ON THE SPECTRUM AND LYAPUNOV EXPONENT OF LIMIT PERIODIC SCHRÖDINGER OPERATORS

ARTUR AVILA

Abstract. We exhibit a dense set of limit periodic potentials for which the corresponding one-dimensional Schrödinger operator has a positive Lyapunov exponent for all energies and a spectrum of zero Lebesgue measure. No example with those properties was previously known, even in the larger class of ergodic potentials. We also conclude that the generic limit periodic potential has a spectrum of zero Lebesgue measure.

1. Introduction

This work is motivated by a question in the theory of one-dimensional ergodic Schrödinger operators. Those are bounded self-adjoint operators of $\ell^2(\mathbb{Z})$ given by

$$(Hu)_n = u_{n+1} + u_{n-1} + v(f^n(x))u_n, \tag{1.1}$$

where $f : X \to X$ is an invertible measurable transformation preserving an ergodic probability measure μ and $v : X \to \mathbb{R}$ is a bounded measurable function, called the potential.

One is interested in the behavior for μ-almost every x. In this case, the spectrum is μ-almost surely independent of x. The Lyapunov exponent is defined as

$$L(E) = \lim \frac{1}{n} \int \ln \|A_n^{(E)}(x)\| d\mu(x), \tag{1.2}$$

where $A_n^{(E)}$ is the n-step transfer matrix of the Schrödinger equation $Hu = Eu$.

Here we will give first examples of ergodic potentials with a spectrum of zero Lebesgue measure such that the Lyapunov exponent is positive throughout the spectrum. This answers a question raised by Barry Simon (Conjecture 8.7 of [S]).

The example we will construct will belong to the class of limit periodic potentials. Those arise from continuous potentials over a minimal translation of a Cantor group (see §2 for a discussion of those notions). In our approach, we fix the underlying dynamics and vary the potential: it turns out that a dense set of such potentials provide counterexamples.

It is actually possible to incorporate a coupling parameter in our construction. Here is a precise version that can be obtained from our technique:

Theorem 1.1. *Let $f : X \to X$ be a minimal translation of a Cantor group. For a dense set of $v \in C^0(X, \mathbb{R})$ and for every $\lambda \neq 0$, the Schrödinger operator with potential λv has a spectrum of zero Lebesgue measure, and the Lyapunov exponent is a continuous positive function of the energy.*

Our result implies, by continuity of the spectrum, that a generic potential over a minimal translation of a Cantor group has a spectrum of zero Lebesgue measure.

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Corollary 1.2. *Let $f : X \to X$ be a minimal translation of a Cantor group. For generic $v \in C^0(X, \mathbb{R})$, and for every $\lambda \neq 0$, the Schrödinger operator with potential λv has a spectrum of zero Lebesgue measure (and the Lyapunov exponent is a continuous function of the energy which vanishes over the spectrum).*

The statements about the Lyapunov exponent in the generic context are rather obvious consequences of upper semicontinuity and density of periodic potentials. They highlight however that the generic approach is too rough and that care must be taken in the proof of Theorem 1.1 in order not to lose the Lyapunov exponent.

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2. Preliminaries

2.1. **From limit periodic sequences to Cantor groups.** Limit periodic potentials are discussed in depth in [AS]. Here we will restrict ourselves to some basic facts used in this paper.

Let σ be the shift operator on $\ell^\infty(\mathbb{Z})$, that is, $(\sigma(x))_n = x_{n+1}$. Let $\mathrm{orb}(x) = \{\sigma^k(x),\, k \in \mathbb{Z}\}$.

We say that x is periodic if $\mathrm{orb}(x)$ is finite. We say that x is limit periodic if it belongs to the closure, in $\ell^\infty(\mathbb{Z})$, of the set of periodic sequences. If x is limit periodic, we let $\mathrm{hull}(x)$ be the closure of $\mathrm{orb}(x)$ in $\ell^\infty(\mathbb{Z})$. It is easy to see that every $y \in \mathrm{hull}(x)$ is limit periodic.

Lemma 2.1. *If x is limit periodic then $\mathrm{hull}(x)$ is compact and it has a unique topological group structure with identity x such that $\mathbb{Z} \to \mathrm{hull}(x)$, $k \mapsto \sigma^k(x)$ is a homomorphism. Moreover, the group structure is Abelian and there exist arbitrarily small compact open neighborhoods of x in $\mathrm{hull}(x)$ which are finite index subgroups.*

Proof. Recall that a metric space is called totally bounded if for every $\epsilon > 0$ it is contained in the ϵ-neighborhood of a finite set. It is easy to see that a totally bounded subset of a complete metric space has compact closure.

If x is limit periodic then $\mathrm{orb}(x)$ is totally bounded: indeed if p is periodic and $\|x - p\| < \epsilon$ then $\mathrm{orb}(x)$ is contained in the ϵ-neighborhood of $\mathrm{orb}(p)$. Since $\ell^\infty(\mathbb{Z})$ is a Banach space, $\mathrm{hull}(x)$ is compact.

Clearly there exists a unique (cyclic) group structure on $\mathrm{orb}(x)$ such that the map $\mathbb{Z} \to \mathrm{orb}(x)$, $k \mapsto \sigma^k(x)$ is a homomorphism.

Let us show that the group structure uniformly continuous. We have

$$
\begin{aligned}
\|\sigma^{k+l}(x) - \sigma^{k'+l'}(x)\|_\infty &= \|\sigma^{k-k'}(x) - \sigma^{l'-l}(x)\|_\infty \\
&\leq \|\sigma^{k-k'}(x) - x\|_\infty + \|x - \sigma^{l'-l}(x)\|_\infty \\
&= \|\sigma^k(x) - \sigma^{k'}(x)\|_\infty + \|\sigma^l(x) - \sigma^{l'}(x)\|_\infty.
\end{aligned} \tag{2.1}
$$

where the inequality is just the triangle inequality and the equalities follow from the fact that σ is an isometry of $\ell^\infty(\mathbb{Z})$. Thus if $y, z, y', z' \in \mathrm{orb}(x)$ then $\|y\cdot z - y'\cdot z'\|_\infty \leq \|y - y'\|_\infty + \|z - z'\|_\infty$, which shows the uniform (even Lipschitz) continuity.

