geq 0} \frac{\lambda_j^k}{k!} \begin{pmatrix} 1 & 0 \\ 0 & (-1)^k \end{pmatrix},$$

and then expand the product to get

$$(5) \qquad A^{(n)}(\theta) = \sum_{m \geq 0} B_{m,n}(\theta),$$

where the coefficients of $B_{m,n}$ are homogeneous polynomials of degree m on the λ_j. Then a direct computation gives, with $\alpha_j = \sum_{j' \leq j} \beta_{j'}$, $B_{0,n} = R_{n\theta + \alpha_n}$ (which thus has unit norm), while

$$(6) \qquad B_{1,n} = R_{n\theta + \alpha_n} \sum_{j=1}^{n} \lambda_j \begin{pmatrix} \cos 4\pi(\alpha_j + j\theta) & -\sin 4\pi(\alpha_j + j\theta) \\ -\sin 4\pi(\alpha_j + j\theta) & -\cos 4\pi(\alpha_j + j\theta) \end{pmatrix},$$

so that Carleson's Theorem is equivalent to the boundedness of the $B_{1,n}$.

One reason to hope for the almost sure boundedness of the sequence $A^{(n)}(\theta)$ is the validity of an analogue of Parseval's Theorem: taking $N(A) = \ln \frac{\|A\| + \|A\|^{-1}}{2}$ (which is asymptotic to $(\ln \|A\|)^2$ when $\|A\|$ is close to 1), we get

$$(7) \qquad \int N(A^{(n)}(\theta)) d\theta = \sum_{j=1}^{n} N(A_j).$$

This presents a quite strict constraint to the construction of any counterexample. Indeed, (7) implies that $\|A^{(n)}(\theta)\|$ is often bounded: any growth one sees in a certain moment must be compensated later. This oscillation is rather hard to achieve in the nonlinear setting: the product of two large $\mathrm{SL}(2,\mathbb{R})$ matrices is usually even larger than each factor, unless there is some rather precise alignement between their polar decompositions. However, any such alignement would appear likely to be destroyed when θ changes. (Another way to see the difficulty it to recall that the Brownian motion in the hyperbolic plane $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ diverges linearly, while in the real line it is recurrent.)

There is nothing sacred about the above setup (which we chose to start with only for the transparency of the various formulas), and there are many alternative settings where a non-linear analogue of Carleson's Theorem is expected to hold. The basic theme to keep in ming is the goal of showing almost sure boundedness of square-summable perturbations of a parametrized infinite product of elliptic matrices, in some setting where some analogue of Parseval's Theorem holds. For a more detailed discussion (with slightly different setup), see the work of Muscalu-Tao-Thiele [MTT1].

1.1.1. *Schrödinger setting.* The eigenfunctions of discrete Schrödinger operators (2) with eigenvalue E satisfy a second-order difference equation which can be expressed in matrix form

$$(8) \qquad A(E, n, n+1) \cdot \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix}$$

where

$$(9) \qquad A(E, n, n+1) = \begin{pmatrix} E - V(n) & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus writing $A(E, m, n) = A(E, n-1, n) \cdots A(E, m, m+1)$, $n > m$, we see that the the boundedness of eigenfunctions is equivalent to the boundedness of the sequences $A(E, 0, n)$ and $A(E, -n, 0)$.

It turns out that if $V \in \ell^2(\mathbb{Z})$ then the essential support of the ac spectrum is $(-2, 2)$ (a result of Deift-Killip [DeK]). This is also the set of E such that the unperturbed matrices $\begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}$ are elliptic. Thus the expected "Carleson's Theorem for Schrödinger operators" is just the Schrödinger Conjecture restricted to potentials in $\ell^2(\mathbb{Z})$. A partial result in this direction, under a stronger decay condition, is obtained in [CKR] (see also [MTT2] for a discussion of the limitations of this approach in the consideration of general potentials in $\ell^2(\mathbb{Z})$).

Why would one want to believe in the Schrödinger Conjecture for ergodic potentials? In our view, it is due to the validity of an inequality, reminiscent of that implied by Parseval's Theorem in the case of decaying potentials. We state it in terms of eigenfunctions: For almost every $x \in X$, and for almost every E in the essential support Λ of the ac spectrum, there is a pair of linearly independent complex conjugate eigenfunctions $\mathfrak{u}(E, x, n)$ and $\overline{\mathfrak{u}(E, x, n)}$, normalized so that the Wronskian $\mathfrak{u}(E, x, n)\overline{\mathfrak{u}(E, x, n-1)} - \overline{\mathfrak{u}(E, x, n)}\mathfrak{u}(E, x, n-1)$ is i, such that

$$(10) \qquad \frac{1}{2\pi} \int_\Lambda |\mathfrak{u}(E, x, n-1)|^2 + |\mathfrak{u}(E, x, n)|^2 dE \leq 1,$$

with equality in the case of pure ac spectrum.[3] Such an inequality is of course all that is needed to deduce a bound on the average size of the transfer matrices:

$$(11) \qquad \frac{1}{4\pi} \int_\Lambda \|A(E, m, n)\| + \|A(E, m, n)\|^{-1} dE \leq 1.$$

The ergodic setup has one advantage and one disadvantage with respect to the decaying setup (as far as constructing counterexamples is concerned):

1. There is no need to "spare potential" in trying to promote eigenfunction growth,
2. But potential we introduce must reappear (infinitely often), hence (by the trend of products of large matrices to get larger) one risks promoting "too much growth", destroying the ac spectrum due to the need to obey (11). (In other words, we have to "spare ac spectrum".)

The effects of recurrence on transfer matrices growth is hard to neglect: in particular, in the ergodic case, eigenfunctions are known [CJ] to be everywhere bounded in any open interval in the essential support of the ac spectrum (in the ℓ^2 case, the essential support is an interval, and one certainly can not hope for boundedness everywhere).

In fact, it is quite difficult to achieve ac spectrum in the ergodic context, which is of course what is behind the formulation of the Kotani-Last Conjecture. One of the known obstructions is Kotani's Determinism Theorem [K], which can be stated as follows: If there is some ac spectrum, then the stochastic process $\{V(n)\}_{n \in \mathbb{Z}}$

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[3]In general, $\frac{1}{2\pi} \int_\Lambda |\mathfrak{u}(E, x, n-1)|^2 + |\mathfrak{u}(E, x, n)|^2 dE$ is half the sum of the weights of the ac part of the spectral measures associated to δ_n and δ_{n-1}.

is deterministic in the sense that perfect knowledge of the past implies perfect knowledge of the future. The Kotani-Last Conjecture can then be seen as an optimistic quantitative generalization of this result: almost periodicity means that approximate knowledge of $\{V(n)\}_{n\in\mathbb{Z}}$ (i.e., up to ℓ^∞-small error) can be obtained by sufficiently precise knowledge of the sequence in a sufficiently long (but finite) interval.

Remark 1.7. Regarding the essential support of the singular part of the spectral measure, it is well known that bounded eigenfunctions can form at most a one-dimensional subspace. The existence of bounded potentials having no non-trivial bounded eigenfunctions was first established by Jitomirskaya in [Z] (for certain explicit ergodic potentials with singular continuous spectrum).

1.2. **Principles of construction.** As discussed above, a key obstacle to the construction of unbounded eigenfunctions in the absolutely continuous spectrum is the need to obey (11). In fact there are other similar constraints that must be satisfied, for instance, $\frac{1}{N}\sum_{k=0}^{N-1}\|A(E,0,k)\|$ is bounded for almost every E.[4]

The need for unbounded eigenfunctions to oscillate shows that if we introduce growth, we must cancel it at some later scale. This demands very careful matching of the transfer matrices: if $x,y \in \mathrm{SL}(2,\mathbb{R})$ then we have $\|yx\| \ge \|y\|\|x\||\sin\omega|$ where ω is the angle between the contracting eigendirection of y^*y and the expanding eigendirection of xx^*. So unless the eigendirections are closely aligned, if x and y are large then yx is even larger. But as energy changes, the eigendirections move, which can easily destroy a precise match, resulting in growth for nearby eigenfunctions, which will result in losses of the ac spectrum. Notice that (11) shows also the need to spread the moments where growth occurs according to the energy, but this creates further complications regarding the interaction of the transfer matrices in those different scales.

Our approach to avoid uncontrolled growth is based on slow deformation of periodic potentials, the spectrum of which consists of bands. In order to create growth in the first place, we must introduce disorder which eats up part of the ac spectrum. In order to lose only an ϵ-proportion of ac spectrum, we must introduce (in our approach) so little disorder that the corresponding growth is of order ϵ in the bulk of the bands (this is clearly not enough to win, due to the need to spare the ac spectrum). However, we can produce slightly more growth near the edges. The disorder is introduced by slow deformation, and then we unwind it. The importance of slowness in the deformation procedure is that any introduced eigenfunction growth is also unwound. We get back to a bounded setting which allows us to iterate the estimates. What we see in the end is that an eigenfunction will tend to pick up oscillation at some time scales. While those oscillations are not absolutely summable, the process is so slow that their sum would still remain bounded unless there is some coherence of the phases. However, at rare random (and it is here one sees the spreading in the energy) time scales the oscillations do become coherent, so the eigenfunction does become unbounded.

We must of course be very careful in our introducing of disorder at each step. Our chosen mechanism is dictated by the setting. In the continuum, it is possible to introduce tiny amounts of rotation (for the transfer matrices), and we proceed

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[4]Indeed if $\mathfrak{u}(E,x,n)$ and $\overline{\mathfrak{u}(E,x,n)}$ is a pair of complex conjugate eigenfunctions with Wronskian i then $\frac{1}{\pi N}\sum_{0\le k\le N-1}|\mathfrak{u}(x,E,n)|^2$ is the derivative of the integrated density of states.

by a large variation on the axis of rotation. This does not work in the discrete case, so we must instead create a tiny disturbance on the axis of rotation. In order to do so, we work all the time with one-parameter families of periodic potentials that remain coherent in a large part of the spectrum.

In order to construct non almost periodic potentials which have ac spectrum, we consider again perturbations of periodic potentials. We construct two distinct deformations which are largely coherent, but which have slightly distinct periods (in the continuum case). Iterating each independently for a long time, they will slowly lose the coherence, until it has a definite magnitude. Later on they will become coherent again, and we can match both to construct a new periodic potential with large ac spectrum. Geometrically, the dynamics has slightly different speeds at nearby orbits of the phase space, creating macroscopic sliding in long time scales (think of the horocycle flow), though at some later time scale everything becomes periodic. Sliding is naturally incompatible with almost periodicity. Technically, the discrete case is much more delicate, since we can not produce a tiny difference of periods (as it must be an integer), so we use slow deformation along a coherent family of periodic potentials to construct coherent periodic potentials with discrepant periods.

2. Continuum case: preliminaries

We will make use of the usual $\mathrm{SL}(2,\mathbb{R})$ action on $\overline{\mathbb{C}}$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$.

Let d be the hyperbolic distance in the upper half-plane $\mathbb{H}$.

Let $R_\theta = \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}$.

If $A \in \mathrm{SL}(2,\mathbb{R})$ satisfies $|\mathrm{tr} A| < 2$, there exists a unique fixed point $\mathbf{u}(A)$ of A in $\mathbb{H}$. Moreover, A is conjugated in $\mathrm{SL}(2,\mathbb{R})$ to a well defined rotation $R_{\Theta(A)}$, where $\Theta(A) \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. The conjugacy matrix B, satisfying $BAB^{-1} = R_{\Theta(A)}$ is not canonical (one may postcompose B with rotations), but can be chosen to have the form

$$(12) \qquad \mathbf{B}(A) = \frac{1}{(\mathrm{Im}\ \mathbf{u}(A))^{1/2}} \begin{pmatrix} 1 & -\Re\mathbf{u}(A) \\ 0 & \mathrm{Im}\ \mathbf{u}(A) \end{pmatrix}.$$

Notice that $\mathbf{u}$ and $\mathbf{B}$ are analytic functions of A.

Given a continuous function $V : \mathbb{R} \to \mathbb{R}$, we define the transfer matrices $A[V](E,t,s) \in \mathrm{SL}(2,\mathbb{R})$ so that $A[V](E,t,t) = id$ and

$$(13) \qquad \frac{d}{ds} A[V](E,t,s) = \begin{pmatrix} 0 & -E - V(s) \\ 1 & 0 \end{pmatrix} A[V](E,t,s).$$

An eigenfunction of the Schrödinger operator with potential V and energy E is a solution $\mathfrak{u} : \mathbb{R} \to \mathbb{R}^2$ of $\mathfrak{u}(s) = A[V](E,t,s) \cdot \mathfrak{u}(t)$.

We have the following basic monotonicity property:

Lemma 2.1. *If $s > t$ and $|\mathrm{tr} A[V](E,t,s)| < 2$ then*

$$(14) \qquad \frac{d}{dE} \Theta(A[V](E,t,s)) > 0.$$

2.1. **Periodic case.** Assume now that V is periodic of period T. In this case we write $A[V](E,t) = A[V](E,t,t+T)$ and $A[V](E) = A[V](E,0)$. Note that $\text{tr}A[V](E,t) = \text{tr}A[V](E)$ for all $t \in \mathbb{R}$.

The spectrum $\Sigma = \Sigma(V)$ of the Schrödinger operator with potential V is the set of all E with $|\text{tr}A[V](E)| \leq 2$. Let also $\Omega = \Omega(V)$ be the set of all E with $|\text{tr}A[V](E)| < 2$. We note that $\Sigma \setminus \Omega = \partial\Omega$ consists of isolated points.

For $E \in \Omega$, let $u[V](E,t) = \mathbf{u}(A[V](E,t))$ and $u[V](E) = u[V](E,0)$. The integrated density of states (i.d.s.) N is absolutely continuous in this case. It satisfies

$$(15) \qquad \frac{d}{dE}N(E) = \frac{1}{2\pi T}\int_0^T \frac{1}{\text{Im } u[V](E,t)}dt,$$

for each $E \in \Omega$.

In the following results, we assume V to be fixed and write $A(\cdot)$ for $A[V](\cdot)$ and $u(\cdot)$ for $u[V](\cdot)$.

Lemma 2.2. *For almost every $E \in \Sigma$, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ with the following property. Let $\tilde{u}(E,t) \in \mathbb{H}$ be some (not necessarily periodic) solution of $A(E,t,s) \cdot \tilde{u}(E,t) = \tilde{u}(E,s)$. Then*

$$(16) \qquad \frac{1}{NT}\int_0^{NT}\frac{1}{\text{Im }\tilde{u}(E,t)}dt > \frac{1}{T}\int_0^T\frac{1}{\text{Im }u(E,t)}dt - \epsilon.$$

Proof. Let $E \in \Omega$. If $d(\tilde{u}(E,t),u(E,t))$ is large, then at least one of $\frac{1}{\text{Im }\tilde{u}(E,t)}$ and $\frac{1}{\text{Im }\tilde{u}(E,t+T)}$ has to be large, so for $N \geq 2$ we have

$$(17) \qquad \frac{1}{N}\sum_{j=0}^{N-1}\frac{1}{\tilde{u}(E,t+jT)} \geq \frac{1}{\text{Im }u(E,t)}.$$

Assume further that $\Theta(A(E))$ is irrational. Then for any $0 \leq t \leq T$, as N grows the sequence $\tilde{u}(E,t+jT)$, $0 \leq j \leq N-1$, is getting equidistributed in the circle of (hyperbolic radius) $d(\tilde{u}(E,t),u(E,t))$ around $u(E,t)$. One directly computes that if $\tilde{z},\tilde{z}' \in \mathbb{H}$ are symmetric points with respect to some $z \in \mathbb{H}$ then $\frac{1}{2}(\frac{1}{\text{Im }\tilde{z}} + \frac{1}{\text{Im }\tilde{z}'}) \geq \frac{1}{\text{Im }z}$. It follows that for N large

$$(18) \qquad \frac{1}{N}\sum_{j=0}^{N-1}\frac{1}{\tilde{u}(E,t+jT)} \geq \frac{1}{\text{Im }u(E,t)} - \epsilon.$$

Since this estimate is uniform on the solution $\tilde{u}(E,t)$ provided $d(\tilde{u}(E,t),u(E,t))$ is bounded, the result follows. $\square$

Lemma 2.3. *For almost every $E \in \Sigma$, for every $t_0 \in \mathbb{R}$ we have*

$$(19) \qquad \inf_{w\in\mathbb{R}^2,\,\|w\|=1}\sup_{t>t_0}\|A(E,t_0,t)\cdot w\| = \sup_t e^{(d(u(E,t),i)-d(u(E,t_0),i))/2}.$$

Proof. Take $E \in \Omega$ such that $\Theta(A(E))$ is irrational. $\square$

2.2. **Uniquely ergodic case.** Let $\{F_t : X \to X\}_{t\in\mathbb{R}}$, be a continuous flow which is minimal and uniquely ergodic with invariant probability measure σ, and let $v : X \to \mathbb{R}$ be a continuous function. Let $A[F,v](E,x,t,s) = A[V](E,t,s)$ with $V(t) = v(F_t(x))$ Let $\Sigma = \Sigma(F,v)$ be the corresponding spectrum (which is x-independent by minimality). Notice that if v is non-negative and non-identically vanishing, then $\Sigma \subset (0,\infty)$.

