ON RIGIDITY OF CRITICAL CIRCLE MAPS

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ABSTRACT. We give examples of analytic critical circle maps which are not $C^{1+\alpha}$ rigid.

1. Introduction

A critical circle map is a C^1 homeomorphism $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ with a unique critical point at 0. Fix $d \geq 3$ odd and let Ω be the space of analytic critical circle maps such that the critical point has order d.

Yoccoz showed that if $f \in \Omega$ has irrational rotation number then f is topologically conjugate to a irrational rotation [Y]. Thus if $f, g \in \Omega$ have the same irrational rotation number then there exists a unique homeomorphism $h = h_{f,g} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ such that $h(0) = 0$ and $h \circ f = g \circ h$. The following result was announced by Teplinsky and Khanin [TK].

Theorem 1.1. *Let $f, g \in \Omega$ have the same irrational rotation number. Then h is C^1.*

This theorem immediately provokes the question of whether one could promote h to $C^{1+\alpha}$ for some $\alpha > 0$. This was known not to be possible if one assumes that f and g are merely C^∞, unless further hypothesis on the rotation number are made [FM1], [FM2]. The following result of Khmelev and Yampolski [KY] seemed to indicate that the analytic case could be different (the question of whether h is always $C^{1+\alpha}$ is explicitly posed in [KY]).

Theorem 1.2. *Let $f, g \in \Omega$ have the same irrational rotation number. Then h is $C^{1+\alpha}$ at 0 for some $\alpha > 0$, that is*

$$h(x) = Dh(0)x + O(|x|^{1+\alpha}). \tag{1.1}$$

Here we show:

Theorem A. *There exists $f, g \in \Omega$ with the same irrational rotation number such that h is not $C^{1+\alpha}$ for any α.*

Several other non-rigidity results in this line can be obtained by similar methods. For instance, analytic unimodal maps with essentially bounded combinatorics are not necessarily $C^{1+\alpha}$-rigid. In another direction, the parameter space of critical circle maps is not $C^{1+\alpha}$ rigid. We will not discuss those issues here, since the principle is always the same: two parabolic maps with different parabolic renormalizations produce, after unfolding, definite oscillations in arbitrarily small scales.

We have not tried to obtain any explicit arithmetic condition for non-rigidity. An heuristic argument, assuming that parabolic renormalization is not more than exponentially convergent, indicates that if the coefficients a_n in the continued fraction expansion of the rotation number do not satisfy $\ln a_n = O(n)$ then $C^{1+\alpha}$ rigidity should not hold.

Acknowledgements: I am indebted to M. Lyubich for asking me the question and to K. Khanin and M. Yampolski for convincing me to write this note.

Date: May 13, 2005.

2. Perturbations of parabolic maps

Let Δ be the space of entire functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+1) = f(x)+1$, and which restrict to a homeomorphisms $\mathbb{R} \to \mathbb{R}$ with negative Schwarzian derivative

$$Sf = \frac{D^3 f}{Df} - \frac{3}{2}\left(\frac{D^2 f}{Df}\right)^2 < 0 \tag{2.1}$$

with critical points of order d at integer points (and no further critical points). We endow it with a complete metric dist, compatible with the natural topology of Δ. Maps in Δ can be seen as critical circle homeomorphisms in the natural way.

Let $\rho(f)$, $f \in \Delta$ be the rotation number, $\rho(f) = \lim \frac{f^n(x)}{n}$, $x \in \mathbb{R}$. For $\alpha \in \mathbb{R}$, let Δ_α be the set of all f with $\rho(f) = \alpha$. If $\rho(f) = \frac{p}{q} \in \mathbb{Q}$, then $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ has a unique periodic orbit in $\{f^i(x)\}_{i=0}^q$, of period q which is non-repelling (by the negative Schwarzian derivative condition [MS]). Moreover, if $Df^q(x) = 1$ we have necessarily $D^2 f(q) \neq 0$.

We write $\Delta_{\frac{p}{q}} = \Delta_{\frac{p}{q},+} \cup \Delta_{\frac{p}{q},-} \cup \Delta_{\frac{p}{q},0}$, corresponding to the cases (+) $Df^q(x) = 1$ and $D^2 f^q(x) > 0$, (-) $Df^q(x) = 1$ and $D^2 f^q(x) < 0$, (0) $Df^q(x) < 1$.