By uniform continuity, the group structure on $\mathrm{orb}(x)$ has a unique continuous extension to $\mathrm{hull}(x)$. Since the group structure on $\mathrm{orb}(x)$ is Abelian, its extension is still Abelian.

For the last statement, fix $\epsilon > 0$ and let p be periodic with $\|x - p\|_\infty < \epsilon/2$. Let k be such that $\sigma^k(p) = p$. Clearly the closure $\mathrm{hull}^k(x)$ of $\{\sigma^{kn}(x),\, n \in \mathbb{Z}\}$ is a compact subgroup of $\mathrm{hull}(x)$ of index at most k. Since $\mathrm{hull}(x)$ is the union of finitely many disjoint translates of $\mathrm{hull}^k(x)$, it follows that $\mathrm{hull}^k(x)$ is also open. Since σ is an isometry, $\mathrm{hull}^k(x)$ is contained in the $\epsilon/2$-ball around p, and hence it is contained in the ϵ-ball around x. □

By the previous lemma, $\mathrm{hull}(x)$ is compact and totally disconnected, so it is either finite (if and only if x is periodic) or it is a Cantor set.

If x is limit periodic but not periodic, we see that every y in $\mathrm{hull}(x)$ (which is also a limit periodic sequence) is of the form $y_n = v(f^n(y))$ where f is a minimal translation of a Cantor group ($f = \sigma|\mathrm{hull}(x)$) and v is continuous ($v(w) = w_0$).

2.2. **From Cantor groups to limit periodic sequences.** Let us now consider a Cantor group X and let $t \in X$. Let $f : X \to X$ be the translation by t. We say that f is minimal if $\{f^n(y),\, n \in \mathbb{Z}\}$ is dense in X for every $y \in X$. This is equivalent to $\{t^n,\, n \in \mathbb{Z}\}$ being dense in X. In this case, since there exists a dense cyclic subgroup, we conclude that X is actually Abelian.

Let $v : X \to \mathbb{R}$ be any continuous function. Let $\phi : X \to \ell^\infty(\mathbb{Z})$, $\phi(x) = (v(f^n(x))_{n\in\mathbb{Z}}$.

Lemma 2.2. *For every $x \in X$, $\phi(x)$ is limit periodic and $\phi(X) = \mathrm{hull}(x)$.*

Proof. It is enough to show that $\phi(x)$ is limit periodic, since $\phi(X)$ is compact and $\mathrm{orb}(\phi(x))$ is the image under ϕ of the set $\{f^n(x),\, x \in X\}$ which is dense in X.

Given $\delta > 0$ we must find a periodic sequence p such that $\|\phi(x) - p\|_\infty \le \delta$. Choose a compact open neighborhood W of the identity of X which is so small that if $y \in W$ then $|v(y \cdot z) - v(z)| \le \delta$.

Introduce a metric d on X, compatible with the topology. Choose an arbitrary compact open neighborhood W of the identity of X. Let $\epsilon > 0$ be such that if $y, z \in X$ are such that $y \in W$ and $z \notin W$ then $d(y, z) > \epsilon$. Choose $m > 0$ such that t^m is so close to the identity that for every $y \in X$, $d(y, f^m(y)) < \epsilon$. Then by induction on $|k|$, $t^{mk} \in W$ for every $k \in \mathbb{Z}$. It follows that the closure of $\{t^{km},\, k \in \mathbb{Z}\}$ is a compact subgroup of X contained in W. Clearly it has index at most m.

Let $p \in \ell^\infty(\mathbb{Z})$ be given by $p_i = v(f^j(w))$ where $0 \le j \le m-1$ is such that $i = j \bmod m$. Then $|\phi(x)_i - p_i| = |v(f^i(w)) - v(f^j(w))| = |v(y \cdot z) - v(z)|$ where $z = f^j(w)$ and $y = t^{i-j}$. Since $i = j \bmod m$, $t^{i-j} \in W$, and by the choice of W we have $|\phi(x)_i - p_i| \le \delta$. It follows that $\|\phi(x) - p\|_\infty \le \delta$ as desired. □

Remark 2.1. By the proof above, there exist arbitrarily small compact subgroups of finite index of X (such subgroups are automatically open as before).

2.3. **Limit periodic Schrödinger operators.** Given $f : X \to X$ a minimal translation of a Cantor group and $v : X \to \mathbb{R}$ a continuous function, we define for every $x \in X$ a Schrödinger operator $H = H_{f,v,x}$ by (1.1). A formal solution of $Hu = Eu$ satisfies

$$A_n^{(E,f,v)}(x) \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix} = \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} \tag{2.2}$$

where

$$A_n^{(E)}(x) = A_n^{(E,f,v)}(x) = S_{n-1} \cdots S_0 \text{ where } S_i = \begin{pmatrix} E - v(f^i(x)) & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.3}$$

The $A_n^{(E)}(x)$ are thus in $\mathrm{SL}(2,\mathbb{R})$, and are called the n-step transfer matrices. The Lyapunov exponent $L(E) = L(E,f,v)$ is defined by (1.2), where we take μ the Haar probability measure on X (this is the only possible choice actually, since minimal translations of Cantor groups are uniquely ergodic). (The limit in (1.2) exists by subadditivity, which also shows that lim may be replaced by inf.)