Let $N(E)$ be the i.d.s., and $L(E)$ be the Lyapunov exponent, defined by

$$(20) \qquad L(E) = \lim_{T \to \infty} \frac{1}{T} \int \ln \|A[F, v](E, x, 0, T)\| d\sigma(x).$$

For a uniquely ergodic flow, the i.d.s. is not necessarily absolutely continuous. However, we have the following result (due to Kotani, see [D]). Let $\Sigma_0 \subset \Sigma$ be the set of E with $L(E) = 0$.

Lemma 2.4. *We have $\frac{d}{dE} N(E) > 0$ for almost every $E \in \Sigma_0$. Moreover, there exists a measurable function $u[F, v] : \Sigma_0 \times X \to \mathbb{H}$, unique up to $\mathrm{Leb} \times \sigma$-zero measure sets, such that $A[F, v](E, x, 0, t) \cdot u[F, v](E, x) = u[F, v](E, F_t(x))$. This function satisfies*

$$(21) \qquad \frac{d}{dE} N(E) = \frac{1}{2\pi} \int \frac{1}{\mathrm{Im}\ u[F, v](E, x)} d\sigma(x).$$

Notice that when F is T-periodic, we have $u[F, v](E, x) = u[V](E)$ for $V = v(F_t(x))$.

The following is due to Kotani, see [D].

Theorem 5. *Let $F_t : X \to X$ be a uniquely ergodic flow, and let $v : X \to \mathbb{R}$ be continuous. If the Lyapunov exponent vanishes in the spectrum and the ids is absolutely continuous, then the spectral measures are absolutely continuous for almost every $x \in X$.*

2.3. **Solenoidal flows.** If K and K' are compact Abelian groups, a projection $K \to K'$ is a continuous surjective homomorphism.

Let K be a totally disconnected compact Abelian group, and let $i : \mathbb{Z} \to K$ be a homomorphism with dense image. The solenoid associated to (K, i) is the compact Abelian group obtained as the quotient of $K \times \mathbb{R}$ by the subgroup $\{(i(-j), j)\}_{j \in \mathbb{Z}}$. It comes with a canonical projection $\pi : S \to \mathbb{R}/\mathbb{Z}$, $\pi(x, t) = t$.

Given S as above, the solenoidal flow on S is $F_t^S : S \to S$, $F_t^S(x, s) = (x, t + s)$.

A time-change of F^S is a flow $F_t : S \to S$ of the form $F_t(x, s) = (x, h(x, s, t))$ with $t \mapsto h(x, s, t)$ C^1 for each x and s, and such that $w_F(x, s) = \frac{d}{dt} h(x, s, t)|_{t=0}$ is a continuous positive function of (x, s). Notice that any continuous positive function in S generates a time-change.

Notice that a time change of a solenoidal flow is uniquely ergodic, and its invariant probability measure is absolutely continuous with respect to the Haar measure, with continuous positive density proportional to $\frac{1}{w_F}$.

If (K, i) and (K', i') are as above, and there is a (necessarily unique) projection $p_{K, K'} : K \to K'$ such that $p_{K, K'} \circ i = i'$, then we can define a projection $p_{S, S'} : S \to S'$ in the natural way.

If F and F' are time-changes of F^S and $F^{S'}$, we say that F is ϵ-close to the lift of F' if $|\ln w_{F'} \circ p_{S, S'} - \ln w| \le \epsilon$. We say that $v : S \to \mathbb{R}$ is ϵ-close to the lift of $v' : S' \to \mathbb{R}$ if $|v' \circ p_{S, S'} - v| \le \epsilon$.

In all cases we will deal with (e.g., $K = \mathbb{Z}/n\mathbb{Z}$), there is a natural choice for the embedding $i : \mathbb{Z} \to K$. Thus we will often omit the embedding i from the notation below.

2.4. **Projective limits.** An increasing sequence of compact Abelian groups is the data given by a sequence K_j of compact Abelian groups, and of projections $p_{j', j} : K_{j'} \to K_j$, $j' > j$ such that $p_{j_2, j_1} \circ p_{j_3, j_2} = p_{j_3, j_1}$.

Given such an increasing family of compact Abelian groups, there exists a compact Abelian group K and a sequence of projections $p_j : K \to K_j$ such that $p_{j',j} \circ p_{j'} = p_j$ for every $j' > j$, and the p_j separate points in K: one takes K as the set of infinite sequences $x_j \in K_j$ with $p_{j',j}(x_{j'}) = x_j$, endowed with the product topology and obvious group structure. We call K the projective limit of the K_j.

When considering pairs (K_j, i_j) as before, we will assume moreover that the projections are compatible with the embeddings, so that $p_{j',j} = p_{K_{j'},K_j}$.

Notice that if the K_j are totally disconnected then the projective limit is totally disconnected. Moreover, if S_j is the solenoid over K_j, then the projective limit S of the S_j is the solenoid over K.

An immediate application of projective limits is the following:

Lemma 2.5. *Let S_j be an increasing sequence of solenoids, and let S be the projective limit. Let F^j be time-changes of the solenoidal flows F^{S_j}. Let $v_j : S_j \to \mathbb{R}$ be continuous functions. Assume that for $j' > j$, $(F^{j'}, v_{j'})$ is ϵ_j-close to the lift of (F^j, v_j), where $\epsilon_j \to 0$. Then there exists a time-change F of the solenoidal flow F^S, and a continuous function $v : S \to \mathbb{R}$ such that (F, v) is ϵ_j-close to the lift of (F^j, v_j) for every j.*

Proof. Define S as a projective limit of the S_j and take $v = \lim v_j \circ p_j$, $w_F = \lim w_{F_j} \circ p_j$. $\qquad\qquad\square$

2.5. **Lifting properties.** The following is a standard "semi-continuity of the spectrum" property:

Lemma 2.6. *Let F' be a time-change of a solenoidal flow $F^{S'}$, and let $v' : S' \to \mathbb{R}$ be a continuous function. Then for every $M > 0$, $\epsilon > 0$, there exists $\kappa > 0$ such that if either (F, v) is κ-close to the lift of (F', v'), or (F', v') is κ-close to the lift of (F, v), then $\Sigma(F, v) \cap (-\infty, M]$ is contained in the ϵ-neighborhood of $\Sigma(F', v')$, and $\Sigma(F', v') \cap (-\infty, M]$ is contained in the ϵ-neighborhood of $\Sigma(F, v)$.*

It easily implies:

Lemma 2.7. *Let F' be a time-change of a solenoidal flow $F^{S'}$, and let $v' : S' \to \mathbb{R}$ be a continuous function. Then for every $M > 0$, $\epsilon > 0$, there exists $\kappa > 0$ such that if (F, v) is κ-close to the lift of (F', v'), then we have $|\Sigma(F, v) \cap (-\infty, M] \smallsetminus \Sigma(F', v')| < \epsilon$.*

We say that (F, v) is (ϵ_1, C_1, M)-crooked if there is a set $\Gamma \subset \Sigma \cap (-\infty, M]$ such that $|(\Sigma \smallsetminus \Gamma) \cap (-\infty, M]| < \epsilon_1$, and for every $E \in \Gamma$, the set of $x \in X$ such that

$$(22) \qquad \inf_{w \in \mathbb{R}^2, \|w\|=1} \sup_{t>0} \|A[F, v](E, x, 0, t) \cdot w\| > C_1$$

has σ-measure (strictly) larger than $1 - \epsilon_1$.

This notion provides a quantification of how large the eigenfunctions are for most of the parameters. Largeness can be checked in many cases by bounding the $u(E, x)$, see Lemma 2.3.

The following is obvious, but convenient:

Lemma 2.8. *Let F' be a time-change of a solenoidal flow $F^{S'}$, and let $v' : S' \to \mathbb{R}$ be a continuous function. Assume that (F', v') is (ϵ_1, C_1, M)-crooked. Then there exists $\kappa > 0$ such that if (F, v) is κ-close to the lift of (F', v') then (F, v) is (ϵ_1, C_1, M)-crooked.*

Proof. Just use that $v \circ F_t$ is close to $v' \circ F'_t \circ p_{S,S'}$ for t (arbitrarily) bounded. $\square$

We say that (F, v) is (ϵ, M)-good if

$$(23) \qquad \sup_{\Sigma \cap (-\infty, M]} L(E) < \epsilon.$$

It also trivially behaves well under lifts:

Lemma 2.9. *Let F' be a time-change of a solenoidal flow $F^{S'}$, and let $v' : S' \to \mathbb{R}$ be a continuous function. Assume that (F', v') is (ϵ, M)-good. Then there exists $\kappa > 0$ such that if (F, v) is κ-close to the lift of (F', v'), then (F, v) is (ϵ, M)-good.*

We say that (F, v) is (ϵ, M)-nice if

$$(24) \qquad N(M) - \int_{-\infty}^{M} \frac{dN}{dE} dE < \epsilon.$$

Niceness provides a measure of how absolutely continuous the i.d.s. is.

The following deserves an argument.

Lemma 2.10. *Let F' be a time-change of a periodic solenoidal flow $F^{S'}$,[5] and let $v' : S' \to \mathbb{R}$ be a continuous function. Assume that (F', v') is (ϵ, M)-nice. Then there exists $\kappa > 0$ with the following property. Assume that (F, v) is κ-close to the lift of (F', v'), the Lyapunov exponent for (F, v) vanishes in the spectrum, and $|(\Sigma(F', v') \smallsetminus \Sigma(F, v)) \cap (-\infty, M]| < \kappa$. Then (F, v) is (ϵ, M)-nice.*

Proof. Let N, N' be the integrated density of states for (F, v), (F', v'). It is clear that $N(M)$ is close to $N'(M)$. Using Lemma 2.2 and Lemma 2.4, we see that for almost every $E' \in \Sigma(F', v')$, for every $\epsilon' > 0$, there exists $\delta > 0$ such that for almost every $E \in \Sigma$ which is δ-close to E', if $\kappa > 0$ is sufficiently small, we have

$$(25) \qquad \frac{d}{dE} N(E) > \frac{d}{dE} N'(E') - \epsilon'.$$

Integrating over $\Sigma(F, v) \cap (-\infty, M]$, and using that the Lebesgue measure of the spectrum is close, we get $\int_{-\infty}^{M} \frac{d}{dE} N dE$ close to $\int_{-\infty}^{M} \frac{d}{dE'} N' dE'$. The result follows. $\square$

2.6. **Slow deformation.** The following is the basic estimate of [FK].

Lemma 2.11. *Let $J \subset \mathbb{R}$ be a closed interval, and let $A : J \times \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2, \mathbb{R})$ be a smooth function such that $|\mathrm{tr} A(E, t)| < 2$ for $(E, t) \in J \times \mathbb{R}/\mathbb{Z}$. Let $B(E, t) = \mathbf{B}(A(E, t))$, $\theta(E, t) = \Theta(A(E, t))$. Then for every $m, k \in \mathbb{N}$, there exists $n(m) \in \mathbb{N}$ and $C_{k,m} > 0$ such that for every $n \geq n(m)$, there exist smooth functions $B_{(m,n)} : J \times \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2, \mathbb{R})$, $\theta_{(m,n)} : J \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ such that*

1. $\|A_{(m,n)} - R_{\theta_{(m,n)}}\|_{C^k} \leq \frac{C_{k,m}}{n^m}$, *where*

$$(26) \qquad A_{(m,n)}(E, t) = B_{(m,n)}(E, t + \frac{1}{n}) A(E, t) B_{(m,n)}(E, t)^{-1},$$

2. $\|B_{(m,n)} - B\|_{C^k} \leq \frac{C_{k,m}}{n}$,
3. $\|\theta_{(m,n)} - \theta\|_{C^k} \leq \frac{C_{k,m}}{n}$.

---

[5]This result still holds without assuming periodicity.

Proof. Consider first the case $m = 1$. In this case, set $B_{(1,n)} = B$, $\theta_{(1,n)} = \theta$, and the estimate is obvious.

Assume we have proved the result for $m \geq 1$. Let

$$
(27) \qquad B_{(m+1,n)}(E,t) = \mathbf{B}(A_{(m,n)}(E,t))B_{(m,n)}(E,t),
$$

$$
(28) \qquad \theta_{(m+1,n)}(E,t) = \Theta(A_{(m,n)}(E,t)).
$$

The estimates follow from the induction hypothesis. $\qquad\qquad\qquad\square$

Under a monotonicity assumption, it yields a parameter estimate:

Lemma 2.12. *Under the hypothesis of the previous lemma, assume moreover that* $\tilde{\theta}(E) = \int_{\mathbb{R}/\mathbb{Z}} \theta(E,t)dt$ *satisfies* $\frac{d}{dE}\tilde{\theta}(E) \neq 0$ *for every* $E \in J$. *For* $n \in \mathbb{N}$, *let* $A^{(n)} : J \times \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ *be given by*

$$
(29) \qquad A^{(n)}(E,t) = A(E, t + \frac{n-1}{n})A(E, t + \frac{n-2}{n})\cdots A(E, t + \frac{1}{n})A(E,t).
$$

Then there exist functions $\tilde{\theta}^{(n)} : J \to \mathbb{R}/\mathbb{Z}$ *such that for every measurable subset* $Z \subset \mathbb{R}/\mathbb{Z}$,

$$
(30) \qquad \lim_{n \to \infty} |\{E \in J, \tilde{\theta}^{(n)}(E) \in Z\}| = |Z||J|,
$$

with the following property. For every $\delta > 0$,

$$
(31) \qquad \lim_{n \to \infty} \|\mathrm{tr}A^{(n)}(E,t) - 2\cos 2\pi\tilde{\theta}^{(n)}(E)\|_{C^0(J \times \mathbb{R}/\mathbb{Z},\mathbb{R})} = 0,
$$

$$
(32) \qquad \lim_{n \to \infty} \sup_{|\sin 2\pi\tilde{\theta}^{(n)}(E)| > \delta} \|\Theta(A^{(n)}(E,\cdot)) - \tilde{\theta}^{(n)}(E)\|_{C^1(\mathbb{R}/\mathbb{Z},\mathbb{R})} = 0,
$$

$$
(33) \qquad \lim_{n \to \infty} \sup_{|\sin 2\pi\tilde{\theta}^{(n)}(E)| > \delta} \|\mathbf{u}(A^{(n)}(E,\cdot)) - \mathbf{u}(A(E,\cdot))\|_{C^1(\mathbb{R}/\mathbb{Z},\mathbb{C})} = 0.
$$

Proof. Let $B_{(m,n)}$, $A_{(m,n)}$ and $\theta_{(m,n)}$ be as in Lemma 2.11, and let

$$
(34) \qquad A^{(m,n)}(E,t) = B_{(m,n)}(E,t)A^{(n)}(E,t)B_{(m,n)}(E,t)^{-1}.
$$

We have

$$
(35) \qquad A^{(m,n)}(E,t) = A_{(m,n)}(E, t + \frac{n-1}{n})\cdots A_{(m,n)}(E,t).
$$

Let $\theta^{(m,n)}(E,t) = \sum_{j=0}^{n-1} \theta_{(m,n)}(E, t + \frac{j}{n})$. Then

$$
(36) \qquad A^{(m,n)} - R_{\theta^{(m,n)}} = \sum_{j=1}^{n} H_{(m,n,j)},
$$

with

$$
(37) \qquad H_{(m,n,j)} = \sum H_{(m,n,\underline{i})}
$$

where the sum runs over all non-empty sequences $\underline{i} = (i_1, ..., i_j)$ satisfying $0 \leq i_1 < ... < i_j < n$, and $H_{(m,n,\underline{i})}$ is a product $T_{n-1}\cdots T_0$ with $T_l(E,t) = R_{\theta_{(m,n)}(E,t+\frac{l}{n})}$ if $l \neq i_r$ for every $1 \leq r \leq j$, and $T_l(E,t) = A_{(m,n)}(E, t+\frac{l}{n}) - R_{\theta_{(m,n)}(E,t+\frac{l}{n})}$ if $l = i_r$ for some $1 \leq r \leq j$. Then

$$
(38) \qquad \|H_{(m,n,\underline{i})}\|_{C^0} \leq \|A_{(m,n)} - R_{\theta_{(m,n)}}\|_{C^0}^j,
$$

$$(39) \quad \|DH_{(m,n,\underline{i})}(E,t)\|_{C^0} \leq j\|D(A_{(m,n)} - R_{\theta_{(m,n)}})\|_{C^0}\|A_{(m,n)} - R_{\theta_{(m,n)}}\|_{C^0}^{j-1}$$
$$+ (n-j)\|DR_{\theta_{(m,n)}}\|_{C^0}\|A_{(m,n)} - R_{\theta_{(m,n)}}\|_{C^0}^{j},$$

where we write D for the total derivative. Thus

$$(40) \quad \|H_{(m,n,\underline{i})}\|_{C^1} \leq \frac{C_m^j}{n^{mj-1}},$$

$$(41) \quad \|A^{(m,n)} - R_{\theta^{(m,n)}}\|_{C^1} \leq \sum_{j=1}^{n} \frac{C_m^j}{n^{(m-1)j-1}},$$

so that for $m \geq 3$ we have

$$(42) \quad \lim_{n\to\infty} \|A^{(m,n)} - R_{\theta^{(m,n)}}\|_{C^1} = 0.$$

Note that

$$(43) \quad \theta^{(m,n)}(E,t) = n \sum_{k \in n\mathbb{Z}} e^{2\pi ikt} \int_{\mathbb{R}/\mathbb{Z}} \theta_{(m,n)}(E,t)e^{-2\pi ikt}dt.$$