Let $f_t(x) = f(x) - t$, $t \in \mathbb{R}$. Then $\rho(f_t)$ is continuous non-increasing, and it is constant in a neighborhood of 0 if and only if $f \in \Delta_{\frac{p}{q},0}$ for some $\frac{p}{q} \in \mathbb{Q}$. If $f \in \Delta_{\frac{p}{q},+}$ then $\rho(f_t)$ is constant in $[0,\epsilon)$ for ϵ small. If $f \in \Delta_{\frac{p}{q},-}$ then $\rho(f_t)$ is constant in $-(\epsilon, 0]$ for ϵ small.

Let T be the set of all C^1 diffeomorphisms $h : \mathbb{R} \to \mathbb{R}$ with $h(x+1) = h(x)$, $h(0) = 0$, endowed with the natural topology.

Theorem 2.1. *Let $f_0, g_0 \in \Delta_{\frac{p}{q},\varepsilon}$, where $\frac{p}{q} \in \mathbb{Q}$ and $\varepsilon \in \{+,-\}$. Let K be a compact subset of T. Then there exists a trigonometric polynomial v, such that $D^k v(0) = 0$ for $0 \le k \le d$, and sequences $\epsilon_n^0, \epsilon_n^1, \epsilon_n^2 \to 0$ such that $f_n = f_0 + \epsilon_n^1$ and $g_n = g_0 + \epsilon_n^0 v + \epsilon_n^2$ have the same irrational rotation number, and there exists no $h \in K$ such that $h \circ f_n = g_n \circ h$.*

We will prove the result only in the case $\frac{p}{q} = 0$, $\varepsilon = -$, the general case being analogous. We will need some results in parabolic renormalization, which we summarize in the next section.

2.1. Parabolic renormalization.

Let Υ be the set of all C^3 critical circle homeomorphisms $f : \mathbb{R} \to \mathbb{R}$, with a unique fixed point $p = p_f \in (-1, 0)$, satisfying $Df(p) = 1$, $D^2 f(p) < 0$.

Fix some $f_0 \in \Upsilon$. The following estimates hold uniformly for maps f in a neighborhood of f_0.

We have $f^n(x) \to p$ and $f^{-n}(x) \to p+1$ if $x \in (p, p+1)$. Near p we have $f(p+h) = f(p) + h + \frac{D^2 f(p)}{2} h^2 + O(h^3)$. It follows that for every $\epsilon > 0$, $f^n(x) = p + \frac{2}{D^2 f(p) n} + O(\frac{\ln n}{n^2})$ and $f^{-n}(x) = p + 1 - \frac{2}{D^2 f(p) n} + O(\frac{\ln n}{n^2})$ if $x \in [p+\epsilon, p+1-\epsilon]$.

Lemma 2.2. *If $n > n_0(\epsilon)$ and $x \in [p+\epsilon, p+1-\epsilon]$ then $|\frac{(n+k)^2}{n^2} Df^k(f^n(x)) - 1| < \epsilon$ for every $k \ge 0$.*

Proof. We have $\ln Df^k(f^n(x)) = \sum_{j=n}^{n+k-1} \ln Df(f^j(x)) = \sum D^2 f(p)(f^j(x) - p) + O((f^j(x)-p)^2) = -2\sum_{j=n}^{n+k-1} \frac{1}{j} + O(\frac{\ln j}{j^2}) = -2 \ln \frac{n+k}{n} + o(1)$. □

Let $f_\epsilon = f - \epsilon$.

Define $\Phi_{f,n,\epsilon,+} : (p, p+1) \to \mathbb{R}$ by

$$\Phi_{f,n,\epsilon,+}(x) = -\frac{2n^2}{D^2 f(p)}(f_\epsilon^n(x) - f_\epsilon^n(0)). \tag{2.2}$$

Let $\Phi_{f,n,+} = \Phi_{f,n,0,+}$.