Remark 2.2. By subadditivity, $\frac{1}{2^k} \int \ln \|A_{2^k}^{(E)}(x)\| d\mu(x)$ is a decreasing sequence converging to $L(E)$. Allowing E to take values in $\mathbb{C}$, we conclude that $E \mapsto L(E)$ is the real part of a subharmonic function.

Lemma 2.3. *If $n \geq 2$, for every non-zero vector $z \in \mathbb{R}^2$, the derivative (with respect to E) of the argument of $A_n^{(E,f,v)}(x)z$ is strictly negative.*

Proof. Let $\rho_n(E,x,z)$ be the derivative (with respect to E) of the argument of $A_n^{(E,f,v)}(x)z$. It is easy to see that $\rho_1(E,x,z)$ is strictly negative whenever z is not vertical, and it is zero if z is vertical. By the chain rule, for $n \geq 2$, $\rho_n(E,x,z) = \sum_{i=1}^n \kappa_i \rho_1(E, f^{i-1}(x), A_{i-1}^{(E,f,v)}(x)z)$, where κ_i are strictly positive (since $A_{n-i}^{(E,f,v)}(f^i(x)) \in \mathrm{SL}(2,\mathbb{R})$ and hence preserves orientation). Since either z or $A_1^{(E,f,v)}(x)z$ is non-vertical, the result follows. $\square$

2.3.1. Let us endow the space $\mathcal{H}$ of bounded self-adjoint operators of $\ell^2(\mathbb{Z})$ with the norm $\|\Phi\| = \sup_{\|u\|_2=1} \|\Phi(u)\|_2$, and the space of compact subsets of $\mathbb{R}$ with the Caratheodory metric ($d(A,B)$ is the infimum of all r such that A is contained in the r-neighborhood of B and B is contained in the r-neighborhood of A). With respect to those metrics, it is easy to see that the spectrum is a 1-Lipschitz function of $\Phi \in \mathcal{H}$. Since the map $C^0(X,\mathbb{R})$, $v \mapsto H_{f,v,x}$ is also 1-Lipschitz, we conclude that the spectrum of $H_{f,v,x}$ is a 1-Lipschitz function of $v \in C^0(X,\mathbb{R})$. It also follows that the spectrum of $H_{f,v,x}$ depends continuously on x.

Since $H_{f,v,x}$ and $H_{f,v,f(x)}$ have obviously the same spectrum, and f is minimal, we conclude that the spectrum is actually x-independent. We will denote it $\Sigma(f,v)$.

2.3.2. We say that v is periodic (of period $n \geq 1$) if $v(f^n(x)) = v(x)$ for every $x \in X$. If v is a periodic potential, then it is locally constant, hence for any compact subgroup $Y \subset X$ contained in a sufficiently small neighborhood of id, the function v is defined over X/Y. If $v \in C^0(X,\mathbb{R})$ and $Y \subset X$ is a compact subgroup of finite index, then we can define another potential v^Y by convolution with Y: $v^Y(x) = \int_Y v(y \cdot x) d\mu_Y$ where μ_Y is the Haar measure on Y. The potential v^Y is then periodic. Since there are compact subgroups with finite index contained in arbitrarily small neighborhoods of id, this shows that the set of periodic potentials is dense in $C^0(X,\mathbb{R})$.

2.3.3. If v is n-periodic then $\mathrm{tr} A_n^{(E,f,v)}(x)$ is x-independent and denoted $\psi(E)$. Then $L(E,f,v)$ is the logarithm of the spectral radius of $A_n^{(E,f,v)}(x)$, for any $x \in X$. This shows that the Lyapunov exponent is a continuous function of both the potential and the energy *when one restricts considerations to potentials of period n.*

2.3.4. We will need some basic facts on the spectrum of periodic potentials, see [AMS], §3, for a discussion with further references.

If v is periodic of period n the spectrum $\Sigma(f,v)$ of H is the set of $E \in \mathbb{R}$ such that $|\psi(E)| \leq 2$. Thus for periodic potentials, we have $\Sigma(f,v) = \{E \in \mathbb{R},\ L(E,f,v) = 0\}$.

The function ψ is a polynomial of degree n. It can be shown that ψ has n distinct real roots and its critical values do not belong to $(-2,2)$, moreover, E is a critical point of ψ with $\psi(E) = \pm 2$ if and only if $A_n^{(E,v,f)}(x) = \pm\,\mathrm{id}$. From this one derives a number of consequences about the structure of periodic spectra:

(1) The set of all E such that $|\psi(E)| < 2$ has n connected components whose closures are called bands,
(2) If E is in the boundary of some band, we obviously have $\mathrm{tr} A_n^{(E,f,v)}(x) = \pm 2$,
(3) Conversely, if $\mathrm{tr} A_n^{(E,f,v)}(x) = \pm 2$, E is in the boundary of some band, thus the spectrum is the union of the bands,
(4) If two different bands intersect then their common boundary point satisfies $A_n^{(E,f,v)}(x) = \pm\,\mathrm{id}$.