Let $\tilde{\theta}_{(m,n)}(E) = \int_{\mathbb{R}/\mathbb{Z}} \theta_{(m,n)}(E,t)dt$. Then for $m \geq 3$, using that $\theta_{(m,n)}$ is uniformly C^3 for fixed m and $n \to \infty$,

$$(44) \quad \lim_{n\to\infty} \sup_{E \in J} \|\theta^{(m,n)}(E,\cdot) - n\tilde{\theta}_{(m,n)}(E)\|_{C^1(\mathbb{R}/\mathbb{Z},\mathbb{R})} = 0.$$

Since

$$(45) \quad \lim_{n\to\infty} \|\tilde{\theta}_{(m,n)}(E) - \tilde{\theta}(E)\|_{C^1} = 0,$$

and the derivative of $\tilde{\theta}(E)$ is non-vanishing, it follows that for $m \geq 3$ and each measurable set $Z \subset \mathbb{R}/\mathbb{Z}$,

$$(46) \quad \lim_{n\to\infty} |\{E \in J, \tilde{\theta}^{(m,n)}(E) \in Z\}| = |Z||J|.$$

Fix $m \geq 3$ and let $\tilde{\theta}^{(n)} = \tilde{\theta}^{(m,n)}$. Then for n large we get

$$(47) \quad \lim_{n\to\infty} \|\text{tr}A^{(m,n)}(E,t) - 2\cos 2\pi\tilde{\theta}^{(n)}(E)\|_{C^0(J\times\mathbb{R}/\mathbb{Z},\mathbb{R})} = 0,$$

$$(48) \quad \lim_{n\to\infty} \sup_{|\sin 2\pi\tilde{\theta}^{(n)}(E)|>\delta} \|\mathbf{u}(A^{(m,n)}(E,\cdot)) - i\|_{C^1(\mathbb{R}/\mathbb{Z},\mathbb{C})} = 0,$$

$$(49) \quad \lim_{n\to\infty} \sup_{|\sin 2\pi\tilde{\theta}^{(n)}(E)|>\delta} \|\Theta(A^{(m,n)}(E,\cdot)) - \tilde{\theta}^{(n)}\|_{C^1(\mathbb{R}/\mathbb{Z},\text{SL}(2,\mathbb{R}))} = 0.$$

Notice that $\text{tr}A^{(m,n)} = \text{tr}A^{(n)}$ and $\Theta(A^{(m,n)}) = \Theta(A^{(n)})$. Moreover, $\mathbf{u}(A^{(n)}(E,t)) = B_{(m,n)}(E,t)^{-1}\mathbf{u}(A^{(m,n)}(E,t))$. Since

$$(50) \quad \lim_{n\to\infty} \|B_{(m,n)} - B\|_{C^1} = 0,$$

and $B(E,t) \cdot \mathbf{u}(A(E,t)) = i$, we conclude that

$$(51) \quad \lim_{n\to\infty} \sup_{|\sin 2\pi\tilde{\theta}^{(n)}(E)|>\delta} \|\mathbf{u}(A^{(n)}(E,\cdot)) - \mathbf{u}(A(E,\cdot))\|_{C^1(\mathbb{R}/\mathbb{Z},\mathbb{C})} = 0.$$

$\square$

3. Continuum case: unbounded eigenfunctions

The potential we will produce will be a suitable projective limit of periodic potentials. The actual work we need to do is to inductively construct suitable periodic potentials.

3.1. Construction of periodic potentials.

Let $V : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}$ be a continuous function with $V(0) = 0$. For $\delta > 0$, $N \in \mathbb{N}$, $n \in \mathbb{N}$, the (δ, N, n)-padding of V is the continuous function $V' : \mathbb{R}/T'\mathbb{Z} \to \mathbb{R}$, $T' = 2NnT + \delta \sum_{j=0}^{2n-1} \sin^{2N} \pi \frac{j}{2n}$, given by the following conditions:

1. $V'(t) = V(t - a_j)$, $a_j \leq t \leq a_j + NT$, $0 \leq j \leq 2n - 1$,
2. $V'(t) = 0$, $a_j + NT \leq t \leq a_{j+1}$, $0 \leq j \leq 2n - 1$,
3. $a_0 = 0$, $a_{j+1} = a_j + NT + \delta \sin^{2N} \pi \frac{j}{2n}$.

In other words, we repeat the periodic potential Nn times, but with an extra "padding" with a small interval of zeroes every N repetitions. The length of those intervals is slowly modulated, but it is always small (at most δ).

The goal of this section is to establish the following estimate:

Lemma 3.1. Let $V^{(0)} : \mathbb{R}/T^{(0)}\mathbb{Z} \to \mathbb{R}$ be a smooth non-constant, non-negative function with $V^{(0)} = 0$ near 0. Then for every $M, \xi > 0$, there exists $C > 0$ such that for every $C_0 > 0$, for every $\delta > 0$ sufficiently small, there exist $0 < P < \xi\delta^{-1}$, and sequences $N^{(j)}, n^{(j)}$, $1 \leq j \leq P$, such that if we define $V^{(j)} : \mathbb{R}/T^{(j)}\mathbb{Z} \to \mathbb{R}$, $1 \leq j \leq P$ so that $V^{(j)}$ is obtained by $(\delta, N^{(j)}, n^{(j)})$-padding of $V^{(j-1)}$, then there exists a compact subset $\Gamma \subset (0, M] \cap \Omega(V^{(P)})$ such that $|\Sigma(V^{(0)}) \cap (-\infty, M] \setminus \Gamma| < \xi$ and for every $E \in \Gamma$ we have

$$(52) \qquad\qquad \sup_t d(u[V^{(P)}](E, t), i) \geq C_0,$$

$$(53) \qquad\qquad \frac{1}{T^{(P)}} \int_0^{T^{(P)}} d(u[V^{(P)}](E, t), i)dt \leq C.$$

We will need a few preliminary results.

Lemma 3.2. For every $C > 0$, $M > 0$, there exist $C' > 0$ and $\delta_0 > 0$ with the following property. Let $V : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}$ be a smooth function with $V(t) = 0$ near 0,[6] and let $A(\cdot) = A[V](\cdot)$ and $u(\cdot) = u[V](\cdot)$. Let $E_0 \in \Omega(V) \cap [M^{-1}, M]$. Assume that $C^{-1} < d(u(E_0), E_0^{1/2}i) < C$. Then there exists $\epsilon_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$, for every $\kappa > 0$, for every $0 < \delta < \delta_0$, for every N sufficiently large, for every n sufficiently large, letting V' be the (δ, N, n)-padding of v, $A'(\cdot) = A[V'](\cdot)$, $u'(\cdot) = u[V'](\cdot)$, we have the following. There exists a compact set $\Lambda \subset \Omega(V') \cap [E_0 - \epsilon, E_0 + \epsilon]$ such that

1. $|\Lambda| > 2(1 - C'\delta)\epsilon$,
2. For $E \in \Lambda$, $d(u'(E), u(E)) < \kappa$ and $C^{-1} < d(u'(E), E^{1/2}i) < C$,
3. For $E \in \Lambda$,

$$(54) \qquad\qquad \sup_{t \in [0,T']} d(u'(E, t), i) \geq \sup_{t \in [0,T]} d(u(E, t), i),$$

_____

[6] This neighborhood can be arbitrarily small, but this will influence the constants below that depend on v.

4. *For any $C'\delta < \gamma < 1/4$, there exists a compact set $\Lambda' \subset \Lambda$ with $|\Lambda'| > \gamma\epsilon$ such that for $E \in \Lambda'$,*

$$(55) \qquad \sup_{t\in[0,T']} d(u'(E,t),i) \geq \sup_{t\in[0,T]} d(u(E,t),i) + C'^{-1}\frac{\delta}{\gamma}.$$

5. *For $E \in \Lambda$,*

$$(56) \qquad \left| \frac{1}{T'} \int_0^{T'} d(u'(E,t),i)dt - \frac{1}{T} \int_0^T d(u(E,t),i)dt \right| < \kappa,$$

Proof. Let $D(E) = \begin{pmatrix} E^{1/4} & 0 \\ 0 & E^{-1/4} \end{pmatrix}$. Let $G : \mathbb{R}_+ \times \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ be given by $G(E,t) = D(E)R_{\delta\frac{E^{1/2}}{2\pi}\sin^{2N}\pi t}D(E)^{-1}A(E)^N$. We have

$$(57) \qquad A'(E) = G(E, \frac{2n-1}{2n})G(E, \frac{2n-2}{2n}) \cdots G(E, \frac{1}{2n})G(E,0).$$

We can write for E near E_0,

$$(58) \qquad B(E)A(E)B(E)^{-1} = R_{\theta(E)},$$

where $B(E) = \mathbf{B}(A(E))$ and $\theta(E) = \Theta(A(E))$. By Lemma 2.1, θ has non-zero derivative.

Thus we can write

$$(59) \qquad G(E,t) = D(E)R_{\delta\frac{E^{1/2}}{2\pi}\sin^{2N}\pi t}D(E)^{-1}B(E)^{-1}R_{N\theta(E)}B(E).$$

Letting $Q(E) = B(E)D(E)$, we get

$$(60) \qquad \mathrm{tr}G(E,t) = \mathrm{tr}Q(E)R_{\delta\frac{E^{1/2}}{2\pi}\sin^{2N}\pi t}Q(E)^{-1}R_{N\theta(E)}$$

Notice that $Q(E) \notin \mathrm{SO}(2,\mathbb{R})$, since $Q(E)\cdot i \neq i$ (here we use that $B(E)^{-1}\cdot i = u(E) \neq E^{1/2}i = D(E)\cdot i$ for E near E_0). Thus we can write $Q = R^{(1)}D^{(0)}R^{(2)}$, a product of rotation, diagonal and rotation matrices, depending analytically on E. Then

$$(61) \qquad \mathrm{tr}G(E,t) = \mathrm{tr}D^{(0)}(E)R_{\delta\frac{E^{1/2}}{2\pi}\sin^{2N}\pi t}D^{(0)}(E)^{-1}R_{N\theta(E)}.$$

Write $D^{(0)}(E) = \begin{pmatrix} \lambda(E) & 0 \\ 0 & \lambda(E)^{-1} \end{pmatrix}$. We may assume that $\lambda(E) > 1$. Then

$$(62) \qquad \lambda(E) = e^{d(u(E),E^{1/2}i)/2},$$

so that $\frac{1}{2C} < \ln \lambda(E) < \frac{C}{2}$. Then

$$(63) \qquad \begin{aligned} \mathrm{tr}G(E,t) = &2\cos((\delta E^{1/2}\sin^{2N}\pi t) + 2\pi N\theta(E)) \\ &- (\lambda(E) - \lambda(E)^{-1})^2 \sin(\delta E^{1/2}\sin^{2N}\pi t)\sin 2\pi N\theta(E). \end{aligned}$$

Thus

$$(64) \qquad |\mathrm{tr}G(E,t) - 2\cos((\delta E^{1/2}\sin^{2N}\pi t) + 2\pi N\theta(E))| \leq C_1\delta\sin 2\pi N\theta(E).$$

We conclude that if $2N\theta(E)$ is at distance at least $C_2\delta$ from $\mathbb{Z}$, then $|\mathrm{tr}G(E,t)| < 2$.

We conclude that for ϵ sufficiently small, for N sufficiently large, the set of $E \in [E_0 - \epsilon, E_0 + \epsilon]$ such that $|\mathrm{tr}G(E,t)| \geq 2$ for some t has Lebesgue measure at most $C_3\delta\epsilon$. By Lemma 2.12, for n large we will have $|\mathrm{tr}A'(E)| < 2$ for a compact set $\Lambda(\epsilon,\delta,N,n) \subset [E_0 - \epsilon, E_0 + \epsilon]$ of Lebesgue measure at least $2(1 - C_4\delta)\epsilon$. We

may further assume that for $E \in \Lambda(\epsilon, \delta, N, n)$, the sequence $\{j\theta(E)\}_{0 \le j \le N-1}$ is $\frac{1}{100}$ dense   mod 1. Thus for such E, and any $w \in \mathbb{H}$, and any $0 \le t \le T$, there exists $0 \le k \le N-1$ such that $d(A(E, 0, t)A(E)^k \cdot w, i) \ge d(u(E, t), i) + \frac{1}{2}d(w, u(E))$. Taking $w = u'(E, a_m)$ for some $0 \le m \le 2n-1$ (where a_j is as in the definition of a (δ, N, n)-padding), we get

$$(65) \qquad d(u'(E, t + kT + a_m, i) \ge d(u(E, t), i) + \frac{1}{2}d(u'(E, a_m), u(E)).$$

In particular,

$$(66) \qquad \sup_t d(u'(E, t), i) \ge \sup_t d(u(E, t), i) + \frac{1}{2} \max_{0 \le m \le 2n-1} d(u'(E, a_m), u(E)).$$

Lemma 2.12 shows that $u'(E, a_m)$ is near $\mathbf{u}(G(E, \frac{m}{2n}))$ for n large. In particular, $u'(E)$ is near $u(E)$, since $G(E, 0) = A(E)$ and $u'(E, a_n)$ is near $w(E) = \mathbf{u}(G(E, 1/2))$.

We want to estimate the hyperbolic distance between $w(E)$ and $u(E)$ in $\mathbb{H}$. Let $w'(E)$ be the fixed point of $D^{(0)}(E)R_{\delta \frac{E^{1/2}}{2\pi}}D^{(0)}(E)^{-1}R_{N\theta(E)}$ in $\mathbb{H}$. Then $w(E) = B^{-1}R^{(1)} \cdot w'(E)$. Since $u(E) = B^{-1}R^{(1)} \cdot i$, it follows that

$$(67) \qquad d(w(E), u(E)) = d(w'(E), i).$$

But $w'(E)$ is the solution $z \in \mathbb{H}$ of the equation $az^2 + bz + c = 0$, where

$$(68) \qquad a = \cos \delta E^{1/2} \sin 2\pi N\theta(E) + \lambda(E)^{-2} \sin \delta E^{1/2} \cos 2\pi N\theta(E),$$

$$(69) \qquad b = (\lambda(E)^2 - \lambda(E)^{-2}) \sin \delta E^{1/2} \sin 2\pi N\theta(E),$$

$$(70) \qquad c = \cos \delta E^{1/2} \sin 2\pi N\theta(E) + \lambda(E)^2 \sin \delta E^{1/2} \cos 2\pi N\theta(E).$$

Then
(71)

$$\operatorname{Im} w'(E) = \left(1 + \frac{(\lambda(E)^2 - \lambda(E)^{-2}) \sin \delta E^{1/2} \cos 2\pi N\theta(E)}{\cos \delta E^{1/2} \sin 2\pi N\theta(E) + \lambda(E)^{-2} \sin \delta E^{1/2} \cos 2\pi N\theta(E)} \right.$$
$$\left. - \frac{(\lambda(E)^2 - \lambda(E)^{-2})^2 \sin^2 \delta E^{1/2} \sin^2 2\pi N\theta(E)}{4(\cos \delta E^{1/2} \sin 2\pi N\theta(E) + \lambda(E)^{-2} \sin \delta E^{1/2} \cos 2\pi N\theta(E))^2} \right)^{1/2}$$

Under the condition that $2N\theta(E)$ is at distance at least $C_2 \delta$ from $\mathbb{Z}$, we have

$$(72) \qquad \left| \frac{(\lambda(E)^2 - \lambda(E)^{-2})^2 \sin^2 \delta E^{1/2} \sin^2 2\pi N\theta(E)}{4(\cos \delta E^{1/2} \sin 2\pi N\theta(E) + \lambda(E)^{-2} \sin \delta E^{1/2} \cos 2\pi N\theta(E))^2} \right| \le C_5 \delta^2,$$

(73)

$$\left| \frac{(\lambda(E)^2 - \lambda(E)^{-2}) \sin \delta E^{1/2} \cos 2\pi N\theta(E)}{\cos \delta E^{1/2} \sin 2\pi N\theta(E) + \lambda(E)^{-2} \sin \delta E^{1/2} \cos 2\pi N\theta(E)} \right| \ge C_6^{-1} \delta \cot 2\pi N\theta(E).$$

If $2N\theta(E)$ is at distance $C_2 \delta < \gamma < 1/4$ from $\mathbb{Z}$ then

$$(74) \qquad \left| \frac{(\lambda(E)^2 - \lambda(E)^{-2}) \sin \delta E^{1/2} \cos 2\pi N\theta(E)}{\cos \delta E^{1/2} \sin 2\pi N\theta(E) + \lambda(E)^{-2} \sin \delta E^{1/2} \cos 2\pi N\theta(E)} \right| \ge C_7^{-1} \frac{\delta}{\gamma},$$

so that

$$(75) \qquad d(w'(E), i) \ge C_8^{-1} \frac{\delta}{\gamma}.$$

It follows that in this case

$$(76) \qquad \sup_t u'(E,t) \geq \sup_t u(E,t) + C_9^{-1}\frac{\delta}{\gamma}.$$

For $C_2\delta < \gamma < 1/4$, let $\Lambda'(\epsilon,\delta,N,n,\gamma)$ be the set of $E \in \Lambda(\epsilon,\delta,N,n)$ such that $2N\theta(E)$ is at distance at most γ from $\mathbb{Z}$. Since θ has non-zero derivative, we have $|\Lambda'(\epsilon,\delta,N,n,\gamma)| \geq \frac{3}{2}\gamma\epsilon$, for ϵ small, N sufficiently large and n sufficiently large.

