# ALMOST REDUCIBILITY AND ABSOLUTE CONTINUITY I

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## 1. Introduction

Here we consider one-frequency analytic $\mathrm{SL}(2,\mathbb{R})$ cocycles, that is, linear skew-products over an irrational rotation $x \mapsto x+\alpha$ of the circle $\mathbb{R}/\mathbb{Z}$ which have the form $(\alpha, A) : (x, w) \mapsto (x+\alpha, A(x)\cdot w)$ with $A \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$. Projectivizing the second coordinate, a one-frequency cocycle can be seen as a map of the two-torus. From the dynamical point of view, interest in the class of one-frequency cocycles is largely motivated by the presence of sufficient complexity to allow for both KAM and nonuniformly hyperbolic types of behavior [H], yet in a setting where one may hope to investigate the natural issues that arise (dynamics of typical parameters, the phase-transition between regimes,...), due to the availability of very effective analytical tools. (See §1.3 for further, non-dynamical, motivation.)

**Nonuniform hyperbolicity.** The iterates of the cocycle have the form $(\alpha, A)^n = (n\alpha, A_n)$ with $A_n(x) = A(x+(n-1)\alpha)\cdots A(x)$, and the *Lyapunov exponent* is defined by

$$L = \lim_{n\to\infty} \frac{1}{n} \int \ln \|A_n(x)\| dx. \tag{1.1}$$

We say that $(\alpha, A)$ is *uniformly hyperbolic* if the cocycle iterates grow exponentially uniformly on $x \in \mathbb{R}/\mathbb{Z}$ (see [Yoc]). Uniform hyperbolicity is robust (it corresponds to an open set of cocycles) and easily analyzed. *Nonuniformly hyperbolic* cocycles are, by definition, those which have a positive Lyapunov exponent but are not uniformly hyperbolic: the corresponding dynamics on the two-torus is quite intrincate and may display such features as "recurrent critical points" similar to chaotic multimodal maps [Y], [Bj]. The theory of nonuniform hyperbolicity is quite developed, through the works of Bourgain, Goldstein, Jitomirskaya and Schlag [BG], [GS1], [BJ1], [GS2], [GS3].

**KAM behavior.** If $A(x)$ is a constant elliptic matrix, the cocycle dynamics on the two-torus is a quasiperiodic motion. The KAM Theorem shows that this behavior tends to persist for many (large measure set of parameters) perturbations [DS], [H]. But perturbations may also become uniformly hyperbolic (as resonances lead to formation of "Arnold tongues" in parameter space). More interestingly, certain oscillating behavior may arise (from the coexistence of infinitely many strong resonances) for a topologically large set of perturbations in the complement of uniform hyperbolicity [E].

**Global theory.** Since neither KAM nor nonuniformly hyperbolic behavior tells the whole story about one-frequency cocycles (after excluding the easy uniformly hyperbolic ones), it is natural to try to incorporate both as parts of a "global theory", and recent breakthroughs in parameter analysis led to a concrete such

program [A2], [A3]. In it, cocycles which are not uniformly hyperbolic are classified in three regimes:

(1) *Supercritical*, or nonuniformly hyperbolic,
(2) *Subcritical*, if the cocycle iterates $\|A_n(z)\|$ are uniformly subexponentially bounded through some strip $\{|\Im z| < \epsilon\}$,
(3) *Critical* otherwise.

A key point of this classification is that (in the complement of uniform hyperbolicity) both supercriticality and subcriticality are stable (respectively, by [BJ1] and [A2]), while criticality is unstable (it is the boundary of supercriticality, see [A3]). Moreover, in [A3] it is shown that criticality is "negligible" in the sense that it does not appear at all in typical one-parameter families (this is quite convenient for the theory since very little is known about the dynamics of critical cocycles, apart that they are rare). Naturally, one still is left with the problem of describing the stable regimes. This is not a problem with supercriticality, which is after all a new name for an old concept with a very well developed theory, as mentioned above. Subcriticality on the other hand, is a relatively new concept, which was first suggested to be relevant in 2006 (see [AJ2]).

**The Almost Reducibility Conjecture.** In fact, [AJ2] basically proposed that the well developed theory of cocycles close to constant ([E], [BJ2], [AJ2], [AFK]), can be applied to all subcritical cocycles, by the application of suitable coordinate changes. Recall that in the cocycle context the natural notion of coordinate change is given by a conjugacy $(x, w) \mapsto (x, B(x) \cdot w)$ with $B : \mathbb{R}/\mathbb{Z} \to \mathrm{PSL}(2, \mathbb{R})$ analytic, which takes $(\alpha, A)$ to $(\alpha, A')$ with $A'(x) = B(x + \alpha)A(x)B(x)^{-1}$. Let us say that $(\alpha, A)$ is *almost reducible* if there exist $\epsilon > 0$ and a sequence of analytic maps $B^{(n)} : \mathbb{R}/\mathbb{Z} \to \mathrm{PSL}(2, \mathbb{R})$, admitting holomorphic extensions to the common strip $\{|\Im z| < \epsilon\}$ such that $B^{(n)}(z+\alpha)A(z)B^{(n)}(z)^{-1}$ converges to a constant uniformly in $\{|\Im z| < \epsilon\}$ (the $B^{(n)}$ themselves are allowed to diverge). Essentially by definition, the concept of almost reducibility prescribes a domain of applicability of "local theories" of cocycles close to constant (this includes whatever can be achieved by KAM techniques). The basic hope expressed by [AJ2] can be thus expressed in the form of the *Almost Reducibility Conjecture* (ARC): subcriticality implies almost reducibility.