2.3.5. We will need some simple estimates on the Lebesgue measure of the bands and of the spectrum.

Lemma 2.4. *Let v be a periodic potential of period n.*

(1) *The measure of each band is at most $\frac{2\pi}{n}$,*
(2) *Let $C \geq 1$ be such that for every E in the union of bands, there exists $x \in X$ and $k \geq 1$ such that $\|A_k^{(E,f,v)}(x)\| \geq C$. Then the total measure of the spectrum is at most $\frac{4\pi n}{C}$.*

Proof. If E belongs to some band, $A_n^{(E,f,v)}(x)$ is conjugate in $\mathrm{SL}(2,\mathbb{R})$ to a rotation: there exists $B^{(E)}(x) \in \mathrm{SL}(2,\mathbb{R})$ such that $B^{(E)}(x)A_n^{(E,f,v)}(x)B^{(E)}(x)^{-1} \in \mathrm{SO}(2,\mathbb{R})$. This matrix is not unique, since $RB^{(E)}(x)$ has the same property for $R \in \mathrm{SO}(2,\mathbb{R})$, but this is the only ambiguity. In particular, the Hilbert-Schmidt norm squared $\|B^{(E)}(x)\|^2_{\mathrm{HS}}$ (the sum of the squares of the entries of the matrix of $B^{(E)}(x)$) is a well defined function $b^{(E)}(x)$, which obviously satisfies $b^{(E)}(f^n(x)) = b^{(E)}(x)$. This allows us to define an x-independent function $\hat{b}(E)$ which is zero if E does not belong to a band and for E in a band is given by

$$\hat{b}(E) = \frac{1}{4\pi n}\sum_{i=0}^{n-1} b^{(E)}(f^i(x)). \tag{2.4}$$

It turns out that $\hat{b}(E)$ is related to the *integrated density of states* by the formula $N(E) = \int_{-\infty}^{E} \hat{b}(E)dE$. As a consequence, we conclude that for any band $I \subset \Sigma(f,v)$, $\int_I \hat{b}(E)dE = \frac{1}{n}$ (in particular $\int_{\mathbb{R}} \hat{b}(E)dE = 1$). See [AD2], §2.4.1 for a discussion of this point of view on the integrated density of states.

The first statement is then an immediate consequence of $\hat{b}(E) \geq \frac{1}{2\pi n}$ which in turn comes from the estimate $\|B\|^2_{\mathrm{HS}} \geq 2$, $B \in \mathrm{SL}(2,\mathbb{R})$.

For the second estimate, it is enough to show that for every E in a band we have $\hat{b}(E) \geq \frac{C}{4\pi n}$. Notice that

$$B^{(E)}(f^k(x))A_k^{(E,f,v)}(x)A_n^{(E,f,v)}(x)A_k^{(E,f,v)}(x)^{-1}B^{(E)}(f^k(x))^{-1} = B^{(E)}(f^k(x))A_n^{(E,f,v)}(f^k(x))B^{(E)}(f^k(x))^{-1} \in \mathrm{SO}(2,\mathbb{R}). \tag{2.5}$$

Thus $B^{(E)}(f^k(x))A_k^{(E,f,v)}(x)$ conjugates $A_n^{(E,f,v)}(x)$ to a rotation so it coincides with $RB^{(E)}(x)$ for some $R \in \mathrm{SO}(2,\mathbb{R})$. Thus

$$C \leq \|A_k^{(E,f,v)}(x)\| \leq \|B^{(E)}(f^k(x))^{-1}\|\|B(x)\|, \tag{2.6}$$

and there exists $y \in X$ (either $y = x$ or $y = f^k(x)$) such that $C \leq \|B^{(E)}(y)\|^2 \leq b^{(E)}(y)$. It follows that $\hat{b}(E) \geq \frac{C}{4\pi n}$. □

2.3.6. We conclude with a weak continuity result for the Lyapunov exponent.

Lemma 2.5. *Let $v^{(n)} \in C^0(X,\mathbb{R})$ be a sequence converging uniformly to $v \in C^0(X,\mathbb{R})$. Then $L(E,f,v^{(n)}) \to L(E,f,v)$ in L^1_{loc}.*

Proof. This follows from the proof of Lemma 1 of [AD1]. Indeed for every compact interval $I \subset \mathbb{R}$, there exists a continuous function $g : I \to \mathbb{R}$, non-vanishing in $\mathrm{int}\, I$, such that

$$\lim_{n\to\infty} \int_I \max\{L(E,f,v^{(n)}) - L(E,f,v), 0\}g(E)dE = 0 \tag{2.7}$$

and

$$\lim_{n\to\infty} \int_I \min\{L(E,f,v^{(n)}) - L(E,f,v), 0\}g(E)dE = 0 \tag{2.8}$$

(see the last two equations in page 396 of [AD1]). The result follows. □

3. Proof of Theorem 1.1

Fix some Cantor group X, and let $f : X \to X$ be a minimal translation. Then the homomorphism $\mathbb{Z} \to X$, $n \mapsto f^n(\mathrm{id})$ is injective with dense image. For simplicity of notation, we identify the integers with its image under this homomorphism.

For a given potential $w \in C^0(X,\mathbb{R})$ and $n \geq 1$, we write $L(E,w) = L(E,f,w)$ for the Lyapunov exponent with energy E corresponding to the potential w.

Since X is Cantor, there exists a decreasing sequence of Cantor subgroups $X_k \subset X$ with finite index such that $\cap X_k = \{0\}$. Let P_k be the set of potentials which are defined on X/X_k. Potentials in P_k are n_k-periodic where n_k is the index of X_k. If $w \in C^0(X,\mathbb{R})$ is a periodic potential, then it belongs to some P_k. Let $P = \cup P_k$ be the set of periodic potentials (which is a dense subset of $C^0(X,\mathbb{R})$, see §2.3.2).

For $n \geq 1$, we write $A_n^{(E,w)}(x) = A_n^{(E,f,w)}(x)$ for the n-step transfer matrix associated with the potential w at x. We also let $A_n^{(E,w)} = A_n^{(E,w)}(0)$. The spectrum will be denoted by $\Sigma(w) = \Sigma(f,w)$.

We will actually work with finite families W of periodic potentials. Here we allow for multiplicity of elements, so the number of elements in W, denoted by $\#W$, may be larger than the number of distinct elements of W. For simplicity of notation, we will often treat W as a set (writing for instance $W \subset P$). We write $L(E,W) = \frac{1}{\#W}\sum_{w\in W} L(E,w)$. (More formally, and generally, one could work with probability measures with compact support contained in P_k for all k sufficiently large.)