To conclude, let us show that if $E \in \Lambda(\epsilon,\delta,N,n)$ and $2N\theta(E)$ is $C_2\delta$-away from $\mathbb{Z}$, then

$$(77) \qquad \frac{1}{T'}\int_0^{T'} d(u'(E,t),i)dt - \frac{1}{T}\int_0^T d(u(E,t),i)dt$$

is small. The formulas for w' imply that $u'(E,t)$ is at bounded hyperbolic distance from some $u(E,t')$. In fact, if $a_j \leq t \leq a_{j+1}$ then $u'(E,t)$ is at bounded hyperbolic distance from $u(E,t-a_j)$. Moreover, if $a_j \leq t \leq a_j + T$, then $A(E,0,t-a_j)^{-1}u'(E,t)$ is near $\mathbf{u}(G(E,\frac{j}{2n}))$. If $\frac{j}{2n}$ is not close to $\frac{1}{2}$, the estimates give that the fixed point of $G(E,\frac{j}{2n})$ is close to $u(E)$, provided N is large. It follows that $u'(E,t)$ is near $u(E,t-a_j)$. The result follows. $\qquad \square$

Lemma 3.3. *For every $C > 0$, $M > 0$, there exist $C' > 0$ and $\delta_0 > 0$ with the following property. Let $V : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}$ be a smooth non-negative function with $V(t) = 0$ near 0. Let $\Xi \subset \Omega(V) \cap [M^{-1},M]$ be a compact subset such that $C^{-1} < d(u[V](E),E^{1/2}i) < C$ for every $E \in \Lambda$. Then for every $\kappa > 0$, $R \in \mathbb{N}$, for every $0 < \delta < \delta_0$, for every N sufficiently large, for every n sufficiently large, if $V' : \mathbb{R}/T'\mathbb{Z} \to \mathbb{R}$ is the (δ,N,n)-padding of v, then there exists a compact subset $\Xi' \subset \Xi \cap \Omega(V')$ such that*

1. *For $j \geq 0$, the conditional probability that $E \in \Xi$ belongs to Ξ', given that $\frac{j}{R} \leq \sup_t d(u[V](E,t),i) < \frac{j+1}{R}$ is at least $1 - 2C'$,*
2. *For every $E \in \Xi'$, $d(u[V'](E),u[V](E)) < \kappa$ and $C^{-1} < d(u[V'](E),E^{1/2}i) < C$,*
3. *For every $E \in \Xi'$,*

$$(78) \qquad \sup_t d(u[V'](E,t),i) \geq \sup_t d(u[V'](E,t),i),$$

4. *For $j \geq 0$, and for every $C'\delta < \gamma < 1/4$, the conditional probability that $E \in \Xi$ belongs to Ξ' and*

$$(79) \qquad \sup_t d(u[V'](E,t),i) > \sup_t d(u[V](E,t),i) + C'^{-1}\frac{\delta}{\gamma},$$

given that $\frac{j}{R} \leq \sup_t d(u[V](E,t),i) < \frac{j+1}{R}$ is at least $\frac{\gamma}{3}$.
5. *For every $E \in \Xi'$,*

$$(80) \qquad \left|\frac{1}{T'}\int_0^{T'} d(u[V'](E,t),i)dt - \frac{1}{T}\int_0^T d(u[V](E,t),i)dt\right| < \kappa.$$

Proof. Follows from the previous lemma by a covering argument. (Notice that the statements about conditional probabilities are automatic for large j, since $\sup_t d(u[V](E,t),E^{1/2}i)$ is bounded by compactness of Ξ.) $\qquad \square$

Proof of Lemma 3.1. Notice that by non-constancy of $V^{(0)}$, $u^{(0)}(E) \neq E^{1/2}i$ for almost every $E \in \Sigma(V^{(0)})$. Up to increasing M, we can assume that $\inf \Sigma(V^{(0)}) >$

M^{-1}. Then for sufficiently large $C > 0$, there exists a compact subset $\Xi^{(0)} \subset \Sigma(V^{(0)}) \cap (-\infty, M] \cap \Omega(V^{(0)})$ such that $|(\Sigma(V^{(0)}) \smallsetminus \Xi^{(0)}) \cap (-\infty, M]| < \frac{\xi}{2}$, and for every $E \in \Xi^{(0)}$ we have

$$(81) \qquad\qquad C^{-1} < d(u[V^{(0)}](E), E^{1/2}i) < C,$$

and

$$(82) \qquad\qquad \sup_t d(u[V^{(0)}](E,t),i) < \frac{C}{2}.$$

Let $\delta_0 = \delta_0(C, M)$ and $C' = C'(C, M)$ be as in Lemma 3.3.

Let P be maximal so that $(1 - 2C'\delta)^P > 1 - \frac{\xi}{2M}$. Choose very small $0 < \delta < \delta_0$, choose $R \in \mathbb{N}$ very large (in particular, much larger than δ^{-1}), and take $\kappa > 0$ very small. Define sequences $V^{(j)}$, $\Xi^{(j)}$, $1 \le j \le P$, so that $V^{(j)}$, $\Xi^{(j)}$ is obtained by applying Lemma 3.3 to $V^{(j-1)}$, $\Xi^{(j-1)}$. It follows that

$$(83) \qquad |\Sigma(V^{(0)}) \cap (-\infty, M] \smallsetminus \Xi^{(P)}| \le \frac{\xi}{2} + M(1 - (1 - 2C'\delta)^P) < \xi.$$

It also follows that

$$(84) \qquad\qquad \frac{1}{T^{(P)}} \int_0^{T^{(P)}} d(u[V^{(P)}](E,t),i)dt \le \frac{C}{2} + \kappa P < C.$$

Let Z_j, $0 \le j \le P$, be random variables on $\Xi^{(0)}$ given as follows. If $E \notin \Xi^{(j)}$, let $Z_j = Z_{j-1} + 1$. Otherwise, let $Z_j = \frac{j}{R}$, where j is maximal with $\sup_t d(u[V^{(j)}](E,t),i) \ge \frac{j}{R}$.

Let $L \subset \mathbb{N}$ be the set of all l with $4\delta C'^{-1}R < l < C'^{-2}R$. We have $Z_0 \ge 0$, and the conditional probability that $Z_j - Z_{j-1} \ge \frac{l}{R} = C'^{-1}\frac{\delta}{\gamma}$, given Z_{j-1} is at least $\frac{\gamma}{3} = \frac{\delta R}{3lC'}$, provided $C'\delta < \gamma < 1/4$, i.e. $l \in L$. Consider i.i.d. random variables W_j, $1 \le j \le P$, taking only values of the form $\frac{l}{R}$ with $l = 0$ or $l \in L$, and such that

$$(85) \qquad\qquad p(W_j \ge \frac{l}{R}) = \frac{\delta R}{3lC'}.$$

whenever $l \in L$. Since $Z_0 \ge 0$ and $p(Z_j - Z_{j-1} \ge \frac{l}{R}|Z_{j-1}) \ge p(W_j \ge \frac{l}{R})$ for every $l \in \mathbb{Z}$, we get

$$(86) \qquad\qquad p(Z^{(P)} \ge \frac{m}{R}) \ge p(\sum_{j=1}^P W_j \ge \frac{m}{R})$$

for every $m \in \mathbb{Z}$.

To conclude, it is enough to show that $p(\sum_{j=1}^P W_j < C_0) < \xi/2$, provided δ is sufficiently small.

By (85), for $C'' < k < -C'' - \ln \delta$, we have $p(2^k\delta < W_j < 2^{k+1}\delta) > C''^{-1}2^{-k}$ (here C'' is an appropriately large constant depending on C', and we are using that R is much larger than δ^{-1}). We also have $P \ge C'''^{-1}\delta^{-1}$ for some constant C''' depending on M, ξ and C'. By the Law of Large Numbers, for each $D \in \mathbb{N}$, and each $C'' < k \le D$, if δ is sufficiently small, then with probability at least $1 - \frac{\xi}{4D}$, we will have $2^k\delta < W_j < 2^{k+1}\delta$ for a set of $1 \le j \le P$ of cardinality at least $C''^{-1}2^{-k-1}P \ge C'''^{-1}C''^{-1}2^{-k-1}\delta^{-1}$. This implies that, with probability at least $1 - \frac{\xi}{4}$, $\sum W_j \ge \frac{D-[C'']}{2C'''C''}$. The result follows by taking $D \ge C'' + 2C'''C''C_0$. $\qquad\square$

3.2. **Passing to the limit.**

Lemma 3.4. *Let $F_t : S \to S$ be a T-periodic time-change of a solenoidal flow, and let $V : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}$, $v : S \to \mathbb{R}$ be continuous functions such that $V(t) = v(F_t(0))$ is smooth. Assume that $V(t) = 0$ for $T - \epsilon_0 \leq t \leq T$ for some $0 < \epsilon_0 < T$. Then for $0 < \delta < \epsilon_0$, and for every $N, n \in \mathbb{N}$, the (δ, N, n)-padding V' of V has the form $V'(t) = v'(F'_t(0))$, where $F'_t : S' \to S'$ is a T'-periodic time-change of a solenoidal flow, $v' : S' \to \mathbb{R}$ is continuous, and (F', v') is $\frac{\delta}{\epsilon_0}$-close to a lift of (F_t, v).*

Proof. Let $U_0 = \{F_t(0), T - \epsilon_0 < t < T\}$. We take S' as the Nn-cyclic cover of S. Let $p : S' \to S$ be the corresponding projection. Let $U'_{0,j}$, $0 \leq j \leq 2nN - 1$, be the connected components of $U'_0 = p^{-1}(U_0)$, labeled so that they are positively cyclically ordered and such that the right boundary of $U'_{0,2nN-1}$ is 0. Then there exists a continuous non-positive function $\rho : S' \to \mathbb{R}$ such that $\rho = 0$ outside U'_0 and any $U'_{0,j}$ with j not divisible by N, and such that

$$(87) \qquad \int_{U'_{0,j}} \frac{1}{e^{\rho(x)} w_F(p(x))} dx = \epsilon_0 + \delta \sin^{2N} \pi \frac{j}{N}$$

is equal to if j is divisible by N. Indeed, we can take $\|\rho\|_{C^0}$ arbitrarily close to $\ln \frac{\epsilon_0 + \delta}{\epsilon_0}$, and hence less than $\frac{\delta}{\epsilon_0}$.

The result then follows with $v' = v \circ p$, and $w_{F'} = e^\rho w_F \circ p$. $\qquad \square$

Lemma 3.5. *Let $F_t : S \to S$ be a periodic time-change of a solenoidal flow, and let $v : S \to \mathbb{R}$ be continuous non-constant non-negative function such that $t \mapsto v(F_t(0))$ is smooth and $v(F_t(0)) = 0$ for t near 0. Then for every $\epsilon_1, C_1, M, \kappa > 0$, there exists a periodic time-change of a solenoidal flow $F'_t : S' \to S'$, and a continuous non-constant non-negative function such that $t \mapsto v'(F'_t(0))$ is smooth, $v'(F'_t(0)) = 0$ for t near 0, (F', v') is κ-close to a lift of (F, v), and (F', v') is (ϵ_1, C_1, M)-crooked.*

Proof. Let $\epsilon_0 > 0$ be such that $v(F_t(0)) = 0$ for $T - \epsilon_0 \leq t \leq T$. Apply Lemma 3.1 to $V^{(0)}(t) = v(F_t(0))$, with parameters C_0 and $\xi < \epsilon_0 \kappa$ to be specified below, to get Γ and P, and then Lemma 3.4 P times, to get (F', v') with $V^{(P)}(t) = v'(F'_t(0))$ such that (F', v') is $\frac{\xi}{\epsilon_0}$-close to a lift of (F, v). By Lemma 2.7, if ξ is small then $|\Sigma(V^{(P)}) \cap (-\infty, M] \setminus \Sigma(V^{(0)})| < \epsilon_1/2$, and if additionally $0 < \xi < \epsilon_1/2$ we conclude that $|\Sigma(V^{(P)}) \cap (-\infty, M] \setminus \Gamma| < \epsilon_1$. Fix such ξ and let C be as in Lemma 3.1.

By (53), for every $E \in \Gamma$, the set Z_E of all $t \in \mathbb{R}/T^{(P)}\mathbb{Z}$ with

$$(88) \qquad d(u[V^{(P)}](E, t), i) \leq \frac{C}{\epsilon_1}$$

has measure at least $(1 - \epsilon_1)T^{(P)}$. By Lemma 2.3, for almost every $E \in \Gamma$, $t_0 \in Z_E$ implies that

$$(89) \qquad \inf_{w \in \mathbb{R}^2, \|w\|=1} \sup_{t > t_0} \|A[V^{(P)}](E, t_0, t)\| \geq \left(\frac{C_0 \epsilon_1}{C} \right)^{1/2},$$

So by taking $C_0 = C_1^2 C / \epsilon_1$, we get that (F', v') is (ϵ_1, C_1, M)-crooked. $\qquad \square$

Proof of Theorem 3. Let $V^{(0)} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be a smooth non-constant non-negative periodic function with $V^{(0)}(t) = 0$ near 0. We can see $\mathbb{R}/\mathbb{Z}$ as the solenoid $S^{(0)}$ corresponding to the trivial group. Let $F^{(0)}$ be the solenoidal flow on $S^{(0)}$.

Let $\kappa_0 = 1$. Apply Lemma 3.5 inductively to obtain a sequence $(F^{(j)}, v^{(j)})$, $j \geq 1$ such that

1. $(F^{(j)}, v^{(j)})$ is $(2^{-j}, 2^j, 2^j)$-crooked,
2. $|(\Sigma(F^{(j)}, v^{(j)}) \smallsetminus \Sigma(F^{(j-1)}, v^{(j-1)})) \cap (-\infty, 2^j]| < \kappa_j$,
3. $(F^{(j)}, v^{(j)})$ is κ_j-close to the lift of $(F^{(j-1)}, v^{(j-1)})$,
4. $\kappa_j < \kappa_{j-1}/2$ is chosen so small that if (F, v) is $2\kappa_j$-close to the lift of $(F^{(j-1)}, v^{(j-1)})$, and $|(\Sigma(F, v) \smallsetminus \Sigma(F^{(j-1)}, v^{(j-1)})) \cap (-\infty, 2^j]| < 2\kappa_j$, then (F, v) is $(2^{-j}, 2^j)$-nice (use Lemma 2.10), $(2^{-j}, 2^j)$-good (use Lemma 2.9), and if $j \geq 2$, $(2^{1-j}, 2^{j-1}, 2^{j-1})$-crooked (use Lemma 2.8).

Let (F, v) be the limit of the $(F^{(j)}, v^{(j)})$ obtained by Lemma 2.5. Then (F, v) is $(2^{-j}, 2^j)$-good for all j, so the Lyapunov exponent vanishes in the spectrum. Moreover, (F, v) is $(2^{-j}, 2^j)$-nice for all j, so the i.d.s. is absolutely continuous.[7] By Theorem 5, for almost every $x \in S$ the spectral measure is purely absolutely continuous. By construction, (F, v) is $(2^{-j}, 2^j, 2^j)$-crooked for all j. Thus for almost every E in the spectrum, for almost every $x \in S$, all non-trivial eigenfunctions are unbounded. $\qquad\square$

4. Continuum case: breaking almost periodicity

The example discussed in the previous section can be verified to be not almost periodic. Here we will discuss a simpler example that will be easier to analyze.