Our first main result establishes a generic version of the ARC. Let us say that $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is exponentially Liouville if $\limsup \frac{\ln q_{n+1}}{q_n} > 0$, where $q_n$ is the sequence of denominators of continued fraction approximants.

**Theorem 1.1.** *If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is exponentially Liouville, then any subcritical cocycle $(\alpha, A)$ is almost reducible.*

*Remark* 1.1. We should mention that the results of this paper (obtained in 2006-2007), precede the results of [A2] and [A3] (obtained in 2008-2009), and (being at the time the only evidence for the ARC), played a large role in motivating those works. On the other hand, we have recently established the ARC for almost every frequency (by very different methods, in particular making use of [A2]).

The proof of Theorem 1.1 depends on the very precise understanding of the behavior of the complexification of the cocycle for rational approximations. Particularly, we would like to highlight the novel application in dynamics of one complex variable results such as the famous Corona Theorem of Carleson [C].

1.1. **Almost reducibility near constants.** Besides the global considerations above, Theorem 1.1 has an important consequence for the local theory of cocycles close to constant. Indeed, a most basic question is whether effective control of the dynamics can be achieved for *all* one-frequency cocycles close to constant, despite the fact that usual KAM techniques involve positive measure restrictions on parameters to avoid the effects of resonances. One most significative advance in this direction was the progress from "positive measure" to "full measure" understanding, achieved through non-KAM methods, the localization-duality technique used in [BJ2], [AJ3]. However, this technique seems to breakdown for generic frequencies. The new approach developed in this paper perfectly complements this previous work, leading to the long-sought:

**Corollary 1.2.** *Any one-frequency cocycle close to constant is almost reducible.*

Our results also allow us to conclude unrestricted *stability* of almost reducibility, a result of much theoretical significance, since it allows for the clean delimitation of sets of parameters with "close to constant" behavior:

**Corollary 1.3.** *Almost reducibility is stable, in the sense that it defines an open set in* $(\mathbb{R}\setminus\mathbb{Q})\times C^\omega(\mathbb{R}/\mathbb{Z},\mathrm{SL}(2,\mathbb{R}))$.

From a more practical point of view, Corollary 1.2 naturally raises the question of whether one really does have a good understanding of almost reducible cocycles, especially for frequencies falling outside the scope of the previously developed techniques. It turns out that we do get very precise estimates for the coordinate changes involved, which allows for the a very effective analysis by periodic shadowing, a theme which we will further explore in the second paper in this series (in connection with the absolutely continuity of spectral measures of one-frequency Schrödinger operators, see §1.3). In the remaining of this introduction, we will exhibit the applicability of our results in addressing a few other natural questions.

1.2. **Rotations reducibility.** Let us now specialize to the case of cocycles $(\alpha, A)$ with $A$ homotopic to a constant, so that the corresponding dynamics in the two-torus is isotopic to the identity. Then (see [H], [JM]), $(\alpha, A)$ has a well defined *rotation vector*, which describes the drift (modulo 1) of the dynamics in the universal cover of the two-torus. The drift in the first coordinate is clearly $\alpha$, while the drift in the second coordinate is called the *fibered rotation number*. The fibered rotation number is called *exactly resonant* if it belongs to $\mathbb{Z}\oplus\alpha\mathbb{Z}$.[1] Uniformly hyperbolic cocycles have exactly resonant fibered rotation numbers [JM].

We say that $(\alpha, A)$ is *reducible* if it is analytically conjugate to a constant. If this constant is elliptic, the dynamics is quasiperiodic. As it turns out, non-ellipticity may only arise if the fibered rotation number is exactly resonant.

If $\alpha$ is very well approximated by rational numbers, reducibility is very rare, even for the simplest cocycles, with values in one-parameter subgroups. Indeed, for such cocycles, reducibility involves the solution of the cohomological equation, which has well known small divisor obstructions. This should not prevent us to try to put cocycles in a simpler form, such as diagonal cocycles, or cocycles of rotations. Indeed, it is well known that uniformly hyperbolic cocycles are "diagonal reducible".