The core of the construction is contained in the following two lemmas.

Lemma 3.1. *Let B be an open ball in $C^0(X,\mathbb{R})$, let $W \subset P \cap B$ be a finite family of potentials, and let $M \geq 1$. Then there exists a sequence $W^n \subset P \cap B$ such that*

(1) $L(E, \lambda W^n) > 0$ *whenever* $M^{-1} \leq |\lambda| \leq M$, $E \in \mathbb{R}$,
(2) $L(E, \lambda W^n) \to L(E, \lambda W)$ *uniformly on compacts (as functions of* $(E, \lambda) \in \mathbb{R}^2$*).*

Lemma 3.2. *Let B be an open ball in $C^0(X,\mathbb{R})$ and let $W \subset P \cap B$ be a finite family of potentials. Then for every K sufficiently large, there exists $W_K \subset P_K \cap B$ such that*

(1) $L(E, \lambda W_K) \to L(E, \lambda W)$ *uniformly on compacts (as functions of* $(E, \lambda) \in \mathbb{R}^2$*),*
(2) *The diameter of* W_K *is at most* n_K^{-10},
(3) *For every* $\lambda \in \mathbb{R}$, *if* $\inf_{E \in \mathbb{R}} L(E, \lambda W) \geq \delta \# W n_k$ *then for every* $w \in W_K$, $\Sigma(\lambda w)$ *has Lebesgue measure at most* $e^{-\delta n_K/2}$.

Before proving the lemmas, let us conclude the proof of Theorem 1.1. First we combine both lemmas:

Lemma 3.3. *Let $B \subset C^0(X,\mathbb{R})$ be an open ball and let $W \subset P \cap B$ be a finite family of potentials. Then for every $M \geq 1$, there exist $\delta > 0$, an open ball B' with closure contained in B, with diameter at most M^{-1} and $W' \subset P \cap B'$ such that*

(1) *The diameter of* B' *is at most* M^{-1},
(2) $|L(E, \lambda W') - L(E, \lambda W)| < M^{-1}$ *for* $|E|, |\lambda| \leq M$,
(3) $L(E, \lambda W) > \delta$ *for every* $M^{-1} \leq |\lambda| \leq M$ *and* $E \in \mathbb{R}$,
(4) *For every* $w \in B'$ *and* $M^{-1} \leq |\lambda| \leq M$ *the Lebesgue measure of* $\Sigma(\lambda w)$ *is at most* M^{-1}.

Proof. First apply Lemma 3.1 to find some $\tilde{W} \subset P \cap B$ such that $L(E, \lambda \tilde{W}) > 0$ for every $E \in \mathbb{R}$ and $M^{-1} \leq |\lambda| \leq M$ (it is easy to see that

$$L(E, \lambda w) \geq 1 \text{ if } |E| \geq \|\lambda w\| + 4, \tag{3.1}$$

so this is really a statement about bounded energies which follows from Lemma 3.1), and $|L(E, \lambda \tilde{W}) - L(E, W)| < M^{-1}/4$ for every $|E|, |\lambda| \leq M$. By continuity of the Lyapunov exponent for periodic potentials (§2.3.3) and compactness (and (3.1) to take care of large energies), we conclude that there exists $\delta > 0$ such that $L(E, \lambda \tilde{W}) > 2\delta$ for every $E \in \mathbb{R}$ and $M^{-1} \leq |\lambda| \leq M$.

Let us now apply Lemma 3.2 to $W = \tilde{W}$ and let $W' = W_K$ for K large. Then W' is contained in a ball $B' \subset B$ with diameter $n_K^{-10} < M^{-1}$ centered around some $w' \in W'$. Both estimates on $L(E, \lambda W')$ are clear from the statement of Lemma 3.2 (using again (3.1) for large $|E|$). To estimate the measure of $\Sigma(\lambda w)$ for $w \in B'$, we notice that $\Sigma(\lambda w)$ is contained in a $M n_K^{-10}$ neighborhood of $\Sigma(\lambda w')$ (by 1-Lipschitz continuity of the spectrum, see §2.3.1). Using that $\Sigma(\lambda w')$ has at most n_K connected components and has measure at most $e^{-\delta (\# \tilde{W} n_k)^{-1} n_K/2}$, the result follows. $\square$

Given an open ball $B_0 \subset C^0(X,\mathbb{R})$ and $W_0 \subset P \cap B_0$, and $\epsilon_1 > 0$, we can proceed by induction, applying the previous lemma, to define, for every $i \geq 1$, open balls B_i with $\overline{B}_i \subset B_{i-1}$, finite families of periodic potentials $W_i \subset P \cap B_i$, and constants $0 < \delta_i < 1$ and $\epsilon_{i+1} = \min\{\epsilon_i, \delta_i\}/10$ such that

(1) $L(E,\lambda W_i) \geq \delta_i$ for $E \in \mathbb{R}$ and $\epsilon_i \leq |\lambda| \leq \epsilon_i^{-1}$,
(2) $|L(E,\lambda W_i) - L(E,\lambda W_{i-1})| < \epsilon_i$ for $|E|, |\lambda| \leq \epsilon_i^{-1}$,
(3) for every $w \in B_i$ and $\epsilon_i \leq |\lambda| \leq \epsilon_i^{-1}$, $\Sigma(\lambda w)$ has measure at most ϵ_i,

Then the common element w_∞ of all the B_i is such that $\Sigma(\lambda w_\infty)$ has zero Lebesgue measure for every $\lambda \neq 0$. Notice that $L(E,\lambda W_i)$ converges uniformly on compacts to a continuous function, positive if $\lambda \neq 0$, which by general considerations must coincide with $L(E,\lambda w_\infty)$. Indeed, if $w_n \to w$ then $L(E,w_n) \to L(E,w)$ in L^1_{loc} by Lemma 2.5. So $L(E,\lambda w_\infty)$ coincides almost everywhere with $\lim L(E,\lambda W_i)$. Since $E \mapsto L(E,\lambda w_\infty)$ is the real part of a subharmonic function (see Remark 2.2) and $E \mapsto \lim L(E,\lambda W_i)$ is continuous, they coincide everywhere.