4.1. **Spectral measure.** Given a bounded continuous function $V : \mathbb{R} \to \mathbb{R}$, we denote by μ_V the spectral measure for the line Schrödinger operator. It has some basic continuity property:

Lemma 4.1. *Let $V : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function, and let $V^{(n)} : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$ be a sequence of uniformly bounded continuous functions such that $V^{(n)} \to V$ uniformly on compact subsets of $\mathbb{R}$. Then $\int \phi d\mu_{V^{(n)}} \to \int \phi d\mu_V$ for every compactly supported continuous function $\phi : \mathbb{R} \to \mathbb{R}$.*

We will only need explicit formulas for the spectral measure in the case of periodic potentials. Let $V : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}$ be continuous, and denote its shifts by $V_s : t \mapsto V(s + t)$. Then μ_{V_s} is absolutely continuous and

$$(90) \qquad\qquad \frac{d}{dE}\mu_{V_s} = \frac{1}{\operatorname{Im} u[V](E, s)}.$$

For $C > 0$, let $\mu_{V_s, C}$ be the restriction of μ_{V_s} to the set of E with $|\frac{d}{dE}\mu_{V_s}| < C$. We say that a periodic v is (ϵ, C, M)-uniform if $\mu_{V_s}(-\infty, M] - \mu_{V_s, C}(-\infty, M] < \epsilon$ for every s.

We clearly have:

Lemma 4.2. *For every periodic V, $\epsilon > 0$, $M > 0$, there exists $C > 0$ such that V is (ϵ, C, M)-uniform.*

4.2. **Weak mixing.** Let $F'_t : S' \to S'$ be a time-change of a periodic solenoidal flow. We say that a time-change $F_t : S \to S$ of a solenoidal flow is (N, F')-mixed if S projects onto S' (through p), and for every $1 \leq j \leq N$, there exists $t_j > 0$ and compact subsets $U_j, V_j \subset S$ with Haar measure strictly larger than $1/3$, such that for each $x \in U_j$ there exists $|t| < \frac{1}{N}$ such that $p(F_{t_j}(x)) = F'_t(p(x))$, and for each $x \in V_j$, there exists $|t - \frac{j}{N}| < \frac{1}{N}$ such that $p(F_{t_j}(x)) = F'_t(p(x))$.

---

[7]Notice that since $\Sigma(F, v)$ contains $\bigcap_{j' \geq j} \Sigma(F^{(j')}, v^{(j')})$ (see Lemma 2.6), we must have $|(\Sigma(F, v) \smallsetminus \Sigma(F^{(j-1)}, v^{(j-1)}) \smallsetminus (-\infty, 2^j]| < 2\kappa_j$.

Lemma 4.3. *Let F be (N, F')-mixed. Then there exists $\kappa > 0$ such that if $\tilde{F}$ is κ-close to the lift of F then $\tilde{F}$ if (N, F') mixed.*

Lemma 4.4. *Let F be the projective limit of $F^{(n)}$, and assume that for every $N \in \mathbb{N}$, for every n sufficiently large, F is $(N, F^{(n)})$-mixed. Then F is weak mixing.*

Proof. If F is not weak mixing, then there exists a non-trivial eigenfunction, i.e., a measurable function $\psi : S \to S^1$ such that $\psi \circ F_t = e^{2\pi i \theta t} \psi$ for some $\theta \in \mathbb{R} \smallsetminus \{0\}$. Let $\psi^{(n)} : S^{(n)} \to \mathbb{C}$ be the expected value of ψ on $p_{S,S^{(n)}}^{-1}(x)$ (with respect to the Haar measure on S). Then $\psi^{(n)} \circ p_{S,S^{(n)}}$ converges to ψ almost everywhere (Martingale Convergence Theorem).

Since ψ is an eigenfunction, $t \mapsto \psi^{(n)}(F_t^{(n)}(0))$ is continuous, uniformly on t and n.

By the definition of projective limit,

$$(91) \qquad \lim_{n \to \infty} \sup_{x \in S^{(n)}} \sup_{0 \le t \le 1} |\psi^{(n)}(F_t^{(n)}(x) - e^{2\pi i \theta t} \psi^{(n)}(x)| = 0.$$

Thus, for $x \in U_j$, $\psi^{(n)}(p_{S,S^{(n)}}(F_{t_j}(x)))$ is close to $\psi^{(n)}(p_{S,S^{(n)}}(x))$. Since the first is close to $\psi(F_{t_j}(x))$ and the second is close to $\psi(x)$ for most $x \in U_j$, this shows that θt_j is close to an integer.

A similar argument using V_j, shows that $\theta(t_j - \frac{j}{N})$ is close to an integer, so that $\theta \frac{j}{N}$ is close to an integer. Since $1 \le j \le N$ is arbitrary, we conclude that $\theta = 0$. $\square$

4.3. **The construction.** Let $V : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}$ be a continuous function with $V(0) = 0$. For $\delta > 0$, $n \in \mathbb{N}$, the (δ, n)-padding (a simplified version of a (δ, N, n)-padding) of V is the continuous function $V' : \mathbb{R}/T'\mathbb{Z} \to \mathbb{R}$, $T' = 2nT + \delta n$, given by the following conditions:

1. $V'(t) = V(t - a_j)$, $a_j \le t \le a_j + T$, $0 \le j \le 2n - 1$,
2. $V'(t) = 0$, $a_j + T \le t \le a_{j+1}$, $0 \le j \le 2n - 1$,
3. $a_j = jT$, $0 \le j \le n$, $a_j = jT + (j - n)\delta$, $n + 1 \le j \le 2n$.

Lemma 4.5. *Let F, v, V, ϵ_0 be as in Lemma 3.4. Then for every $\delta > 0$ sufficiently small, for every $N \in \mathbb{N}$, for every n sufficiently large, the (δ, n)-padding V' of V has the form $V'(t) = v'(F_t'(0))$, where $F_t' : S' \to S'$ is a T'-periodic time-change of a solenoidal flow, $v' : S' \to \mathbb{R}$ is continuous, (F', v') is $\frac{\delta}{\epsilon_0}$-close to a lift of (F, v), and (F', v') is (N, F)-mixed.*

Proof. Let $N_S \in \mathbb{N}$ be the period of the solenoidal flow F_t^S. Define S' as the $2n$-cover of S. Define a continuous function $\rho : S \to \mathbb{R}$ supported on $\{F_t(0), T - \epsilon_0 < t < T\}$ such that $\int_0^T e^{-\rho(F_t(0))} dt = T + \delta$. As in Lemma 3.4, we can choose ρ with $\|\rho\|_{C^0} < \frac{\delta}{\epsilon_0}$. Let $\rho' : S' \to \mathbb{R}$ be defined so that $\rho' = 0$ on $[0, nN_S]$ and $\rho' = \rho \circ p_{S',S}$ on $[nN_S, 2nN_S]$.

Let F' be the solenoidal flow with $w_{F'} = e^\rho w \circ p_{S',S}$, and let $v' = v \circ p_{S',S}$. All properties, but the last one, follow as in Lemma 3.4. For the last property, notice that if $t_j = [\frac{j}{\delta N}](T + \delta)$ then for $x \in \{F_s'(0), 0 \le s \le nT - t_j\}$ we have $p_{S',S} \circ F_{t_j}'(x) = F_{[j/(\delta N)]\delta}(p_{S',S}(x))$, which belongs to $\{F_s(p_{S',S}(x)), \frac{j}{N} - \delta \le s \le \frac{j}{N}\}$, and for $x \in \{F_s'(0), nT \le s \le T' - t_j\}$ we have $p_{S',S} \circ F_{t_j}'(x) = p_{S',S}(x)$. $\square$

Lemma 4.6. *Let $V : \mathbb{R}/T\mathbb{Z}$ be a continuous function with $V(0) = 0$. If V is (ϵ, C, M)-uniform then for $\delta > 0$ sufficiently small, for every $n \in \mathbb{N}$, if V' is the (δ, n)-padding of V, then V' is (ϵ, C, M)-uniform.*

Proof. Let $A(\cdot) = A[V](\cdot)$. Let $J \subset \Omega(V) \cap (-\infty, M]$ be a finite union of closed intervals such that

$$\text{(92)} \qquad \sup_s \mu_{V_s}(-\infty, M] - \mu_{V_s, C}(J) < \epsilon_0 < \epsilon,$$

where V_s is the shift of V and $\mu_{V_s, C}$ is the truncation of the spectral measure.

If $E \in \Omega(V)$, then $B(E)A(E)B(E)^{-1} = R_{\theta(E)}$, where $B = \mathbf{B}(A(E))$ and $\theta(E) = \Theta(A(E))$ are analytic functions and $\frac{d}{dE}\theta(E) > 0$. Let

$$\text{(93)} \qquad A_\delta(E) = D(E)R_{\delta\frac{E^{1/2}}{2\pi}}D(E)^{-1}A(E),$$

where $D(E) = \begin{pmatrix} E^{1/4} & 0 \\ 0 & E^{-1/4} \end{pmatrix}$. Then

$$\text{(94)} \qquad A'(E) = A_\delta(E)^n A(E)^n,$$

where $A'(\cdot) = A[V'](\cdot)$.

For every $\kappa > 0$, for $\delta > 0$ sufficiently small, it is clear that for every $0 \le t \le \delta$, for every $E \in J$, we have $d(H(t) \cdot u(E), u(E)) < \kappa$, where $H(t) = D(E)R_{t\frac{E^{1/2}}{2\pi}}D(E)^{-1}$ is the exponential of $t \begin{pmatrix} 0 & -E \\ 1 & 0 \end{pmatrix}$ and $u(\cdot) = u[V](\cdot)$.

Notice that for $\delta > 0$ sufficiently small, we have $|\text{tr}A_\delta(E)| < 2$ for $E \in J$. Moreover, $B_\delta(E) = \mathbf{B}(A_\delta(E))$ and $\theta_\delta(E) = \Theta(A_\delta(E))$ converge to $B(E)$ and $\theta(E)$, when $\delta \to 0$, as analytic functions of $E \in J$. In particular,

$$\text{(95)} \qquad \limsup_{\delta \to 0} \sup_n \sup_{E \in J} \|B(E)A'(E)B(E)^{-1} - R_{n(\theta(E)+\theta_\delta(E))}\| = 0.$$

For $0 < \eta < 1/2$, let $J_{\delta,n,\eta} \subset J$ be the set of all E such that $2n(\theta(E) + \theta_\delta(E))$ is at distance at least η from $\mathbb{Z}$. Since $\frac{d}{dE}\theta(E) > 0$ and $\frac{d}{dE}\theta_\delta(E) > 0$, we get, for every $\delta > 0$ small,

$$\text{(96)} \qquad \lim_{n \to \infty} |J_{\delta,n,\eta}| = (1 - 2\eta)|J|.$$

For every $0 < \eta < 1/2$ and $\kappa > 0$, if δ is sufficiently small, then for every n and for every $E \in J_{\delta,n,\eta}$, we have $E \in \Omega(V')$ and $d(u'(E), u_\delta(E)) + d(u_\delta(E), u(E)) < \kappa$, where $u'(\cdot) = u[V'](\cdot)$ and $u_\delta(E) = \mathbf{u}(A_\delta(E))$. Notice that $A(E)^j \cdot u'(E) = u'(E, a_j)$ for $0 \le j \le n$, and $A_\delta(E)^{2n-j} \cdot u'(E, a_j) = u'(E)$ for $n \le j \le 2n$. In particular,

$$\text{(97)} \qquad d(u'(E, a_j), u(E)) < \kappa.$$

Thus for $0 \le j \le 2n - 1$ and $a_j \le t \le a_j + T$, we get

$$\text{(98)} \qquad d(u'(E, t), u(E, t - a_j)) < \kappa.$$

For $n \le j \le 2n - 1$ and $a_j + T \le t \le a_j + T + \delta$, we have $H(a_j + T + \delta - t)u'(E, t) = u'(E, a_{j+1})$, so that

$$\text{(99)} \qquad d(u'(E, t), u(E)) = d(u'(E, a_{j+1}), H(a_j + T + \delta - t) \cdot u(E))$$
$$\le d(u'(E, a_{j+1}), u(E)) + d(H(a_j + T + \delta - t) \cdot u(E), u(E)) < 2\kappa.$$

It follows that for each $0 \le t' \le T'$ we can find some $0 \le t(t') \le T$, defined by $t(t') = t' - a_j$ if $a_j \le t' \le a_j + T$ for some j, and $t(t') = 0$, if $a_j + T \le t \le a_{j+1}$ for some j, such that for every $E \in J_{\delta,n,\eta}$, we have $d(u'(E, t'), u(E, t)) < 2\kappa$.

It follows that for every $0 < \eta < 1/2$ and $\kappa > 0$, for $\delta > 0$ sufficiently small, for every n sufficiently large,

$$(100) \qquad \mu_{V_{t'}',C}(J_{\delta,n,\eta}) \geq e^{-2\kappa} \mu_{V_t,C}(J_{\delta,n,\eta})$$
$$\geq \mu_{V_t,C}(J) - (1 - e^{-2\kappa})C|J| - e^{-2\kappa}C|J \smallsetminus J_{\delta,n,\eta}|$$
$$\geq \mu_{V_t,C}(J) - 2(\kappa + \eta)C|J|$$

(where $\mu_{V_{t'}',C}$ denotes the truncation of the spectral measure $\mu_{V_{t'}'}$ for the shift of V'). Thus, if $\eta + \kappa$ is sufficiently small, we get

$$(101) \qquad \mu_{V_{t'}',C}(-\infty, M] \geq \mu_{V_t,C}(J) - \frac{\epsilon - \epsilon_0}{2}.$$

Notice that $t(t')$ is such that for every $C_1 > 0$, $\epsilon_1 > 0$, we have, for every $\delta > 0$ sufficiently small, for every $n \in \mathbb{N}$,

$$(102) \qquad \sup_{|s| \leq C_1} |V'(t' + s) - V(t + s)| < \epsilon_1.$$

By Lemma 4.1, if $\delta > 0$ is sufficiently small we have

$$(103) \qquad \mu_{V_{t'}}(-\infty, M] < \mu_{V_t}(-\infty, M] + \frac{\epsilon - \epsilon_0}{2}.$$

Together with (92), it follows that

$$(104) \qquad \mu_{V_{t'}'}(-\infty, M] < \mu_{V_{t'}',C}(-\infty, M] + \epsilon,$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 4.7. The construction also gives that for every $\kappa > 0$, for a subset of $\Omega(V)$ whose complement has arbitrarily small measure, we have

$$(105) \qquad \sup_{t'} d(u[V'](E,t'),i) \leq \sup_{t} d(u[V](E,t),i) + 2\kappa.$$

Proof of Theorem 4. Define a sequence of $T^{(n)}$-periodic time-changes of solenoidal flows $F_t^{(n)} : S^{(n)} \to S^{(n)}$ and a sequence of continuous functions $v^{(n)} : S^{(n)} \to \mathbb{R}$ in the following way.

First take $T^{(0)} = 1$, $S^{(0)} = \mathbb{R}/\mathbb{Z}$, $F_t^{(0)} = F_t^{S^{(0)}}$, and $v^{(0)} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ a non-constant smooth function with $v^{(0)} = 0$ near 0. Let $\kappa_0 = 1$. Then for $j \geq 1$,

1. Choose $C_{j-1} > 0$ so that $t \mapsto v^{(j-1)}(F_t^{(j-1)}(0))$ is $(2^{1-j}, C_{j-1}, 2^{j-1})$-uniform,
2. Choose $(F^{(j)}, v^{(j)})$ so that it is $(2^{-j'}, C_{j'}, 2^{j'})$-uniform for all $0 \leq j' \leq j - 1$, $F^{(j)}$ is $(2^{j-1}, F^{(j-1)})$-mixed, and $(F^{(j)}, v^{(j)})$ is κ_{j-1}-close to a lift of $(F^{(j-1)}, v^{(j-1)})$
3. Let $0 < \kappa_j < \kappa_{j-1}/2$ be such that if (F, v) is $2\kappa_j$-close to the lift of $(F^{(j)}, v^{(j)})$ then (F, v) is $(2^{j-1}, F^{(j-1)})$-mixed.

The first step is an application of Lemma 4.2, the second is an application of Lemmas 4.6 (notice that by the previous choices, $(F^{(j-1)}, v^{(j-1)})$ is $(2^{-j'}, C_{j'}, 2^{j'})$-uniform for all $0 \leq j' \leq j - 1$) and 4.5, and the third is an application of Lemma 4.3.