[1]Notice that our definition of the fibered rotation number is twice the most commonly used in the litterature. This is because we define the two-torus dynamics by interpreting the second coordinate as a projective coordinate, instead of taking it as the normalized angle in $\mathbb{R}^2$.

More recently, [AFK] showed that cocycles near constant are *rotations reducible* for a positive measure set of parameters, independently of $\alpha$. More precisely [AFK], Theorem 1.3, shows that there exists an explicit full measure set of "non-resonant" fibered rotation numbers $\mathcal{NR}(\alpha)$ such that if the fibered rotation number $\rho$ of $(\alpha, A)$ belongs to $\mathcal{NR}(\alpha)$ then $(\alpha, A)$ is rotations reducible (i.e., it is conjugated to $(\alpha, A')$ where $A'$ takes values in $\mathrm{SO}(2,\mathbb{R})$), provided $A$ is sufficiently close to constant, *depending on* $\rho$.

Due to the dependence of the closeness quantifier on $\rho$, this result corresponds to positive measure, but not full measure, results in parameter space. However, for an almost reducible cocycle, the closeness quantifier may be improved arbitrarily by conjugation. Thus it is not surprising that it is possible to show that *any* almost reducible cocycle $(\alpha, A)$ with fibered rotation number in $\mathcal{NR}(\alpha)$ is rotations reducible. There is a caveat in this argument, since conjugacy may change the fibered rotation number by an element of $\mathbb{Z} \oplus \alpha\mathbb{Z}$, if the change of coordinates is not isotopic to a constant. However, it turns out that almost reducibility implies "almost reducibility with changes of coordinates isotopic to a constant", for cocycles that are not uniformly hyperbolic:[2]

**Theorem 1.4.** *If $(\alpha, A)$ is almost reducible and not uniformly hyperbolic, then there exist $\epsilon' > 0$, a sequence $B^{(n)} : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ with bounded analytic extensions to $\{|\Im z| < \epsilon'\}$ and a rotation $R_* \in \mathrm{SO}(2,\mathbb{R})$, such that $B^{(n)}$ is homotopic to a constant and*

$$\lim_{n\to\infty} \sup_{|\Im z|<\epsilon'} \|B^{(n)}(z+\alpha)A(z)B^{(n)}(z)^{-1} - R_*\| = 0. \tag{1.2}$$

Thus we conclude:

**Corollary 1.5** (Rotations reducibility under a full measure condition)**.** *If $(\alpha, A)$ is sufficiently close to a constant, or more generally, if it is almost reducible, and its fibered rotation number belongs to $\mathcal{NR}(\alpha)$, then $(\alpha, A)$ is rotations reducible.*

1.3. **One-frequency Schrödinger operators.** The main non-dynamical motivation to study $\mathrm{SL}(2,\mathbb{R})$ cocycles comes from its applications to the theory of an important class of one-dimensional Schrödinger operators.

A one-frequency Schrödinger operator is a bounded self-adjoint opeator $H = H_{\alpha,v,\theta}$ on $\ell^2(\mathbb{Z})$ of the form

$$(Hu)_n = u_{n+1} + u_{n-1} + v(\theta + n\alpha)u_n, \tag{1.3}$$

where $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$ (the frequency), $v \in C^\omega(\mathbb{R}/\mathbb{Z},\mathbb{R})$ (the potential) and $\theta \in \mathbb{R}/\mathbb{Z}$ (the phase) are all important parameters.

The analysis of one-frequency Schrödinger operators is intimately connected to that of a particular family of one-frequency cocycles, with the special form

$$A(x) = A^{(E-v)}(x) = \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix}, \tag{1.4}$$

where $E \in \mathbb{R}$ (the energy) is a parameter: Indeed, a formal solution of the eigenvalue equation $Hu = Eu$ satisfies $\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = A_n(\theta) \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix}$. One basic connection

[2]For uniformly hyperbolic cocycles, there is an obvious obstruction, since diagonal cocycles have fibered rotation number 0.

between dynamics and spectral theory is that an energy $E$ is in the spectrum of $H_{\alpha,v,\theta}$ if and only if $(\alpha, A^{(E-v)})$ is not uniformly hyperbolic.

One-frequency Schrödinger operators are particularly interesting for reasons that mirror our early dynamical considerations: It is a class that allows for two very distinct behaviors, transport and localization (roughly corresponding to KAM and nonuniformly hyperbolic dynamics). Accordingly, much of the theory of $\mathrm{SL}(2,\mathbb{R})$ cocycles has been in fact developed having in mind the Schrödinger applications.

Key information about the spectral theory of the Schrödinger operator comes from understanding the Lebesgue decomposition of the spectral measures into absolutely continuous, singular continuous, and atomic (pure point) components. Particularly, the absolutely continuous (ac) part is associated with the strongest transport properties (ballistic motion).