Lemma 2.3. *The sequence $\Phi_{f,n,+}$ converges C^1 uniformly on compact sets to a C^1-homeomorphism $\Phi_{f,+} : (p, p+1) \to \mathbb{R}$ with critical points in $\{f^k(0)\}_{k \ge 0}$. Moreover $\Phi_{f,+}(f(x)) = \Phi_{f,+}(x) - 1$.*

Proof. It follows from the previous lemma that $D\Phi_{f,n,+}$ converges uniformly on compacts, and indeed $\ln D\Phi_{f,n,+}(x)$ converges if x is not in $\{f^k(x)\}_{k\leq 0}$. Since $\Phi_{f,n,+}(0)=0$, the sequence $\Phi_{f,n,+}$ has a limit $\Phi_{f,+}$ which is C^1, monotonically increasing, and whose critical points are in $\{f^k(0)\}_{k\leq 0}$. Notice that $\Phi_{f,n,+}(f(x))-\Phi_{f,n,+}(x)=\frac{-2n^2}{D^2f(p)}(f^{n+1}(x)-f^n(x))\to 1$. This implies also that $\Phi_{f,+}$ is a homeomorphism. □

Define $\Phi_{f,n,\epsilon,-}:(p,p+1)\to\mathbb{R}$ by

$$\Phi_{f,n,\epsilon,-}(x)=-\frac{2n^2}{D^2f_\epsilon(p)}(f_\epsilon^{-n}(x)-f_\epsilon^{-n}(0)). \tag{2.3}$$

Let $\Phi_{f,n,-}=\Phi_{f,n,0,-}$.

As before we have the following.

Lemma 2.4. *The sequence $\Phi_{f,n,-}^{-1}$ converges C^1 uniformly on compacts to a C^1-homeomorphism $\Phi_{f,-}^{-1}:\mathbb{R}\to(p,p+1)$ with critical values in $\{f^k(0)\}_{k<0}$. Moreover $\Phi_{f,-}(f(x))=\Phi_{f,-}(x)-1$.*

The previous two lemmas imply easily the following.

Lemma 2.5. *There exists $\epsilon(n)>0$ such that if $0<\epsilon_n<\epsilon(n)$ then $\Phi_{f,n,\epsilon_n,+}\to\Phi_{f,+}$ and $\Phi_{f,n,\epsilon_n,-}^{-1}\to\Phi_{f,-}^{-1}$ C^1 uniformly on compacts.*

We define $R_0f=\Phi_+\circ\Phi_-^{-1}$. Then R_0f is a C^1 critical circle homeomorphism $\mathbb{R}\to\mathbb{R}$ with $R_0f(0)=0$.

Let $n\geq n_0$, $0<\epsilon<\epsilon_0(n)$. Let $m=m(f,n,\epsilon)>0$ be the first moment such that $f_\epsilon^{n+m}(0)\in -1+[f_\epsilon^{n+1}(0),f_\epsilon^n(0))$. Let $c=c_{f,n,\epsilon}=f_\epsilon^{-m+n+1}(-1)$ be the critical point of f_ϵ^m in $(f_\epsilon^{n+1}(0),f_\epsilon^n(0)]$.

Let $F_{f,n,\epsilon}(x)=\frac{2n^2}{D^2f(p)}(f_\epsilon^m(c+\frac{D^2f(p)}{2n^2}x)-f_\epsilon^m(c))$.

Let $\Psi_{f,n,\epsilon}(x)=\frac{2n^2}{D^2f(p)}(f_\epsilon^{m-2n-1}(c+\frac{D^2f(p)}{2n^2}x)-f_\epsilon^{m-2n-1}(c))$.

Lemma 2.6. *There exists $\epsilon(n)>0$ such that if $0<\epsilon_n<\epsilon(n)$ then $\Psi_{f,n,\epsilon_n}\to\mathrm{id}$ C^1 uniformly on compacts.*

Proof. Since $\Psi(0)=0$, we only need to show that $D\Psi(x)$ is δ close to 1 for $|x|<C$, provided that $n>n_0$ and $0<\epsilon<\epsilon(n)$. This is equivalent to showing that $Df_\epsilon^{m-2n-1}(x)$ is δ close to 1 when $|x-c|<\frac{D^2f(p)}{2n^2}C$. We may replace this last condition by the weaker $x\in I$, where $I=[f_\epsilon^k(c),f_\epsilon^{-k}(c)]$, for k large depending on C. Then $|I|$ is of order $2k/n^2$. Also, $f_\epsilon^{m-2n-1}(I)=[f^{-n+k}(-1),f^{-n-k}(-1)]$ has size also about $2k/n^2$. By the classical Schwartz estimate on distortion, $\ln\frac{Df(y)}{Df(x)}$ for $x,y\in I$ is at most $2k$ times the total variation of $\ln Df_\epsilon$ in $[f_\epsilon^{-k}(c),f_\epsilon^{-n+k}(-1)]$. Since f is C^3, this goes to 0 as n goes to ∞. The result follows. □