Let S be the projective limit of the $S^{(j)}$ and let (F, v) be the projective limit of the $(F^{(j)}, v^{(j)})$. Then F is $(2^j, F^{(j)})$-mixed for all $j \geq 1$, so it is weak mixing by Lemma 4.4. We also have that for every $x \in S$, $V_x^{(j)} : t \mapsto v^{(j)}(F_t^{(j)}(p_{S,S^{(j)}}(x)))$ converges to $V_x : t \mapsto v(F_t(x))$ uniformly on compacts. It follows that the spectral measure $\mu = \mu_{V_x}$ is the limit of the spectral measures $\mu_{V_x^{(j)}}$. For every $C > 0$, and

up to taking a subsequence, the truncations $\mu_{V_x^{(j)},C}$ converge to a measure $\mu_C \leq \mu$ which is absolutely continuous with density bounded by C.

Then we have

$$(106) \quad \mu(-\infty, 2^j) - \int_{-\infty}^{2^j} \frac{d\mu}{dE} dE \leq \lim_{k \to \infty} \mu_{V_x^{(k)}}(-\infty, 2^j) - \mu_{V_x^{(k)}, C_j}(-\infty, 2^j) \leq 2^{-j}.$$

The result follows.   □

Remark 4.8. Notice that the construction allows us to obtain (by Remark 4.7), that for $k \in \mathbb{N}$ there exists a subset $\Gamma^{(k)} \subset \Omega(V^{(k)}) \cap \Omega(V^{(k+1)})$ such that $|\Omega^{(k)} \smallsetminus \Gamma^{(k)}| \leq 2^{-k}$ and

$$(107) \quad \sup_t d(u[V^{(k+1)}](E, t), i) \leq \sup_t d(u[V^{(k)}](E, t), i) + 2^{-k}.$$

Moreover, by Lemma 2.7, we may also assume that $|\Sigma(F, v) \smallsetminus \Omega(V^{(k)})| \leq 2^{-k}$. It follows that for almost every $E \in \Sigma(F, v)$, there exists $C(E) > 0$ such that $E \in \Omega(V^{(k)})$ for every k sufficiently large and $\sup_t d(u[V^{(k)}](E, t), i) \leq C(E)$. This implies that $\sup_t \sup_s \|A[V^{(k)}](E, t, s)\| \leq e^{C(E)}$ and hence

$$(108) \quad \sup_t \sup_s \|A[F, v](E, x, t, s)\| \leq e^{C(E)},$$

so that every eigenfunction with such an energy must be bounded.

5. Discrete case: unbounded eigenfunctions

5.1. **Schrödinger cocycles.** Given a function $V : \mathbb{Z} \to \mathbb{R}$, we define the transfer matrices $A[V](E, m, n)$ so that $A[V](E, m, m) = id$,

$$(109) \quad A[V](E, m, n+1) = \begin{pmatrix} E - V(n) & -1 \\ 1 & 0 \end{pmatrix} A[V](E, m, n).$$

An eigenfunction of the Schrödinger operator with potential V and energy E is a solution of $\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = A[V](E, m, n) \cdot \begin{pmatrix} u_m \\ u_{m-1} \end{pmatrix}$.

Lemma 5.1. *If $n > m$ and $|\mathrm{tr} A[V](E, m, n)| < 2$ then*

$$(110) \quad \frac{d}{dE} \Theta(A[V](E, m, n)) < 0.$$

Assume now that V is periodic of period N. In this case we write $A[V](E, n) = A[V](E, n, n+N)$ and $A[V](E) = A[V](E, 0)$. Note that $\mathrm{tr} A[V](E, n) = \mathrm{tr} A[V](E)$ for all $n \in \mathbb{N}$. Then the spectrum $\Sigma = \Sigma(V)$ of the Schrödinger operator with potential V is the set of all E with $|\mathrm{tr} A[V](E)| \leq 2$. Let also $\Omega = \Omega(V)$ be the set of all E with $|\mathrm{tr} A[V](E)| < 2$. We note that $\Sigma \smallsetminus \Omega = \partial \Omega$ consists of finitely many points. For $E \in \Omega(V)$, put $u[V](E, n) = \mathbf{u}(A[V](E, n))$ and $u[V](E) = u[V](E, 0)$.

Let $f : X \to X$ be a minimal uniquely ergodic map with invariant probability measure σ. Given $v : X \to \mathbb{R}$ continuous, we let $A[f, v](E, x, m, n) = A[V](E, m, n)$ where $V(n) = v(f^n(x))$. We define the Lyapunov exponent

$$(111) \quad L(E) = \lim_{n \to \infty} \frac{1}{n} \int \ln \|A[f, v](E, x, 0, n)\| d\sigma(x).$$

We will use the following criterion for the existence of ac spectrum (Ishii-Pastur, Kotani, Last-Simon).

Theorem 6 (see [D]). *The ac part of the spectral measure of the discrete Schrödinger operator with potential $V(n) = v(f^n(x))$ is equivalent to the restriction of Lebesgue measure to $\{L(E) = 0\}$.*

Remark 5.2. The fact that the essential support of the ac spectrum is contained in $\{L(E) = 0\}$, for almost every x, is the Ishii-Pastur Theorem. Kotani's Theorem gives the reverse inclusion, still for almost every x. Those results apply for general ergodic dynamics. Last-Simon proved that for minimal dynamics the essential support of the ac spectrum is constant everywhere (and not only almost everywhere).

5.2. **Construction of families of periodic potentials.** Let $\mathcal{V} : \mathbb{R}/N_0\mathbb{Z} \times \mathbb{Z}/N_1\mathbb{Z} \to \mathbb{R}$ be a continuous function. We think of $\mathcal{V}$ as a one-parameter family (parametrized by $\mathbb{R}/N_0\mathbb{Z}$) of periodic potentials $\mathcal{V}_t(\cdot) = \mathcal{V}(t, \cdot)$ (of period N_1).

We define some basic operations on such a $\mathcal{V}$. First, for $n \in \mathbb{N}$, the n-repetition $\mathcal{V}' : \mathbb{R}/N_0\mathbb{Z} \times \mathbb{Z}/nN_1\mathbb{Z} \to \mathbb{R}$ of $\mathcal{V}$ is given by $\mathcal{V}'(t, j) = \mathcal{V}(t, j)$. We obviously have

(112) $$A[\mathcal{V}'_t](E, m) = A[\mathcal{V}_t](E, m)^n.$$

Secondly, given some $n \in \mathbb{N}$, we define the n-twist $\mathcal{V}' : \mathbb{R}/N_0\mathbb{Z} \times \mathbb{Z}/nN_1\mathbb{Z} \to \mathbb{R}$ of $\mathcal{V}$ by $\mathcal{V}'(t, j) = \mathcal{V}(t + N_0\frac{k}{n}, l)$, whenever $j = kN_1 + l$ with $0 \leq j \leq n - 1$ and $0 \leq l \leq N_1 - 1$. We notice that

(113) $$A[\mathcal{V}'_t](E) = A[\mathcal{V}_{t+N_0\frac{n-1}{n}}](E) \cdots A[\mathcal{V}_t](E).$$

For the third operation, we will make use of some fixed smooth function $\Psi : [-1, 2] \to [0, 1]$, with $\Psi = 0$ in a neighborhood of -1 and 2, and $\Psi = 1$ in a neighborhood of $[0, 1]$. We also assume that $N_1 \geq 3$. Then for $\delta > 0$ and $n \in \mathbb{N}$, we define the (δ, n)-slide $\mathcal{V}' : \mathbb{R}/2nN_0\mathbb{Z} \times \mathbb{Z}/3N_1\mathbb{Z} \to \mathbb{R}$ of v by

(114) $$\mathcal{V}'(t, j) = \mathcal{V}(t, j), \quad 0 \leq j \leq 2N_1 - 1,$$

(115) $\mathcal{V}'(t, j) = \mathcal{V}(t, j), \quad 2N_1 \leq j \leq 3N_1 - 1 \quad t \in [0, nN_0 - 1] \cup [nN_0 + 2, 2nN_0],$

and

(116) $\mathcal{V}'(t, j) = \mathcal{V}(t + \delta\Psi(t - nN_0), j), \quad 2N_1 \leq j \leq 3N_1 - 1 \quad t \in [nN_0 - 1, nN_0 + 2].$

Notice that we have

(117) $$A[\mathcal{V}'_t](E) = A[\mathcal{V}_t](E)^3, \quad t \in [0, nN_0 - 1] \cup [nN_0 + 2, 2nN_0],$$

(118) $$A[\mathcal{V}'_t](E) = A[\mathcal{V}_{t+\delta\Psi(t-nN_0)}](E) \cdot A[\mathcal{V}_t](E)^2, \quad t \in [nN_0 - 1, nN_0 + 2].$$

Lemma 5.3. *Fix some closed interval $J \subset \mathbb{R}$ and let $u_0 : J \times [-1, 2] \to \mathbb{H}$ be a smooth function with*

(119) $$\sup_{t \in [0,1]} \left| \frac{d}{dt} u_0(E, t) \right| > 0$$

for every $E \in J$. There exists $\epsilon_1 > 0$, $C' > 0$ and $\delta_0 > 0$ with the following property. Let $\mathcal{V} : \mathbb{R}/N_0\mathbb{Z} \times \mathbb{Z}/N_1\mathbb{Z} \to \mathbb{R}$ be a smooth function. Let $E_0 \in \mathrm{int}\, J \cap \bigcap_t \Omega(\mathcal{V}_t)$. Assume that $[-1, 2] \ni t \mapsto u[\mathcal{V}_t](E_0)$ is (strictly) ϵ_1-close in the C^1-topology to $[-1, 2] \mapsto u_0(E_0, t)$. Then there exists $\epsilon_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$, for every $\kappa > 0$, for every $0 < \delta < \delta_0$, for every N_2 sufficiently large, for every N_3 sufficiently large, for every N_4 sufficiently large, for every N_5 sufficiently large,

if $\mathcal{V}' : \mathbb{R}/2N_4N_0\mathbb{Z} \times \mathbb{Z}/3N_5N_3N_2N_1\mathbb{Z}$ *is the* N_5-*twist of the* (δ, N_4)-*slide of the* N_2-*repetition of the* N_3-*twist of* $\mathcal{V}$, *then there exists a compact set*

$$\text{(120)} \qquad \Lambda \subset [E_0 - \epsilon, E_0 + \epsilon] \cap \bigcap_t \Omega(\mathcal{V}'_t)$$

such that

1. $|\Lambda| > 2(1 - C'\delta)\epsilon$,
2. *For* $E \in \Lambda$, $[-1, 2] \ni t \mapsto u[\mathcal{V}'_t](E)$ *is (strictly)* ϵ_1-*close in the* C^1-*topology to* $[-1, 2] \ni t \mapsto u_0(E, t)$.
3. *For* $E \in \Lambda$,

$$\text{(121)} \qquad \left| \frac{1}{6N_5N_4N_3N_2N_1N_0} \sum_{j \in \mathbb{Z}/3N_5N_3N_2N_1\mathbb{Z}} \int_0^{2N_4N_0} d(u[\mathcal{V}_t](E, j), i)\,dt \right.$$

$$\left. - \frac{1}{N_1N_0} \sum_{j \in \mathbb{Z}/N_1\mathbb{Z}} \int_0^{N_0} d(u[\mathcal{V}_t](E, j), i)\,dt \right| < \kappa,$$

4. *For* $E \in \Lambda$,

$$\text{(122)} \qquad \inf_t \sup_j d(u[\mathcal{V}'_t](E, j), i) \geq \sup_t \sup_j d(u[\mathcal{V}_t](E, j), i) - \kappa,$$

5. *For any* $C'\delta < \gamma < C'^{-1}$, *there exists a compact set* $\Lambda' \subset \Lambda$ *with* $|\Lambda'| > \gamma\epsilon$ *such that for* $E \in \Lambda'$,

$$\text{(123)} \qquad \inf_t \sup_j d(u[\mathcal{V}'_t](E, j), i) \geq \sup_t \sup_j d(u[\mathcal{V}_t](E, j), i) + C'^{-1}\frac{\delta}{\gamma} - \kappa.$$

Proof. Write $\mathcal{V}''''$ for the N_3-twist of $\mathcal{V}$, $\mathcal{V}'''$ for the N_2-repetition of $\mathcal{V}''''$, $\mathcal{V}''$ for the (δ, N_4)-slide of $\mathcal{V}'''$.

In the first step, going from $\mathcal{V}$ to $\mathcal{V}''''$, we obtain, using Lemma 2.12, a set of good energies $E \in [E_0 - \epsilon, E_0 + \epsilon] \cap \bigcap_t \Omega(\mathcal{V}''''_t)$ of measure at least $2\epsilon(1 - \delta)$, such that $t \mapsto u[\mathcal{V}''''_t](E)$ is C^1 close to $t \mapsto u[\mathcal{V}_t](E)$, and letting $B(E, t) = B(A[\mathcal{V}''''_t](E))$, $\theta(E, t) = \Theta(A[\mathcal{V}''''_t](E))$, we have $N_2(\sup_t \theta(E, t) - \inf_t \theta(E, t))$ arbitrarily small. Moreover, the random variables $\theta(E, 0)$ near E_0 are becoming equidistributed in $\mathbb{R}/\mathbb{Z}$ as N_3 grows. We also have that

$$\text{(124)} \qquad \inf_t \sup_j d(u[\mathcal{V}''''_t](E, j)) \geq \sup_t \sup_j d(u[\mathcal{V}_t](E, j), i) - \frac{\kappa}{3}.$$

The second step does not change much, since $E \in \Omega(\mathcal{V}'''_t)$ provided $N_2\theta(E, t)$ is not an integer, and in this case $u[\mathcal{V}'''_t](E, j) = u[\mathcal{V}''''_t](E, j)$. On the other hand, if $\{j\theta(E, 0)\}_{0 \leq j \leq N_2 - 1}$ is $\frac{1}{100}$ dense   mod 1, then for any $w \in \mathbb{H}$, and for every t,

$$\text{(125)}$$
$$\sup_{0 \leq j \leq N_3N_2N_1 - 1} d(A[\mathcal{V}'''_t](E, 0, j) \cdot w, i) \geq \sup_l d(u[\mathcal{V}'''_t](E, l), i) + \frac{1}{2}d(w, u[\mathcal{V}'''_t](E)).$$

The condition on $\theta(E, 0)$ demands the exclusion of some energies, but of arbitrarily small measure (which we can take less than $2\delta\epsilon$), provided N_2 is large.

Going from $\mathcal{V}'''$ to $\mathcal{V}''$, for such good energies E we obtain

$$\text{(126)} \quad \text{tr} A[\mathcal{V}''_t](E) = 2\cos 2N_2\pi(2\theta(E, t) + \theta(E, t + \delta\tilde{\Psi}(t)))$$

$$- (\lambda(E, t) - \lambda(E, t)^{-1})^2 \sin 4N_2\pi\theta(E, t)\sin 2N_2\pi\theta(E, t + \delta\tilde{\Psi}(t)),$$

with

(127) $$\lambda(E,t) = e^{d(u[\mathcal{V}_t'''](E), u[\mathcal{V}_{t+\delta\tilde\Psi(t)}'''](E))/2},$$

and $\tilde\Psi : \mathbb{R}/2N_4N_0\mathbb{Z} \to [0,1]$ is given by $\tilde\Psi(t) = 0$ if $t \in [0, N_4N_0 - 1] \cup [N_4N_0 + 2, 2N_4N_0]$ and $\tilde\Psi(t) = \Psi(t - N_4N_0)$ if $t \in [N_4N_0 - 1, N_4N_0 + 2]$.

Notice that $\lambda(E,t) - 1$ vanishes if $t \in [0, N_4N_0 - 1] \cup [N_4N_0 + 2, 2N_4N_0]$, is at most of order δ everywhere, and gets to be of precisely order δ for some $t \in [N_0, N_0 + 1]$ (here we use (119)).

We now exclude E with $\sin 6N_2\pi\theta(E,0)$ of order δ. The excluded set of energies has measure of order $2\delta\epsilon$. For the remaining energies, $|\mathrm{tr}A[\mathcal{V}_t''](E)| < 2 - \delta^2$ for all t.