A fundamental result ([LS], [K]) is that the ac part of spectral measures is supported in the set $\{L(E) = 0\}$ of energies for which the cocycle $(\alpha, A^{(E-v)})$ has zero Lyapunov exponent (in other words, the ac part gives zero weight to $\{L > 0\}$). However, $\{L = 0\}$ may also support non-ac spectrum (this is the case of the so-called critical almost Mathieu operator [AK]). This raises the question of what is the exact dynamical counterpart of absolutely continuous spectrum. One of our goals in this work is to establish that the counterpart is precisely almost reducibility. This has the important theoretical consequence of the *stability* of absolutely continuous spectrum (in view of Corollary 1.3). Especially, it follows that ac spectrum lives in an open set "separate" from singular spectrum.

Establishing the link between absolutely continuous spectrum and almost reducibility is the main result in the second part of this series [A4], and it involves a considerable amount of spectral theory preparation (this is the reason we have split the presentation). However, here we will still be able to conclude the following version for almost every phase.

**Corollary 1.6.** *For any $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, $v \in C^\omega(\mathbb{R}/\mathbb{Z},\mathbb{R})$, and for almost every $\theta \in \mathbb{R}/\mathbb{Z}$, the ac (respectively, singular) component of any spectral measure of (1.3) gives full (respectively, zero) weight to the set of almost reducible energies.*

**Corollary 1.7.** *If $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and the potential $v \in C^\omega(\mathbb{R}/\mathbb{Z},\mathbb{R})$ is close to constant, then for almost every $\theta \in \mathbb{R}\setminus\mathbb{Q}$, the spectral measures of (1.3) are ac.*

This result is closely related to a regularity result about another important object in the spectral theory of (1.3), the *integrated density of states* (i.d.s.). The i.d.s. $N : \mathbb{R} \to [0,1]$ is the asymptotic distribution of eigenvalues of large block restrictions of (1.3), and it is a continuous non-decreasing function onto $[0,1]$. As it turns out, the i.d.s. is directly related to the fibered rotation number [AS]: $1 - N(E)$ is a determination of the fibered rotation number of $(\alpha, A^{(E-v)})$. It is known that the i.d.s. is Lipschitz at any energy for which the cocycle $(\alpha, A^{(E-v)})$ is rotations reducible (see, e.g., the proof of [AJ1], Theorem 6.1). By Corollary 1.5, the image under $N$ of the set of almost reducible energies at which the i.d.s. is not Lipschitz has zero Lebesgue measure (it is contained in the complement of $\mathcal{NR}(\alpha)$). Thus we conclude:

**Corollary 1.8.** *The restriction of the i.d.s. of (1.3) to the open set of almost reducible energies is absolutely continuous. In particular, if the potential $v \in C^\omega(\mathbb{R}/\mathbb{Z},\mathbb{R})$ is close to constant then the whole i.d.s. is absolutely continuous.*

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## 2. Rational approximation

Let $C^\omega_\delta(\mathbb{R},*)$, $*=\mathbb{R},\mathrm{SL}(2,\mathbb{R}),...$, be the space of bounded analytic functions with values in $*$ which admit a bounded analytic extension to the strip $\{|\Im z|<\delta\}$, with the norm $\|a\|_\epsilon=\sup_{|\Im z|<\delta}|a(z)|$. We let $C^\omega_\delta(\mathbb{R}/\mathbb{Z},*)\subset C^\omega_\delta(\mathbb{R},*)$ be the subspace of 1-periodic functions, with the same norm. Let $R_\theta=\begin{pmatrix}\cos 2\pi\theta & -\sin 2\pi\theta\\ \sin 2\pi\theta & \cos 2\pi\theta\end{pmatrix}$.

Theorem 1.1 will be obtained as a consequence of an estimate for *periodic* cocycles with large period.

**Theorem 2.1.** *For every $0<\epsilon<\epsilon_0$ there exists $C>0$ such that if $\delta_1>0$ is sufficiently small, then for every $p/q\in\mathbb{Q}$ with $q$ sufficiently large, if $A\in C^\omega_{\epsilon_0}(\mathbb{R}/\mathbb{Z},\mathrm{SL}(2,\mathbb{R}))$ is such that*

$$\max_{0\le k\le q}\ln\|A_k\|_{\epsilon_0}\le\delta_1 q,\quad \textit{with}\quad A_k(x)=A(x+(k-1)p/q)\cdots A(x), \tag{2.1}$$

*then there exist $B\in C^\omega_\epsilon(\mathbb{R}/\mathbb{Z},\mathrm{PSL}(2,\mathbb{R}))$ and a rotation or diagonal matrix $L_*\in\mathrm{SL}(2,\mathbb{R})$ such that $\|B\|_\epsilon\le e^{C\delta_1 q}$, and $\|B(z+p/q)A(z)B(z)^{-1}-L_*\|_\epsilon\le e^{-\delta_1 q}$.*