Theorem 2.7. *There exists $\epsilon(n)\to 0$ such that if $0<\epsilon_n<\epsilon(n)$ then F_{f,n,ϵ_n} converges to R_0f, C^1 uniformly on compacts of $\mathbb{R}$.*

Proof. Notice that

$$\begin{aligned} F_{f,n,\epsilon}(x)&=\frac{2n^2}{D^2f(p))}(f_\epsilon^{n+1}(f_\epsilon^{m-n-1}(c+\frac{D^2f(p)}{2n^2}x))-f_\epsilon^{n+1}(-1)) \\ &=\frac{n^2}{(n+1)^2}\Phi_{f,n+1,\epsilon,+}(\Phi_{f,n,\epsilon,-}^{-1}(\Psi_{f,n,\epsilon}(x))). \end{aligned} \tag{2.4}$$

The result follows. □

Lemma 2.8. *Let $q \in (f(0),0)$. There exists $\kappa > 0$ with the following property. Let $v : \mathbb{R} \to \mathbb{R}$ be C^3 such that $v(x+1) = v(x)$, $D^k v(0) = D^k v(p) = 0$, $0 \le k \le 2$, $f + \epsilon v$ is a critical circle map for ϵ small, $|D^3 v(x)| < \kappa$, $x \in [p, p+1] \setminus [q-\kappa, q+\kappa]$, $v(q) = 1$. Then for ϵ sufficiently small and $g = f + \epsilon v$, $R_0 g \ne R_0 f$.*

Proof. We first note that a simple induction shows that for $x \in (f^2(0), f(0))$, $|g^n(x) - f^n(x)| \le C\epsilon\kappa n^{-2} \ln n$. This easily implies the bound $|\ln D\Phi_{g,+}(x) - \ln D\Phi_{f,+}(x)| < C\epsilon\kappa$. Since $g(0) = f(0)$ and $\Phi_{g,+}(g(0)) = \Phi_{f,+}(f(0)) = 1$, we have $|\Phi_{g,+}(g(q)) - \Phi_{f,+}(g(q))| \le C\epsilon\kappa$.

Similar estimates yield $|\Phi_{g,-}(q) - \Phi_{f,-}(q)| \le C\epsilon\kappa$.

We have $|\Phi_{g,+}(g(q)) - \Phi_{f,+}(f(q))| \ge \epsilon D\Phi_{f,+}(q) - C\epsilon\kappa > \frac{D\Phi_{f,+}(q)}{2}\epsilon$. This gives $|\Phi_{g,+}(q) - \Phi_{f,+}(q)| > \frac{D\Phi_{f,+}(q)}{2}\epsilon$.

Since $R_0 g(\Phi_{g,-}(q)) = \Phi_{g,+}(q)$, if $R_0 g = R_0 f$ we would have $|\Phi_{g,+}(q) - \Phi_{f,+}(q)| = |R_0 f(\Phi_{g,-}(q)) - \Phi_{f,+}(q)| \le DR_0 f(\Phi_{f,-}(q)) C\epsilon\kappa$, a contradiction. $\square$

Lemma 2.9. *Let $f, g \in \Upsilon$, and let $\epsilon_n, \delta_n \to 0$ be such that there exists a C^1 diffeomorphism $h_n \in T$ such that $h_n \circ f_{\epsilon_n} = g_{\delta_n} \circ h_n$. Assume that the h_n converge in the C^1 topology to a C^1 diffeomorphism h. Then $R_0 f = R_0 g$.*

Proof. It is easy to see that $Dh(p) = D^2 g(h(p))/D^2 f(p)$. Let $c_n = c_{f,n,\epsilon_n}$. Then

$$
\begin{aligned}
F_{f,n,\epsilon_n}(x_n) &= \frac{2n^2}{D^2 f(p)} \left(f^m_{\epsilon_n}\left(c_n + \frac{D^2 f(p)}{2n^2} x_n\right) - f^m_{\epsilon_n}(c_n)\right) \\
&= \frac{2n^2}{D^2 g(h(p))} \left(h_n \circ f^m_{\epsilon_n}\left(c_n + \frac{D^2 f(p)}{2n^2} x\right) - h_n(f^m_{\epsilon_n}(c_n))\right) + o(1) \\
&= \frac{2n^2}{D^2 g(h(p))} \left(g^m_{\delta_n} \circ h_n\left(c_n + \frac{D^2 f(p)}{2n^2} x\right) - h_n(f^m_{\epsilon_n}(c_n))\right) + o(1) \\
&= \frac{2}{D^2 g(h(p))} \left(g^m_{\delta_n}\left(h_n(c_n) + \frac{D^2 g(h_n(p))}{2n^2} x\right) - h_n(f^m_{\epsilon_n}(c_n))\right) + o(1) \\
&= F_{g,n,\delta_n}(x) + o(1).
\end{aligned} \tag{2.5}
$$