By (125), for every t, using that $u[\mathcal{V}_t''](E, j) = A[\mathcal{V}_t''](E, 0, j) \cdot u[\mathcal{V}_t''](E)$ and also $A[\mathcal{V}_t''](E, 0, j) = A[\mathcal{V}_t'''](E, 0, j)$ for $0 \le j \le 2N_3N_2N_1$, we have

(128) $$\sup_j d(u[\mathcal{V}_t'''](E, j), i) \ge \sup_j d(u[\mathcal{V}_t''](E, j), i) + \frac{1}{2}d(u[\mathcal{V}_t''](E), u[\mathcal{V}_t'''](E)),$$

and together with (124) we get
(129)
$$\sup_j d(u[\mathcal{V}_t''](E, j), i) \ge \sup_t \sup_j d(u[\mathcal{V}_t](E, j), i) + \frac{1}{2}d(u[\mathcal{V}_t''](E), u[\mathcal{V}_t'''](E)) - \frac{\kappa}{3}.$$

In particular, we always have

(130) $$\sup_j d(u[\mathcal{V}_t''](E, j), i) \ge \sup_t \sup_j d(u[\mathcal{V}_t](E, j), i) - \frac{\kappa}{3}.$$

We compute the distance from $u[\mathcal{V}_t'''](E)$ to $u[\mathcal{V}_t''](E)$. It is equal to the distance from $w'(E,t)$ to i where $w'(E,t)$ is the solution $z \in \mathbb{H}$ of the equation $az^2 + bz + c = 0$, where

(131) $$a = \cos 2N_2\pi\theta(E, t + \delta\tilde\Psi(t)) \sin 4N_2\pi\theta(E, t)$$
$$+ \lambda(E, t)^{-2} \sin 2N_2\pi\theta(E, t + \delta\tilde\Psi(t)) \cos 4N_2\pi\theta(E, t),$$

(132) $$b = (\lambda(E, t)^2 - \lambda(E)^{-2}) \sin 2N_2\pi\theta(E, t + \delta\tilde\Psi(t)) \sin 4N_2\pi\theta(E, t)),$$

(133) $$c = \cos 2N_2\pi\theta(E, t + \delta\tilde\Psi(t)) \sin 4N_2\pi\theta(E, t)$$
$$+ \lambda(E, t)^2 \sin 2N_2\pi\theta(E, t + \delta\tilde\Psi(t)) \cos 4N_2\pi\theta(E, t) = 0.$$

If the distance from $N_2\theta(E, 0)$ to $\frac{1}{3} + \mathbb{Z}$ is exactly γ, with $C_2\delta < \gamma < C_2^{-1}$, then

(134) $$C_3^{-1} \frac{\lambda(E, t)^2 - \lambda(E, t)^{-2}}{\gamma} \le d(w'(E, t), i) \le C_3 \frac{\lambda(E, t)^2 - \lambda(E, t)^{-2}}{\gamma}.$$

Using that $\lambda(E, t) - 1$ does become of order δ for some t, we get, for such E,

(135) $$\sup_t \sup_j d(u[\mathcal{V}_t''](E, j), i) \ge \sup_t \sup_j d(u[\mathcal{V}_t](E, j), i) + C_4^{-1}\frac{\delta}{\gamma} - \frac{\kappa}{3}.$$

On the other hand, if one only excludes the energies with $\sin 6N_2\pi\theta(E, 0)$ of order δ, we still get that $d(w'(E, t), i)$ is uniformly bounded as N_4 grows, which implies that $\sup_t \sup_j d(u[\mathcal{V}_t''](E, j), i)$ is uniformly bounded as N_4 grows.

Proceeding with the last step, we get

(136) $$\inf_t \sup_j d(u[\mathcal{V}_t'](E, j), i) \ge \sup_t \sup_j d(u[\mathcal{V}_t](E, j), i) - \frac{2\kappa}{3},$$

while for $2C_2\delta < \gamma < C_2^{-1}$ and a set of E of probability of order γ we get

$$(137) \qquad \inf_t \sup_j d(u[\mathcal{V}_t'](E,j),i) \geq \sup_t \sup_j d(u[\mathcal{V}_t](E,j),i) + C_4^{-1}\frac{\delta}{\gamma} - \frac{2\kappa}{3}.$$

It remains to check that the average of $d(u[\mathcal{V}_t'](E,j),i)$ is close to the average of $d(u[\mathcal{V}_t](E,j),i)$. The first, second, and fourth steps clearly do not increase the average significantly. For the third step, we have $u[\mathcal{V}_t''](E,j) = u[\mathcal{V}_t'''](E,j)$ except when $t \in [N_4N_0 - 1, N_4N_0 + 2]$. Since $d(u[\mathcal{V}_t''](E,j),i)$ remains bounded as N_4 grows, we conclude that the average can not be increased significantly in the third step as well. $\square$

With this result in hands, analogues of Lemmas 3.3 and 3.1 can be easily obtained. We state the conclusion:

Lemma 5.4. *Fix some closed interval $J \subset \mathbb{R}$ and let $u_0 : J \times [-1,2] \to \mathbb{H}$ be a smooth function with*

$$(138) \qquad \sup_{t\in[0,1]} \left|\frac{d}{dt}u_0(E,t)\right| > 0$$

for every $E \in J$. There exists $\epsilon_1 > 0$ with the following property. Let $\mathcal{V}^{(0)} : \mathbb{R}/N_0^{(0)}\mathbb{Z} \times \mathbb{Z}/N_1^{(0)}\mathbb{Z} \to \mathbb{R}$ be a smooth function. Let $\Gamma_0 \subset J \cap \bigcap_t \Omega(\mathcal{V}_t^{(0)})$ be a compact set of E such that $[-1,2] \ni t \mapsto u[\mathcal{V}_t^{(0)}](E)$ is (strictly) ϵ_1-close to $[-1,2] \ni t \mapsto u_0(E,t)$ in the C^1-topology.
Let $C > 0$ be such that

$$(139) \qquad \sup_{E\in\Gamma_0} \frac{1}{N_0N_1} \sum_{j=0}^{N_1-1} \int_0^{N_0} d(u[\mathcal{V}^{(0)}](E,j),i)dt < C.$$

Then for every $\xi > 0$, $C_0 > 0$, for every $\delta > 0$ sufficiently small, there exist $0 < P < \xi\delta^{-1}$, and sequences $N_l^{(j)}$, $1 \leq j \leq P$, $2 \leq l \leq 5$, such that if we define $\mathcal{V}^{(j)}$, $1 \leq j \leq P$ so that $\mathcal{V}^{(j)}$ is obtained by $N_5^{(j)}$-twist of the $(\delta, N_4^{(j)})$-slide of the $N_2^{(j)}$-repetition of the $N_3^{(j)}$-twist of $\mathcal{V}^{(j-1)}$, then there exists a compact subset $\Gamma \subset \Gamma_0 \cap \bigcap_t \Omega(\mathcal{V}_t^{(P)})$ such that $|\Gamma_0 \setminus \Gamma| < \xi$, and for every $E \in \Gamma$, letting $N_0' = N_0 2^P \prod_{j=1}^P N_4^{(j)}$ and $N_1' = N_1 3^P \prod_{j=1}^P N_5^{(j)} N_3^{(j)} N_2^{(j)}$, we have

$$(140) \qquad \inf_t \sup_j d(u[\mathcal{V}_t^{(P)}](E,j),i) \geq C_0,$$

$$(141) \qquad \frac{1}{N_0'N_1'} \sum_{j=0}^{N_1'-1} \int_0^{N_0'} d(u[\mathcal{V}_t^{(P)}](E,j),i) < C.$$

Moreover, for $E \in \Gamma$, $[-1,2] \ni t \mapsto u[\mathcal{V}_t^{(P)}](E)$ is (strictly) ϵ_1-close to $[-1,2] \ni t \mapsto u_0(E,t)$ in the C^1-topology.

Remark 5.5. In the setting of the previous lemma, we have the following extra information on $\mathcal{V}^{(P)}$. There exists $n \in \mathbb{N}$ such that for every $E \in \Gamma$ we have

$$(142) \qquad \inf_{w\in\mathbb{R}^2, \|w\|=1} \sup_{0\leq l\leq n} A[\mathcal{V}_t^{(P)}](E,j,j+l) > e^{(C_0-2C)/4}/2,$$

except for a set of (t,j) of measure less than $C_0^{-1/2}$. Indeed, if (t,j) is such that $d(u[\mathcal{V}_t^{(P)}](E,j),i) \leq C_0^{1/2}C$ and $l \in \mathbb{N}$ is such that $d(u[\mathcal{V}_t^{(P)}](E,j+l),i) \geq C_0$,

then $A[\mathcal{V}_t^{(P)}](E, j, j+l+kN_1')$ decomposes as a product $B(t, j+l)^{-1} R_{\tilde{\theta}+k\theta} B(t, j)$, where $B(t, m) = \mathbf{B}(A[\mathcal{V}_t^{(P)}](E, m))$, and $\theta = \Theta(A[\mathcal{V}_t^{(P)}](E, j))$. Since $2\theta \notin \mathbb{Z}$, this implies that for any w we can find k such that $R_{\tilde{\theta}+k\theta} B(t, j) \cdot w$ has angle at most $\pi/4$ with the direction most expanded by $B(t, j+l)^{-1}$, which gives the estimate $\|B(t, j+l)^{-1} R_{\tilde{\theta}+k\theta} B(t, j) \cdot w\| \geq \|B(t, j+l)\|\|B(t, j)\|^{-1}/\sqrt{2}$.

5.3. **Construction of almost periodic dynamics.** Let $N_0, N_1 \in \mathbb{N}$, and let $a \in \mathbb{Q}$ be an integer multiple of $\frac{N_0}{N_1}$. Consider a smooth family of periodic potentials $\mathcal{V} : \mathbb{R}/N_0\mathbb{Z} \times \mathbb{Z}/N_1\mathbb{Z} \to \mathbb{R}$. It is natural to consider this periodic family as arising from the non-ergodic dynamics $(t, j) \mapsto (t, j+1)$ on $\mathbb{R}/N_0\mathbb{Z} \times \mathbb{Z}/N_1\mathbb{Z}$, in the obvious way. But we can also think of it as arising from the dynamics $(t, j) \mapsto (t+a, j+1)$, by considering the sampling function $v : \mathbb{R}/N_0\mathbb{Z} \times \mathbb{Z}/N_1\mathbb{Z} \to \mathbb{R}$ defined by $\mathcal{V}(t, j) = v(t + ja, j)$.

Such a point of view is advantageous in that it allows to consider our three operations on potentials as "small perturbations".

Take $\mathcal{V}'$ to be the n-repetition of $\mathcal{V}$. Defining $v' : \mathbb{R}/N_0\mathbb{Z} \times \mathbb{Z}/nN_1\mathbb{Z} \to \mathbb{R}$ by $\mathcal{V}'(t, j) = v'(t + ja, j)$, we obviously still get $v'(t, j) = v(t, j)$.

Take $\mathcal{V}'$ to be the n-twist of $\mathcal{V}$. Set $a' = a + \frac{N_0}{nN_1}$. Defining $v' : \mathbb{R}/N_0\mathbb{Z} \times \mathbb{Z}/nN_1\mathbb{Z} \to \mathbb{R}$ by $\mathcal{V}'(t, j) = v'(t + ja', j)$, we see that $\sup_t \sup_j |v'(t, j) - v(t, j)|$ becomes small for large n.

Take $\mathcal{V}'$ to be the (δ, n)-slide of $\mathcal{V}$. Defining $v' : \mathbb{R}/2nN_0\mathbb{Z} \times \mathbb{Z}/3N_1\mathbb{Z} \to \mathbb{R}$ by $\mathcal{V}'(t, j) = v'(t + ja, j)$, we see that $\sup_t \sup_j |v'(t, j) - v(t, j)| \leq \delta \sup_t \sup_j |\frac{d}{dt} v(t, j)|$. Moreover, we also have $\sup_t \sup_j |\frac{d}{dt} v'(t, j)| \leq (1 + K_1\delta) \sup_t \sup_j |\frac{d}{dt} v(t, j)|$, where $K_1 = \sup_t |\frac{d}{dt} \Psi(t)|$ is a fixed constant.

Given those observations, we can proceed with the inductive construction.

Proof of Theorem 1. Choose $0 < \lambda_0 < 1/2$, and let $J = [-2 + 4\lambda_0, 2 - 4\lambda_0]$. Let $u_0(E, t)$ be the fixed point of $\begin{pmatrix} E - 2\lambda_0 \cos 2\pi t & -1 \\ 1 & 0 \end{pmatrix}$. Let $\epsilon_1 > 0$ be as in Lemma 5.4. Let $C_1 > \int_0^1 d(u_0(E, t), i) dt$.

We now produce sequences $\mathcal{V}^j, v^j : \mathbb{R}/N_{0,j}\mathbb{Z} \times \mathbb{Z}/N_{1,j}\mathbb{Z} \to \mathbb{R}$, $a_j \in \mathbb{Q}$, and compact sets Γ_j as follows.

First set $N_0 = N_1 = 1$, $\mathcal{V}^0(t, j) = 2\lambda \cos 2\pi t$, $v^0 = \mathcal{V}^0$, $a_0 = 0$, $\Gamma_0 = J$.

We now apply Lemma 5.4, to obtain $\Gamma_1 \subset \Gamma_0$ with $|\Gamma_0 \setminus \Gamma_1|$ arbitrarily small, and some $\mathcal{V}^1 : \mathbb{R}/N_{0,1}\mathbb{Z} \times \mathbb{Z}/N_{1,1}\mathbb{Z} \to \mathbb{R}$ such that for $E \in \Gamma_1$

$$(143) \qquad \inf_t \sup_j d(u[\mathcal{V}_t^1](E, j), i) \geq 2C_1 + 4,$$

$$(144) \qquad \frac{1}{N_{0,1} N_{1,1}} \sum_{j=0}^{N_{1,1}-1} \int_0^{N_{0,1}} d(u[\mathcal{V}_t^1](E, j), i) < C_1,$$

and moreover, $[-1, 2] \ni t \mapsto u[\mathcal{V}_t^1](E, j))$ is (strictly) ϵ_1-close to $[-1, 2] \ni t \mapsto u_0(E, t)$ in the C^1-topology. Using Remark 5.5, we see that there exists $q_1 \in \mathbb{N}$ such that for every $E \in \Gamma_1$,

$$(145) \qquad \inf_{w \in \mathbb{R}^2, \|w\|=1} \sup_{0 \leq l \leq q_1} \|A[\mathcal{V}_t^1](E, j, j+l) \cdot w\| > \frac{e}{2},$$

except for a set of (t, j) of measure less than $(2C_1 + 4)^{-1/2}$.

Moreover, we can alternatively realize $\mathcal{V}^1(t,j) = v^1(t + a_1 j, j)$ for appropriate a_1, so that $\sup_t \sup_j |v^1(t,j) - v^0(t,j)|$ is arbitrarily small. Notice that $|a_1 - a_0|$ can be also taken arbitrarily small but non-zero.

We continue by induction, obtaining a decreasing sequence Γ_k, and $\mathcal{V}^k, v^k, a_k$ such that for $k \geq 2$ and $E \in \Gamma_k$ we have

1. $\inf_t \sup_j d(u[\mathcal{V}^k_t](E,j), i) \geq 2C_1 + 4k$,
2. $\frac{1}{N_{0,k}N_{1,k}} \sum_{j=0}^{N_{1,k}-1} \int_0^{N_{0,k}} d(u[\mathcal{V}^k_t](E,j), i) < C_1$,
3. $[-1,2] \ni t \mapsto u[\mathcal{V}^k_t](E,j)$ is (strictly) ϵ_1-close to $[-1,2] \ni t \mapsto u_0(E,t)$ in the C^1-topology,
4. $\sup_t \sup_j |v^k(t,j) - v^{k-1}(t,j)| < 2^{-k}$,
5. There exists $q_k \in \mathbb{N}$ such that for every $E \in \Gamma_k$,

$$(146) \qquad \inf_{w \in \mathbb{R}^2, \|w\|=1} \sup_{0 \leq l \leq q_k} \|A[\mathcal{V}^k_t](E,j,j+l) \cdot w\| > \frac{e^k}{2},$$

except for a set of (t,j) of measure (strictly) less than $(2C_1 + 4k)^{-1/2}$.
6. $|a_k - a_{k-1}|$ is non-zero but smaller than $\frac{1}{2^{k-1} N_{1,k-1}}$.

We now construct the sampling function and the dynamics.

Let K be the projective limit of $\mathbb{Z}/N_{1,j}\mathbb{Z}$ (a Cantor group), and let S be the projective limit of $\mathbb{R}/N_{0,j}\mathbb{Z}$ (a solenoid). Then $v(t,j) = \lim v^k(t,j)$ defines a continuous function on $S \times K$ (for simplicity, we ommit the projections $S \times K \to \mathbb{R}/N_{0,k}\mathbb{Z} \times \mathbb{Z}/N_{1,k}\mathbb{Z}$ from the notation). This is the sampling function.

Notice that $a = \lim a_k$ is irrational, since a_k are rational with denominators at most $N_{1,k}$ and $0 < |a - a_k| \leq \frac{1}{2^{k-1} N_{1,k}}$. Thus $f(t,j) = (t+a, j+1)$ is a uniquely ergodic translation in the compact Abelian group $S \times K$. This is the base dynamics.