*Proof of Theorem 1.1.* Let $(\alpha,A)$ be subcritical. By definition, there exists $\epsilon_0>0$ such that

$$\lim_{n\to\infty}\sup_{|\Im z|<\epsilon_0}\frac{1}{n}\ln\|A(z+(n-1)\alpha)\cdots A(z)\|=0. \tag{2.2}$$

Thus for every $\delta>0$, there exists $n\ge 1$ such that

$$\sup_{0\le k\le n}\sup_{|\Im z|<\epsilon_0}\|A(z+(k-1)\alpha)\cdots A(z)\|\le e^{\delta n}. \tag{2.3}$$

We may assume that $\epsilon_0$ is chosen so that $A$ extends holomorphically to a band $\{|\Im z|<\epsilon_0'\}$ for some $\epsilon_0'>\epsilon_0$. In particular, if $q\ge n$ and $p/q$ is close to $\alpha$, then

$$\sup_{0\le k\le q}\|A_k\|_{\epsilon_0}\le e^{2\delta q},\quad \text{with}\quad A_k(x)=A(x+(k-1)p/q)\cdots A(x), \tag{2.4}$$

which implies (2.1) with $\delta_1=2\delta$.

Assume now that $\alpha$ is exponentially Liouville. Then there exists $\delta'>0$ such that we may choose $p/q$ arbitrarily close to $\alpha$ and satisfying $|\alpha-\frac{p}{q}|<e^{-\delta' q}$. Fix $0<\epsilon'<\epsilon<\epsilon_0$ and let $C$ be as in Theorem 2.1. Selecting $0<\delta_1<\frac{1}{10C}\delta'$ and letting $B$ and $L_*$ be as in Theorem 2.1, we get

$$\begin{aligned}\|B(z+\alpha)A(z)B(z)^{-1}-L_*\|_{\epsilon'}&\\ \le\|B(z+p/q)A(z)B(z)^{-1}-L_*\|_\epsilon+\|A\|_\epsilon\|B\|_\epsilon|\alpha-\frac{p}{q}|\|\partial B\|_{\epsilon'}&\\ \le\|B(z+p/q)A(z)B(z)^{-1}-L_*\|_\epsilon+\|A\|_\epsilon\|B\|_\epsilon^2C(\epsilon,\epsilon')|\alpha-\frac{p}{q}|&\\ \le e^{-\delta_1 q}+C(\epsilon,\epsilon')e^{-\delta' q/2}\le 2e^{-\delta_1 q}.&\end{aligned} \tag{2.5}$$

Since $q$ can be taken arbitrarily large, the result follows. □

The analysis of periodic cocycles will be carried out in the next two sections. Indeed, we can actually obtain much more information than what is described in Theorem 2.1. For further applications (see [A4]), we will need such stronger

estimates, but only in a particular case (which is not the hardest to prove). Below, we use the notation $\hat{\phi}_k$ for the coefficients in the Fourier series of a function $\phi$, thus $\phi(x) = \sum \hat{\phi}_k e^{2\pi i k x}$.

**Theorem 2.2.** *For every $0 < \epsilon < \epsilon_0$ there exist $\delta_2 > 0$ and $C > 0$ such that if $\delta_1 > 0$ is sufficiently small, then for every $p/q \in \mathbb{Q}$ with $q$ sufficiently large, if $A \in C^\omega_{\epsilon_0}(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ is such that (2.1) holds and $t = trA_q$ satisfies $|\hat{t}_0| < 2$ and $|2 - |\hat{t}_0|| \geq e^{-\delta_1 q}$ then there exists $B \in C^\omega_\epsilon(\mathbb{R}/\mathbb{Z}, \mathrm{PSL}(2,\mathbb{R}))$ and $\theta \in C^\omega_\epsilon(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ such that $\theta$ is $1/q$-periodic, $\|\theta - \hat{\theta}_0\|_\epsilon \leq e^{-\delta_2 q}$, $\|B\|_\epsilon \leq e^{C\delta_1 q}$, and*

$$B(z+p/q)A(z)B(z)^{-1} = \begin{cases} R_{\theta(z)}, & \text{if } |\hat{t}_0| < 2, \\ \begin{pmatrix} e^{\theta(z)} & 0 \\ 0 & e^{-\theta(z)} \end{pmatrix}, & \text{if } |\hat{t}_0| > 2. \end{cases} \tag{2.6}$$