Since B_0 was arbitrary, the denseness claim of Theorem 1.1 follows.

3.1. **Proof of Lemma 3.1.** Let k be such that $W \subset P_k$. For every $K > k$, choose $N_1(K) > 0$ such that if $|E| \leq K$, $|\lambda| \leq K$, $w \in W$ and $w' \in P_K$ are such that w' is $\frac{2n_k+1}{N_1(K)}$ close to w then $|L(E,\lambda w') - L(E,\lambda w)| < \frac{1}{K}$. Here we use the continuity of the Lyapunov exponent for periodic potentials, see §2.3.3.

For $w \in W$, $K > k$, $1 \leq j \leq 2n_k+1$, we define potentials $w^{K,j} \in P_K$ by

$$w^{K,j}(i) = w(i),\ 0 \leq i \leq n_K - 2 \text{ and } w^{K,j}(n_K-1) = w(n_K-1) + \frac{j}{N_1(K)}. \tag{3.2}$$

(This uniquely defines $w^{K,j}$ by periodicity.)

Claim 3.4. *For every $\lambda \neq 0$, $K > k$ there exists $1 \leq j \leq 2n_k+1$ such that $\Sigma(\lambda w^{K,j})$ has exactly n_K components.*

Proof. Recall that for every $w' \in P_m$, there exist exactly $2n_m$ values of E such that $\mathrm{tr} A_m^{(E,w')} = \pm 2$, if one counts the *exceptional* energies such that $A_m^{(E,w')} = \pm\,\mathrm{id}$ with multiplicity 2, see §2.3.4.

For each j such that $\Sigma(\lambda w^{K,j})$ does not have exactly n_K components, there exists at least one energy $E_j \in \Sigma(\lambda w^{K,j})$ with $A_{n_K}^{(E_j,\lambda w^{K,j})} = \pm\,\mathrm{id}$. Then

$$A_{n_K}^{(E_j,\lambda w)} = \pm \begin{pmatrix} 1 & \frac{-\lambda j}{N_1(K)} \\ 0 & 1 \end{pmatrix}. \tag{3.3}$$

But since w is n_k-periodic, this means that

$$A_{n_k}^{(E_j,\lambda w)} = \pm \begin{pmatrix} 1 & \frac{-\lambda j n_k}{N_1(K) n_K} \\ 0 & 1 \end{pmatrix}. \tag{3.4}$$

This implies, in particular, that $A_{n_k}^{(E_j,\lambda w)} \neq A_{n_k}^{(E_{j'},\lambda w)}$ for $j \neq j'$, thus we must also have $E_j \neq E_{j'}$ for $j \neq j'$. But there can be at most $2n_k$ values of E such that $\mathrm{tr} A_{n_k}^{(E,\lambda w)} = \pm 2$. Thus there must be some $1 \leq j \leq 2n_k+1$ such that $\Sigma(\lambda w^{K,j})$ has exactly n_K connected components. □

By the previous claim and compactness, there exists $\delta = \delta(W,K,M) > 0$ such that for $w \in W$ and $M^{-1} \leq |\lambda| \leq M$, there exists $1 \leq j = j(K,\lambda,w) \leq 2n_k+1$ such that $\Sigma(\lambda w^{K,j})$ has n_K components and the measure of the smallest gap is at least δ. Choose an integer $N_2(K)$ with $N_2(K) > \frac{4\pi M}{\delta n_K}$.

For $0 \leq l \leq N_2(K)$ and $w^{K,j}$ as above, let $w^{K,j,l} \in P_K$ be given by $w^{K,j,l} = w^{K,j} + \frac{4\pi M l}{n_K N_2(K)}$.

Claim 3.5. *For every $M^{-1} \leq |\lambda| \leq M$, $w \in W$, $K > k$,*

$$\cap_{0 \leq l \leq N_2(K)} \Sigma(\lambda w^{K,j(K,\lambda,w),l}) = \emptyset. \tag{3.5}$$

Proof. Each of the connected components of $\Sigma(\lambda w^{K,j})$ has measure at most $\frac{2\pi}{n_K}$, see Lemma 2.4. Since $N_2(K) > \frac{4\pi M}{\delta n_K}$, for every E there exists at least some l with $0 \leq l \leq N_2(K)$ such that $E - \lambda \frac{4\pi M l}{n_K N_2(K)} \notin \Sigma(\lambda w^{K,j})$, that is, $E \notin \Sigma(\lambda w^{K,j,l})$, which gives the result. □

Let W^K be the family obtained by collecting the $w^{K,j,l}$ for different $w \in W$, $1 \leq j \leq 2n_k + 1$ and $0 \leq l \leq N_2(K)$. By the second claim, $L(E, \lambda W^K) > 0$ for every $M^{-1} \leq |\lambda| \leq M$ and $E \in \mathbb{R}$ (since $L(E, \lambda w) > 0$ if $E \notin \Sigma(\lambda w)$, see §2.3.4).