Let $\Gamma = \bigcap \Gamma_k$. By construction, Γ is a compact set of positive Lebesgue measure. Notice that

$$(147) \qquad \sup_x \sup_{E \in \Gamma} \sup_{0 \leq l \leq n} \|A[f,v](E,x,0,l) - A[\mathcal{V}^k_t](E,j,j+l)\|$$

(with (t,j) the projection of x) can be made arbitrarily small, for any n chosen after v^k is constructed, but before v^{k+1} is constructed. Choosing parameters growing sufficiently fast we get

$$(148) \qquad \lim_{n \to \infty} \sup_x \frac{1}{n} \ln \|A[f,v](E,x,0,n)\| = 0,$$

i.e., the Lyapunov exponent vanishes for $E \in \Gamma$, so that Γ is contained in the essential support of the ac spectrum for every x, and moreover, for every $k \geq 1$ and $E \in \Gamma_k$,

$$(149) \qquad \inf_{w \in \mathbb{R}^2, \|w\|=1} \sup_{0 \leq l \leq q_k} \|A[f,v](E,x,0,l) \cdot w\| \geq \frac{e^k}{4},$$

except for a set of x of measure less than $(2C_1 + 4k)^{-1/2}$. Thus, for every $E \in \Gamma$, for almost every x we have

$$(150) \qquad \inf_{w \in \mathbb{R}^2, \|w\|=1} \limsup_{l \to \infty} \|A[f,v](E,x,0,l) \cdot w\| = \infty,$$

which is the desired eigenfunction growth.   □

6. Discrete case: breaking almost periodicity

6.1. **Slow deformation.** The following are variations of Lemmas 2.11 and 2.12, and we leave the proofs for the reader.

Lemma 6.1. *Let $J \subset \mathbb{R}$ be a closed interval, let $N \in \mathbb{N}$, and let $A : J \times \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ be a smooth function such that $|\mathrm{tr}A^{(N)}(E,t)| < 2$ for $(E,t) \in J \times \mathbb{R}/\mathbb{Z}$, where*

$$(151) \qquad A^{(N)}(E,t) = A(E, t + \frac{N-1}{N}) \cdots A(E,t).$$

Let $B(E,t) = \mathbf{B}(A^{(N)}(E,t))$ and let $\theta(E,t)$ be a smooth function satisfying

$$(152) \qquad B(E, t + \frac{1}{N})A(E,t)B(E,t)^{-1} = R_{\theta(E,t)}.$$

Then for every $m, k \in \mathbb{N}$, there exists $n(m) \in \mathbb{N}$ and $C_{k,m} > 0$ such that for every $n \geq n(m)$, there exist smooth functions $B_{(m,n)} : J \times \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$, $\theta_{(m,n)} : J \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ such that

1. $\|A_{(m,n)} - R_{\theta_{(m,n)}}\|_{C^k} \leq \frac{C_{k,m}}{n^m}$, *where*

$$(153) \qquad A_{(m,n)}(E,t) = B_{(m,n)}(E, t + \frac{n+1}{nN})A(E,t)B_{(m,n)}(E,t)^{-1},$$

2. $\|B_{(m,n)} - B\|_{C^k} \leq \frac{C_{k,m}}{n}$,
3. $\|\theta_{(m,n)} - \theta\|_{C^k} \leq \frac{C_{k,m}}{n}$.

Lemma 6.2. *Under the hypothesis of the previous lemma, assume moreover that $\tilde{\theta}(E) = \int_{\mathbb{R}/\mathbb{Z}} \theta(E,t)dt$ satisfies $\frac{d}{dE}\tilde{\theta}(E) \neq 0$ for every $E \in J$. For $n \in \mathbb{N}$, let $A^{(N*n)} : J \times \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ be given by*

$$(154) \quad A^{(N*n)}(E,t) = A(E, t+(nN-1)\frac{n+1}{nN})A(E, t+(nN-2)\frac{n+1}{nN}) \cdots A(E,t).$$

Then there exist functions $\tilde{\theta}^{(n)} : J \to \mathbb{R}/\mathbb{Z}$ such that for every measurable subset $Z \subset \mathbb{R}/\mathbb{Z}$,

$$(155) \qquad \lim_{n \to \infty} |\{E \in J, \tilde{\theta}^{(n)}(E) \in Z\}| = |Z||J|,$$

$$(156) \qquad \lim_{n \to \infty} |\{E \in J, \tilde{\theta}^{(n)}(E) + \tilde{\theta}^{(2n)}(E) \in Z\}| = |Z||J|,$$

with the following property. For every $\delta > 0$,

$$(157) \qquad \lim_{n \to \infty} \|\mathrm{tr}A^{(N*n)}(E,t) - 2\cos 2\pi\tilde{\theta}_{(m,n)}(E)\|_{C^0(J \times \mathbb{R}/\mathbb{Z}, \mathbb{R})} = 0,$$

$$(158) \qquad \lim_{n \to \infty} \sup_{|\sin 2\pi\tilde{\theta}^{(n)}(E)|>\delta} \|\Theta(A^{(N*n)}(E, \cdot)) - \tilde{\theta}^{(n)}(E)\|_{C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R})} = 0,$$

$$(159) \qquad \lim_{n \to \infty} \sup_{|\sin 2\pi\tilde{\theta}^{(n)}(E)|>\delta} \|\mathbf{u}(A^{(N*n)}(E, \cdot)) - \mathbf{u}(A^{(N)}(E, \cdot))\|_{C^1(\mathbb{R}/\mathbb{Z}, \mathbb{C})} = 0.$$

6.2. **The construction.** In this section, we will interpret a continuous function $\mathcal{V} : \mathbb{R}/N\mathbb{Z} \to \mathbb{R}$ as a one-parameter family of N-periodic potentials $\mathcal{V}_t(j) = \mathcal{V}(t+j)$.

For $n \in \mathbb{N}$, we define the n-crumbling $\mathcal{V}' : \mathbb{R}/3nN\mathbb{Z} \to \mathbb{R}$ of v by

1. $\mathcal{V}'(t) = \mathcal{V}(\frac{n+1}{n}t)$, $t \in [0, nN]$,
2. $\mathcal{V}'(t) = \mathcal{V}(\frac{2n+1}{2n}(t - nN))$, $t \in [nN, 3nN]$.

Lemma 6.3. *Let $\mathcal{V} : \mathbb{R}/N\mathbb{Z}$ be a smooth function which is constant near 0. Then for every $\delta > 0$, for every n sufficiently large, letting $\mathcal{V}'$ be the n-crumbling of $\mathcal{V}$, we have $|\bigcap_t \Omega(\mathcal{V}_t) \smallsetminus \bigcap_t \Omega(\mathcal{V}'_t)| < \delta$.*

Proof. Fix a compact interval $J \subset \bigcap_t \Omega(\mathcal{V}_t)$. Apply Lemma 6.2 to $A(E,t) = \begin{pmatrix} E - v(Nt) & -1 \\ 1 & 0 \end{pmatrix}$. It yields a sequence $\tilde{\theta}^{(n)}(E)$.

If $\mathcal{V}'$ is the n-crumbling of v, then for $t \in [0,1]$, we get $A[\mathcal{V}'_t](E, 0, nN) = A^{(N*n)}(E, \frac{n+1}{nN}t)$ and $A[\mathcal{V}_t](E, nN, 3nN) = A^{(N*2n)}(E, \frac{2n+1}{2nN}t)$. Thus for $t \in [0,1]$ we have

$$(160) \qquad A[\mathcal{V}_t](E) = A^{(N*2n)}(E, \frac{2n+1}{2nN}t)A^{(N*n)}(E, \frac{n+1}{nN}t)$$

As long as $|\sin 2\pi\tilde{\theta}^{(n)}|$ and $|\sin 2\pi\tilde{\theta}^{(2n)}|$ are not too small, we can write, for $t \in [0,1]$,
$$(161)$$
$$A[\mathcal{V}'_t](E) = B^{(2n)}(E,t)^{-1}R_{\theta^{(2n)}(E,t)}B^{(2n)}(E,t)B^{(n)}(E,t)^{-1}R_{\theta^{(n)}(E,t)}B^{(n)}(E,t),$$

where $B^{(m)} = \mathbf{B}(A^{(N*m)}(E,t))$ and $\theta^{(m)}(E,t) = \Theta(A^{(N*m)}(E,t))$. Notice that $B^{(n)}$ and $B^{(2n)}$ are both C^1-close to $\mathbf{B}(A[\mathcal{V}_t](E))$ as functions of $t \in [0,1]$. Moreover, $\theta^{(n)}(E,t)$ is close to $\tilde{\theta}^{(n)}(E)$ and $\theta^{(2n)}(E,t)$ is close to $\tilde{\theta}^{(2n)}(E)$.

It follows that for $t \in [0,1]$, $\mathrm{tr}A[\mathcal{V}'_t](E)$ is close to $2\cos 2\pi(\tilde{\theta}^{(2n)}(E) + \tilde{\theta}^{(n)}(E))$. Thus, as long as $|\sin 2\pi(\tilde{\theta}^{(n)} + \tilde{\theta}^{(2n)})|$ is not small, we have $|\mathrm{tr}A[\mathcal{V}'_t](E)| < 2$ for every $t \in [0,1]$. Since $\mathrm{tr}A[\mathcal{V}'_t](E)$ is 1-periodic, this implies that $|\mathrm{tr}A[\mathcal{V}'_t](E)| < 2$ for all t. $\qquad \square$

Remark 6.4. One also easily gets from this construction,

$$(162) \qquad \sup_t d(u[\mathcal{V}'_t](E), i) < \sup_t d(u[\mathcal{V}_t](E), i) + \delta$$

except for a set of $E \in \bigcap_t \Omega(\mathcal{V}_t) \cap \bigcap_t \Omega(\mathcal{V}'_t)$ of arbitrarily small measure.

Proof of Theorem 4. Starting with a smooth non-constant function $\mathcal{V}^{(0)} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$, apply Lemma 6.3 successively to obtain a sequence $\mathcal{V}^{(k)} : \mathbb{R}/N^{(k)}\mathbb{Z} \to \mathbb{R}$ such that $\mathcal{V}^{(k)}$ is the n_k-crumbling of $\mathcal{V}^{(k-1)}$, and compact sets $\Gamma^{(k)} \subset \bigcap_t \Omega(\mathcal{V}^{(k)}_t)$ with $\Gamma^{(k)} \subset \Gamma^{(k-1)}$ and $\lim_{k \to \infty} |\Gamma^{(k)}| > 0$. By taking parameters n_k growing sufficiently fast, we ensure that for $E \in \Gamma^{(k+1)}$ we have

$$(163) \qquad \sup_{1 \leq j \leq N^{(k)}} |\sup_t \ln \|A[\mathcal{V}^{(k+1)}_t](E, 0, j)\| - \sup_t \ln \|A[\mathcal{V}^{(k)}_t](E, 0, j)\|| \leq \frac{1}{2^k},$$

$$(164) \qquad \sup_t \frac{1}{N^{(k+1)}} \ln \|A[\mathcal{V}^{(k+1)}_t](E, 0, N^{(k+1)})\| \leq \frac{1}{2^k}.$$

We now turn to the dynamical realization. Let $\tilde{N}^{(k)}$ be defined by $\tilde{N}^{(0)} = 1$, $\tilde{N}^{(k)} = (3n_k + 2)\tilde{N}^{(k-1)}$. We first construct $N^{(k)}$-periodic time changes $F^{(k)}$ of the

solenoidal flow on $S^{(k)} = \mathbb{R}/\tilde{N}^{(k)}\mathbb{Z}$ such that $\mathcal{V}^{(0)}(p_{S^{(k)},S^{(0)}}(F_t^{(k)}(0))) = \mathcal{V}^{(k)}(t)$. We first take $F^{(0)}$ to be just the solenoidal flow on $S^{(0)}$. Now define inductively

$$(165) \qquad w_{F^{(k+1)}}(t) = w_{F^{(k)}}(t)e^{\rho^{(k+1)}(t)}$$

for a suitable function $\rho^{(k+1)}$. Here it is enough to take $\rho^{(k+1)} = \ln\frac{n_{k+1}+1}{n_k}$ on $[0, (n_{k+1}+1)\tilde{N}^{(k)}]$, $\rho^{(k+1)} = \ln\frac{2n_{k+1}+1}{2n_{k+1}}$ on $[(n_{k+1}+1)\tilde{N}^{(k)} + \epsilon, (3n_{k+1}+2)\tilde{N}^{(k)} - \epsilon]$, for suitably small ϵ, and such that

$$(166) \qquad \int_{(n_{k+1}+1)\tilde{N}^{(k)}}^{(n_{k+1}+1)\tilde{N}^{(k)}+\epsilon} \frac{1}{w_{F^{(k)}}(t)e^{\rho^{(k+1)}(t)}}dt = \frac{2n_{k+1}}{2n_{k+1}+1}\int_0^\epsilon \frac{1}{w_{F^{(k)}}(t)}dt,$$

$$(167) \qquad \int_{(3n_{k+1}+2)\tilde{N}^{(k)}-\epsilon}^{(3n_{k+1}+2)\tilde{N}^{(k)}} \frac{1}{w_{F^{(k)}}(t)e^{\rho^{(k+1)}(t)}}dt = \frac{2n_{k+1}}{2n_{k+1}+1}\int_{\tilde{N}_k-\epsilon}^{\tilde{N}^{(k)}} \frac{1}{w_{F^{(k)}}(t)}dt.$$

Notice that by taking parameters growing sufficiently fast, we can take $F^{(k+1)}$ close to the lift of $F^{(k)}$.

Let S be the projective limit of $\mathbb{R}/\tilde{N}^{(k)}\mathbb{Z}$, and let $v : S \to \mathbb{R}$ be given by $v(x) = \mathcal{V}^{(0)}(p_{S,S^{(0)}}(x))$. Let $F_t : S \to S$ be the projective limit of the $F_t^{(k)}$. The base dynamics will be the time-one map F_1 and the sampling function will be v.

By (163), for every k, if $E \in \Gamma = \bigcap \Gamma^{(k)}$,

$$(168) \qquad \sup_{1 \leq j \leq N^{(k)}} |\sup_x \ln\|A[F_1, v](E, x, 0, j)\| - \sup_t \ln\|A[\mathcal{V}_t^{(k)}](E, 0, j))\|| \leq \frac{1}{2^{k-1}},$$

and together with (164) we get, for $E \in \Gamma$

$$(169) \qquad \sup_x \frac{1}{N^{(k+1)}}\ln\|A[F_1, v](E, x, 0, N^{(k+1)})\| \leq \frac{1}{2^k} + \frac{1}{N^{(k+1)}2^k} \leq \frac{1}{2^{k-1}},$$

so that the Lyapunov exponent (with respect to any F_1-invariant measure) must vanish over Γ.

To conclude, let us show that the flow F is weak mixing: This implies that the discrete dynamics F_1 is weak mixing as well, and since F is minimal and uniquely ergodic, it also implies that F_1 is minimal and uniquely ergodic, so that Γ is contained in the essential support of the absolutely continuous spectrum for every x. In order to do this, we notice that for $0 \leq j \leq n_{k+1} - 1$

$$(170) \qquad p_{S^{(k+1)},S^{(k)}}(F_{jN^{(k)}}^{(k+1)}(t)) = F_{j/n_{k+1}}^{(k)}(p_{S^{(k+1)},S^{(k)}}(t)),$$

as long as $t \in [0, (n_{k+1}+1)(1-\frac{j}{n_{k+1}})\tilde{N}^{(k)}]$. On the other hand, for $0 \leq j \leq 2n_{k+1}-1$

$$(171) \qquad p_{S^{(k+1)},S^{(k)}}(F_{jN^{(k)}}^{(k+1)}(t)) = F_{j/2n_{k+1}}^{(k)}(p_{S^{(k+1)},S^{(k)}}(x)),$$

as long as $t \in [(n_{k+1}+1)\tilde{N}^{(k)} + 1, \tilde{N}^{(k+1)} - \frac{2n_{k+1}+1}{2n_{k+1}}j\tilde{N}^{(k)} - 1]$.

The conclusion proceeds along the same line as in Lemma 4.4. Take a measurable eigenfunction ψ taking values on the unit circle, associated to an eigenvalue $\theta \neq 0$, so that $\psi \circ F_t = e^{2\pi i\theta t}\psi$. Taking conditional expectations, we obtain $\psi^{(j)}$ on $S^{(j)}$, taking values on the closed unit disk, with $\lim \psi^{(j)}(p_{S,S^{(j)}}(x)) = \psi(x)$ for almost every x. We then conclude from (170) and (171) that $\frac{\theta j}{2n_{k+1}}$ is close to an integer for $1 \leq j \leq [n_{k+1}/2]$. This contradicts $\theta \neq 0$. $\qquad\square$

Remark 6.5. Using Remark 6.4, we can ensure in the construction that

$$(172) \qquad C = \sup_k \sup_{E \in \Gamma_k} \sup_t d(u[\mathcal{V}_t^{(k)}](E), i) < \infty.$$

This implies that

$$(173) \qquad \sup_k \sup_{E \in \Gamma_k} \sup_t \sup_j \|A[\mathcal{V}_t^{(k)}](E, 0, j)\| \le e^C,$$

and by (168),

$$(174) \qquad \sup_{E \in \Gamma} \sup_x \sup_j \|A[F_1, v](E, x, 0, j)\| \le e^C,$$

so all eigenfunctions with energies in Γ are bounded.

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