The starting observation in our analysis of periodic cocycles is that the trace function $t = \mathrm{tr}A_q$ is $1/q$-periodic, since $A_q(x+p/q)$ is conjugate to $A_q(x)$ in $\mathrm{SL}(2,\mathbb{R})$ (by $A(x)$). Assuming that $\|A_q(x)\|$ is under subexponential control in a given band, this implies that $t$ actually oscillates exponentially little in a smaller band. Indeed, a general $1/q$ periodic function $\phi \in C^\omega_{\epsilon_0}(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ has only non-vanishing Fourier coefficients at frequencies multiple of $q$, so that

$$\|\phi - \hat{\phi}_0\|_\epsilon \leq \|\phi\|_{\epsilon_0} \sum_{k \in \mathbb{Z} \smallsetminus \{0\}} e^{-2\pi |k| q(\epsilon_0 - \epsilon)}, \quad 0 < \epsilon < \epsilon_0, \tag{2.7}$$

so for large $q$ (depending on $\epsilon$ and $\epsilon_0$) we have

$$\|\phi - \hat{\phi}_0\|_\epsilon \leq 3 e^{-2\pi(\epsilon_0 - \epsilon)q} \|\phi\|_{\epsilon_0}. \tag{2.8}$$

Notice that, while its trace is under very good control, the matrix $A_q(x)$ itself is still allowed to oscillate a lot. Much of our analysis will center around the dependence of the eigendirections of $A_q(x)$ (which is particularly complicated in the case, complementary to the one consider in Theorem 2.2, where $|\hat{t}_0|$ is close to 2, due to the possible development of singularities) through a complex band, and will need a number of estimates on holomorphic functions of one-complex variable.

## 3. Proof of Theorem 2.2

### 3.1. Preliminary estimates.

**Lemma 3.1.** *Let $P(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$ with $a, b, c, d \in C^\omega_{\epsilon_0}(\mathbb{R}/\mathbb{Z}, \mathbb{C})$. Assume that $\det P$ is identically vanishing and $\delta \leq \|P(z)\| \leq 1$ through $\{|\Im z| < \epsilon_0\}$. Then there exists $u \in C^\omega_\epsilon(\mathbb{R}/\mathbb{Z}, \mathbb{C}^2)$ such that $P(z)u(z) = 0$ and $C^{-1}\delta^C \leq \|u(z)\| \leq 1$ through $\{|\Im z| < \epsilon\}$. Here $C = C(\epsilon_0, \epsilon)$.*

*Proof.* If $c$ or $d$ vanishes identically, the result is obvious. Indeed, if $c$ vanishes identically, for instance, then either $a$ vanishes identically (and $u = (1,0)$ will do) or $d$ vanishes identically (and $u = (-b, a)$ will do).

Let us assume that both $c$ and $d$ are not identically vanishing. Define a meromorphic function (not identically $\infty$) $\phi(x) = \frac{a(x)}{b(x)} = \frac{c(x)}{d(x)}$. All estimates below are for $|\Im x| < \epsilon$, and $C = C(\epsilon_0, \epsilon)$.

If $1/4 < |\phi(x)| < 1$ then $|D\phi(x)| \leq C/\delta$. Thus the $C^{-1}\delta$-neighborhood of $\phi(\{|\Im x| = \epsilon\})$ intersects $\{1/2 < |\kappa| < 3/4\}$ in a set of $\kappa$ of Lebesgue measure at

most $1/10$. This implies that there exists $|\kappa| < 3/4$ such that $|\phi(x) - \kappa| > C^{-1}\delta$ for every $x$ with $|\Im x| = \epsilon$, and such that for every $y$ with $|\Im y| < \epsilon$ and $\phi(y) = \kappa$ we have $|D\phi(y)| > C^{-1}$. Up to replacing $P$ by $P\begin{pmatrix} 1 & -\overline{\kappa} \\ -\kappa & 1 \end{pmatrix}$, we may suppose that $\kappa = 0$. In particular, the zeros of $\phi$ are simple. Let us estimate the number of zeros of $\phi$ in $|\Im x| < \epsilon$.

If $\phi(x) = 0$, then either for $\psi_0 = a$ and $\psi_1 = c$ or $\psi_0 = b$ and $\psi_1 = d$ we have $|D\psi_0(x)| > C^{-1}\delta$, $|\psi_1(x)| > C^{-1}\delta$. This implies that we can cover the zeros of $\phi$ in $\{|\Im x| < \epsilon\}$ with disjoint disks $D$ of radius $C^{-1}\delta$, such that $\max_{\psi=a,b} \inf_{x\in\partial D} |\psi(x)| > C^{-1}\delta^2$. The zeros (of $a$ or $b$) in such disks persist truncation of the Fourier series keeping frequencies at most $-C\ln\delta$, hence $\phi$ has at most $-C\ln\delta$ zeros in $|\Im x| < \epsilon$.

Let $p(z) = \prod_{s=1}^{N}(z - z_s)$ where $z_s = e^{2\pi i\theta_s}$ and $\theta_s$, $1 \le s \le N$ are the zeros of $\phi$. Let $u_1(\theta) = p(e^{2\pi i\theta})$, and let $u_2(x) = u_1(x)/\phi(x)$. Since the zeros of $\phi$ are simple, $u_1$ and $u_2$ are bounded holomorphic functions in $|\Im x| < \epsilon$. Let $\lambda = \left\| \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\|_\epsilon$. We claim that $u = \lambda^{-1}\begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}$ has the desired properties.