To conclude, it is enough to show that

$$\max_{1 \leq j \leq 2n_k+1} \max_{0 \leq l \leq N_2(K)} |L(E, \lambda w^{K,j,l}) - L(E, \lambda w)| \to 0 \tag{3.6}$$

uniformly on compacts of $(E, \lambda) \in \mathbb{R}^2$. Write

$$\begin{aligned} |L(E, \lambda w^{K,j,l}) - L(E, \lambda w)| \leq & |L(E, \lambda w^{K,j,l}) - L(E - \lambda \frac{4\pi M l}{n_K N_2(K)}, \lambda w)| \\ & + |L(E - \lambda \frac{4\pi M l}{n_K N_2(K)}, \lambda w) - L(E, \lambda w)|. \end{aligned} \tag{3.7}$$

Then the first term in the right hand side is smaller than K^{-1} provided $K \geq |E| + 4\pi M^2$ (by the choice of $N_1(K)$), while the second term in the right hand side is bounded by $\max_{w \in W} \sup_{|t| \leq \frac{4\pi M^2}{n_K}} |L(E + t, \lambda w) - L(E, \lambda w)|$ which converges to zero uniformly on compacts of $(E, \lambda) \in \mathbb{R}^2$ as $K \to \infty$ (by continuity of the Lyapunov exponent for periodic potentials, §2.3.3).

3.2. **Proof of Lemma 3.2.** Assume that $W \subset P_k$, $n_k \geq 2$, and let $K > k$ be large. Order the elements $w^1, ..., w^m$ of W. Let $r = [n_K / m n_k]$.

First consider a potential $w \in P_K$ obtained as follows. It is enough to define $w(l)$ for $0 \leq l \leq n_K - 1$. Let $I_j = [j n_k, (j+1) n_k - 1] \subset \mathbb{Z}$ and let $0 = j_0 < j_1 < ... < j_{m-1} < j_m = n_K / n_k$ be a sequence such that $j_{i+1} - j_i - r \in \{0, 1\}$. Given $0 \leq l \leq n_K - 1$, let j be such that $l \in I_j$, let i be such that $j_{i-1} \leq j < j_i$ and let $w(l) = w^i(l)$.

For any sequence $t = (t_1, ..., t_m)$ with $t_i \in \{0, ..., r-1\}$, let $w^t \in P_K$ be the potential defined as follows. Let $0 \leq l \leq n_K - 1$, and let j be such that $l \in I_j$. If $j = j_i - 1$ for some $1 \leq i \leq m$, we let $w^t(l) = w(l) + r^{-20} t_i$. Otherwise we let $w^t(l) = w(l)$.

Let W_K be the family consisting of all the w^t. The claimed diameter estimate is obvious for large K.

Let us show that $L(E, \lambda W_K) \to L(E, \lambda W)$ uniformly on compacts. It is enough to restrict ourselves to compact subsets of $(E, \lambda) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$, since it is easy to see that $L(E, \lambda w) - L(E, 0) \to 0$ uniformly as $\|\lambda w\| \to 0$.

For fixed E and λ, we write

$$A_{n_K}^{(E, \lambda w^t)} = C^{(t_m, m)} B^{(m)} \cdots C^{(t_1, 1)} B^{(1)}, \tag{3.8}$$

where $C^{(t_i, i)} = A_{n_k}^{(E - \lambda r^{-20} t_i, \lambda w^i)}$ and $B^{(i)} = (A_{n_k}^{(E, \lambda w^i)})^{j_i - j_{i-1} - 1}$. Notice that, for E and λ in a compact set, the norm of the $C^{(t_i, i)}$-type matrices stays bounded as r grows, while the $B^{(i)}$ matrices may get large.

Find some cutoff $(\ln\ln r)^{-m} \leq c \leq (\ln\ln\ln r)^m/(\ln\ln r)^m$ such that if $\|B^{(i)}\| < e^{cr}$ then $\|B^{(i)}\| < e^{(\ln\ln\ln r)^{-1} c n_K}$ (this is possible since there are only m different $B^{(i)}$).

Call i good if $\|B^{(i)}\| \geq e^{cr}$. If no $B^{(i)}$ is good, then $L(E,\lambda W_K)$ and $L(E,\lambda W)$ are close since they are both $o(1)$ with respect to r. So we can assume that there exists at least one good $B^{(i)}$. Let $i_1 < ... < i_d$ be the list of all good i. Write $A^{(E,\lambda w^t)}(0) = \hat{C}^{(d)}\hat{B}^{(d)} \cdots \hat{C}^{(1)}\hat{B}^{(1)}$, where for $1 \leq j \leq d$ we let $\hat{C}^{(j)} = C^{(t_{i_j}, i_j)}$ and $\hat{B}^{(j)} = B^{(i_j)} D^{(j)}$, where we denote $D^{(j)} = C^{(i_j-1, t_{i_j-1})} B^{(i_j-1)} \cdots C^{(i_{j-1}+1, t_{i_{j-1}+1})} B^{(i_{j-1}+1)}$ (denoting also $i_0 = 0$).

By the choice of the cutoff, we have $\|D^{(j)}\| \leq e^{cr/2}$ for r large (uniformly on compacts of $(E,\lambda) \in \mathbb{R}^2$), so $\|\hat{B}^{(j)}\| \geq e^{cr/2}$.

Claim 3.6. *As r grows,*

$$\frac{1}{n_K} \sum_{j=1}^{d} \ln \|\hat{B}^{(j)}\| \to L(E,\lambda W) \tag{3.9}$$

uniformly on compacts of E and λ.

Proof. Notice that this is equivalent to showing that

$$\frac{1}{n_K} \sum_{i=1}^{m} \ln \|B^{(i)}\| \to L(E,\lambda W) \tag{3.10}$$

(uniformly), which in turn is equivalent to

$$\frac{1}{m} \sum_{i=1}^{m} \frac{1}{n_k(j_i - j_{i-1} - 1)} \ln \|B^{(i)}\| \to L(E,\lambda W) \tag{3.11}$$

(uniformly). Thus it is enough to show that

$$\frac{1}{n_k(j_i - j_{i-1} - 1)} \ln \|B^{(i)}\| \to L(E,\lambda w^i) \tag{3.12}$$

(uniformly). But $B^{(i)}$ is just the $j_i - j_{i-1} - 1$ iterate of the matrix $A_{n_k}^{(E,\lambda w^i)}$, whose spectral radius is precisely the exponential of $n_k L(E,\lambda w^i)$. But it is easy to see that $\|T^n\|^{1/n}$ converges to the spectral radius of T uniformly on compacts of $T \in \mathrm{SL}(2,\mathbb{R})$. This gives (3.12) and the result. □

Let s_j be the most contracted direction of $\hat{B}^{(j)}$ and let u_j be the image under $\hat{B}^{(j)}$ of the most expanded direction.