The result follows. $\square$

2.2. **Proof of Theorem 2.1.** Recall that we are restricting to the case $\frac{p}{q} = 0$, $\varepsilon = -$ for simplicity. Let v be a trigonometrical polynomial satisfying the bounds given in Lemma 2.8, and also the condition $D^k v(0) = 0$, $0 \le k \le d$.

If $R_0 f_0 \ne R_0 g_0$, let $\epsilon^0_n = 0$. Otherwise, let $\epsilon^0_n > 0$ be any sequence converging to 0. Fix n large. Then $R_0 f_0 \ne R_0(g_0 + \epsilon^0_n)$. Let $f = f_0$, $g = g_0 + \epsilon^0_n$. Let ϵ_s, δ_s be sequences converging to 0 such that $f + \epsilon_s$ and $g + \delta_s$ have the same irrational rotation number. By Lemma 2.9, for large s there exists no $h \in K$ such that $h \circ (f + \epsilon_s) = (g + \delta_s) \circ h$. We take $\epsilon^1_n = \epsilon_s$ and $\epsilon^2_n = \delta_s$ for such an s.

3. Proof of Theorem A

The set of $h \in T$ which are $C^{1+\alpha}$ for some $\alpha > 0$ can be written as a union of compact sets K_n, $K_{n+1} \supset K_n$.

Lemma 3.1. *Let $f, g \in \Delta$, $\rho(f) = \rho(g) \in \mathbb{R} \setminus \mathbb{Q}$. For every $\epsilon > 0$, $k > 0$, there exists $\hat f, \hat g \in \Delta$ such that $\rho(\hat f) = \rho(\hat g) \in \mathbb{R} \setminus \mathbb{Q}$, $\mathrm{dist}(f, \hat f), \mathrm{dist}(g, \hat g) < \epsilon$, and if $\mathrm{dist}(\tilde f, \hat f), \mathrm{dist}(\tilde g, g) < \delta$ then $k!\rho(\tilde f) \notin \mathbb{Z}$, and there exists no $h \in K_k$ such that $h \circ \tilde f = \tilde g \circ h$.*

Proof. Let $f_0, g_0 \in \Delta_{\frac{p}{q},\varepsilon}$ be such that $\mathrm{dist}(f, f_0), \mathrm{dist}(g, g_0) < \frac{\epsilon}{2}$. Let $K = K_k$, and let $\hat f = f_n$, $\hat g = g_n$ be as in Theorem 2.1, for n large. Then $\mathrm{dist}(f, \hat f), \mathrm{dist}(g, \hat g) < \epsilon$. If the result does not hold then there exists $\tilde{f}_n \to \hat{f}$, $\tilde{g}_n \to \hat{g}$ and $h_n \in K$ such that $h_n \circ \tilde{f}_n = \tilde{g}_n \circ h_n$. But then $h \circ \hat{f} = \hat{g} \circ h$ for some $h \in K$, contradiction. $\square$

Let us define a sequence $f_n, g_n \in \Delta$, $\epsilon_n > 0$ by induction as follows. Take $f_0, g_0 \in \Delta$ arbitrary with $\rho(f_0) = \rho(g_0)$ irrational, let $\epsilon_0 = \frac{1}{10}$.

If f_n, g_n, ϵ_n are defined, take $f = f_n$, $g = g_n$, $\epsilon = \epsilon_n/10$, $k = n$ in the previous lemma, and set $\epsilon_{n+1} = \min\{\frac{\epsilon_n}{10}, \delta\}$, $f_{n+1} = \hat{f}$, $g_{n+1} = \hat{g}$. Let $f = \lim f_n$, $g = \lim g_n$. It follows that $\rho(f) = \rho(g)$ is irrational and $h_{f,g} \notin K_n$, $n \geq 0$, so $h_{f,g}$ is not $C^{1+\alpha}$.

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