Clearly $-au_2 + bu_1 = u_2(-a + b\phi) = 0$ and similarly $-cu_2 + du_1 = 0$, so that $Pu = 0$. We also have $\|u\|_\epsilon = 1$. We need to show that $C^{-1}\delta^C \le \max\{\|u_1(x)\|, \|u_2(x))\|\} \le C\delta^{-C}$ in $|\Im x| < \epsilon$.

Since the number $N$ of zeros of $\phi$ in $|\Im x| < \epsilon$ is bounded by $-C\ln\delta$, we have $|u_1(x)| \le C\delta^{-C}$ in $|\Im x| < \epsilon$, and since $|\phi(x)| > C^{-1}\delta$ in $|\Im x| = \epsilon$, we also have $|u_2(x)| \le C\delta^{-C}$ in $|\Im x| < \epsilon$. This gives the upper estimate.

To conclude, let us show that $a(x)/u_1(x) \le C^{-1}\delta^C$ and $c(x)/u_1(x) \le C^{-1}\delta^C$ for $|\Im x| < \epsilon$. This implies the lower estimate, since $(a, b)$ and $(c, d)$ are multiples of $(u_1, u_2)$ and $\|P(x)\| \ge \delta$.

Since $a(x)/u_1(x)$ and $c(x)/u_1(x)$ are holomorphic in $|\Im x| < \epsilon_0$ and $\|P(x)\| \le 1$ in $|\Im x| < \epsilon_0$, it is enough to show that $u_1(x) \ge C\delta^{-C}$ for $|\Im x| = \epsilon_0$. But clearly $|u_1(x)| \ge |e^{\mp 2\pi\epsilon} - e^{\mp 2\pi\epsilon_0}|^N$ if $\pm\Im x = \epsilon_0$, where $N < -C\ln\delta$ is the number of zeros of $\phi$ in $|\Im x| < \epsilon$. The result follows. □

**Lemma 3.2.** *For every $0 < \epsilon_3 < \epsilon_2$, there exists $\delta > 0$ and $C > 0$ such if $\delta_1$ is sufficiently small and $p/q \in \mathbb{Q}$ with $q$ sufficiently large, then the following property holds. Let $\mu \in C^\omega_{\epsilon_2}(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, and let $\mu_k = \prod_{j=0}^{k-1} \mu(x + jp/q)$. Assume that $\|\mu_k\|_{\epsilon_2} \le e^{\delta_1 q}$, $1 \le k \le q$ and that $\|\mu_q^{-1}\|_{\epsilon_2} \le e^{\delta_1 q}$. Then there exist $\psi, \theta \in C^\omega_{\epsilon_3}(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ such that*

$$\mu(z) = e^{2\pi i\theta(z)} \frac{e^{2\pi i\psi(z+p/q)}}{e^{2\pi i\psi(z)}} \tag{3.1}$$

*and $\|\psi\|_{\epsilon_3} \le C\delta_1 q$, $\ln\|\theta - \hat{\theta}_0\|_{\epsilon_3} \le -\delta q$ and $\theta$ is $1/q$-periodic. Moreover,*

1. *If $|\mu(x)| = 1$ for every $x \in \mathbb{R}$, then $\Im\psi(x) = 0$ for every $x \in \mathbb{R}$,*
2. *If $\mu(x) \in \mathbb{R}$ for every $x \in \mathbb{R}$, then $\Re\psi(x) = 0$ for every $x \in \mathbb{R}$.*

*Proof.* Fix $\epsilon_3 < \epsilon_3' < \epsilon_2' < \epsilon_2$. Let $\mu(x) = e^{2\pi i(dx + \phi(x))}$, where $\phi \in C^\omega_{\epsilon_2'}(\mathbb{R}/\mathbb{Z}, \mathbb{C})$. Then $\mu_q(x) = e^{2\pi i(dqx) + \tilde{\phi}(x)}$ with $\tilde{\phi} \in C^\omega_{\epsilon_2'}(\mathbb{R}/\mathbb{Z}, \mathbb{C})$. Let $\lambda$ be the average of $\mu_q$ over $\mathbb{R}/\mathbb{Z}$. Since $\mu_q$ is $1/q$-periodic and $\|\mu_q\|_{\epsilon_2} \le e^{\delta_1 q}$ with small $\delta_1$, $\sup_{x\in\mathbb{R}/\mathbb{Z}} |\mu_q(x) - \lambda| \le e^{-\delta_2 q}$, for some $\delta_2 = \delta_2(\epsilon_2)$. Since $|\mu_q(x_0)^{-1}| \ge e^{-\delta_1 q}$ for each $x_0 \in \mathbb{R}$, this

implies that $|\lambda| \geq e^{-\delta_1 q}/2$. In particular, $\sup_{x\in\mathbb{R}/\mathbb{Z}} |\mu_q(x) - \lambda| < |\lambda|/2$, so that $d = 0$.