Let us say that t is j-nice, $1 \leq j \leq d$, if the absolute value of the angle between $\hat{C}^{(j)} u_j$ and s_{j+1} is at least r^{-70} (with the convention that $j+1 = 1$ for $j = d$).

Claim 3.7. *Let r be sufficiently large, and let t be j-nice. If z is a non-zero vector making an angle at least r^{-80} with s_j, then $z' = \hat{C}^{(j)}\hat{B}^{(j)} z$ makes an angle at least r^{-80} with s_{j+1} and $\|z'\| \geq \|\hat{B}^{(j)}\| r^{-100} \|z\|$.*

Proof. Let $0 \leq \theta \leq \pi/2$ be the angle between z and s_j, and let $0 \leq \theta' \leq \pi/2$ be the angle between $z'' = \hat{B}^{(j)} z$ and u_j.

The orthogonal projection of z'' on u_j has norm $\|z\|\|\hat{B}^{(j)}\| \sin\theta$. Since $\|\hat{C}^{(j)}\|$ stays bounded as r grows, we conclude that $\|z'\| \geq \|\hat{B}^{(j)}\| r^{-100} \|z\|$.

On the other hand, $\tan\theta'\tan\theta = \|\hat{B}^{(j)}\|^{-2}$. Since $\|\hat{B}^{(j)}\| \geq e^{cr/2} \geq r^{400}$ for r large, it follows that $\theta' < r^{-100}$. The boundedness of $\hat{C}^{(j)}$ again implies that the angle between z' and $\hat{C}^{(j)}u_j$ is at most r^{-90}. Since t is j-nice, z' makes an angle at least r^{-80} with s_{j+1}. □

It follows that if t is very nice in the sense that it is j-nice for every $1 \leq j \leq d$, then if z is a non-zero vector making an angle at least r^{-80} with s_1 then $z' = A_{n_K}^{(E,\lambda w^t)} z$ also makes an angle at least r^{-80} with s_1, and moreover $\|z'\| \geq \prod_{j=1}^d r^{-100}\|\hat{B}^{(j)}\|$. By (3.9), it follows that $L(E,\lambda w^t) - L(E,\lambda W) \to 0$ as r grows, at least for very nice t.

To conclude the estimate on the Lyapunov exponent, it is thus enough to show that most t are nice, in the sense that for every $\epsilon > 0$, for every r sufficiently large, the set of $t \in \{0,...,r-1\}^m$ which are not very nice has at most ϵr^m elements. A more precise estimate is provided below.

Claim 3.8. *For every r sufficiently large, the set of t which are not very nice has at most mr^{m-1} elements.*

Proof. We will show in fact that, for every $1 \leq j \leq d$, if for every $1 \leq k \leq m$ with $k \neq i_j$ one chooses $t_k \in \{0,...,r-1\}$, there exists at most one "exceptional" $t_{i_j} \in \{0,...,r-1\}$ such that $t = (t_1,...,t_m)$ is not j-nice. Thus the set of t which are not j-nice has at most r^{m-1} elements and the estimate follows.

Once t_k is fixed for $1 \leq k \leq m$ with $k \neq i_j$, both u_j and s_{j+1} become determined, but $\hat{C}^{(j)} = C^{(t_{i_j},i_j)} = A_{n_k}^{(E-\lambda r^{-20}t_{i_j},\lambda w^{i_j})}$ depends on t_{i_j}.

Since $n_k \geq 2$, we can apply Lemma 2.3 to conclude that for any non-zero vector $z \in \mathbb{R}^2$, the derivative of the argument of the vector $A_{n_k}^{(E',\lambda w^{i_j})} z$ as a function of E' is strictly negative, and hence bounded away from zero and infinity, uniformly on z and on compacts of $(E',\lambda) \in \mathbb{R}^2$, and independently of r.

If r is sufficiently large, we conclude that for every $0 \leq l \leq r-2$, there exists a rotation R_l of angle θ with $r^{-21} < \theta < r^{-19}$ such that $C^{(l+1,i_j)}u_j = R_l C^{(l,i_j)}u_j$. It immediately follows that there exists at most one choice of $0 \leq t_{i_j} \leq r-1$ such that $C^{(t_{i_j},i_j)}u_j$ has angle at most r^{-90} with s_{j+1}, as desired. □

We now estimate the measure of the spectrum. Let $w^i \in W$ be such that $L(E,\lambda w^i) \geq \delta n_k m$. Then $\|A_{(r-1)n_k}^{(E,\lambda w^t)}((j_{i-1})n_k)\| \geq e^{\delta m(r-1)n_k^2}$. Since E is arbitrary, we can apply Lemma 2.4 to conclude that the measure of the spectrum is at most $4\pi n_K e^{-\delta m(r-1)n_k^2} \leq e^{-\delta n_K/2}$ for r large. The result follows.

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CNRS UMR 7599, Laboratoire de Probabilités et Modèles aléatoires, Université Pierre et Marie Curie–Boîte courrier 188, 75252–Paris Cedex 05, France

Current address: IMPA, Estrada Dona Castorina 110, Rio de Janeiro, 22460-320, Brazil

E-mail address: artur@math.sunysb.edu