Let $\phi^{(k)} = \sum_{j=0}^{k-1} \phi(x + jp/q)$. By hypothesis, $\|e^{2\pi i\phi^{(k)}}\|_{\epsilon_2'} \leq e^{\delta_1 q}$, $1 \leq k \leq q$ and $\|e^{-2\pi i\phi^{(q)}}\|_{\epsilon_2'} \leq e^{\delta_1 q}$. This implies that

$$\|\phi^{(k)} - k\hat{\phi}_0\|_{\epsilon_3'} \leq C\delta_1 q. \tag{3.2}$$

Indeed, through $\{|\Im z| < \epsilon_2'\}$, it is obvious that $-\Im\phi^{(k)} \leq \delta_1 q$, $1 \leq k \leq q$ and $\Im\phi^{(q)} \leq \delta_1 q$, and since for $1 \leq k \leq q-1$ we have $\phi^{(k)}(z) + \phi^{(q-k)}(z + kp/q) = \phi^{(q)}(z)$, we can conclude that we have the estimate $|\Im\phi^{(k)}(z)| \leq 2\delta_1 q$ through the same band. To estimate the real part, one just uses that harmonic conjugation in $\{|\Im z| < \epsilon_2'\}$ composed with restriction to $\{|\Im z| < \epsilon_3'\}$ is a bounded operator on bounded harmonic functions.

For every $k \in \mathbb{Z}\smallsetminus q\mathbb{Z}$, let $1 \leq j_k \leq q-1$ be such that $|1-e^{2\pi i j_k kp/q}| \geq |1-e^{2\pi i/3}|$. Then

$$\frac{|\hat{\phi}_k|}{|1-e^{2\pi ikp/q}|} = \frac{|\hat{\phi}_k^{(j_k)}|}{|1-e^{2\pi ij_k kp/q}|} \leq C\delta_1 q e^{-2\pi\epsilon_3'|k|}. \tag{3.3}$$

Let

$$\psi(x) = \sum_{k\in\mathbb{Z}\smallsetminus q\mathbb{Z}} \frac{\hat{\phi}_k}{e^{2\pi ikp/q} - 1}, \tag{3.4}$$

so that $\|\psi\|_{\epsilon_3} \leq C\delta_1 q$. Let $\theta = \phi^{(q)}/q$, so that $\|\theta\|_{\epsilon_3'} = \|\phi^{(q)}\|_{\epsilon_3'}/q \leq C\delta_1$. Then $\phi(z) = \theta(z) + \psi(z + p/q) - \psi(z)$ (check the Fourier series). Since $\|\theta - \hat{\theta}_0\|_{\epsilon_3'} \leq C\delta_1$ and $\theta$ is $1/q$-periodic, we have $\|\theta - \hat{\theta}_0\|_{\epsilon_3} \leq e^{-\delta q}$ (c.f. (2.8)).

The last statement follows automatically from the construction. □

3.2. **Construction of the conjugacy.** Fix $\epsilon < \epsilon_3 < \epsilon_2 < \epsilon_1 < \epsilon_0$. Below, $C$ is a large constant depending on $\epsilon, \epsilon_2, \epsilon_1, \epsilon_0$ that may increase (finitely many times) along the argument. Clearly $\ln\|t\|_{\epsilon_0} \leq \ln(2\|A_q\|_{\epsilon_0}) \leq \delta_1 q + \ln 2$. Since $t$ is $1/q$-periodic, it follows (c.f. (2.8)) that $\|t - \hat{t}_0\|_{\epsilon_1} < e^{-\delta_3 q}$ for some $\delta_3 = \delta_3(\epsilon_0, \epsilon)$.

3.2.1. *Case 1.* Assume first that $|\hat{t}_0| < 2$. Then $t(x) = \lambda(x) + \lambda(x)^{-1}$ with $\lambda \in C^\omega_{\epsilon_1}(\mathbb{R}/\mathbb{Z}, \mathbb{C})$. Notice also that $|\lambda(x)| = 1$ for $x \in \mathbb{R}$.

Applying Lemma 3.1 to $P = A_q - \lambda\,\mathrm{id}$, we conclude that there exists $u \in C^\omega_{\epsilon_2}(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ with $0 \leq -\ln\|u(z)\| \leq C\delta_1 q$, $|\Im z| < \epsilon_2$, such that $A_q(z)\cdot u(z) = \lambda(z)u(z)$. Notice that $A(z)\cdot u(z)$ is a multiple of $u(z + p/q)$ for every $z$, $A(z) = \mu(z)u(z + p/q)$. Let $\mu_k$ be as in Lemma 3.2. We clearly have

$$\max_{1\leq k\leq q} \ln\|\mu_k\|_{\epsilon_2}, \ln\|\mu_k^{-1}\|_{\epsilon_2} \leq C\delta_1 q + \ln\|A_k\|_{\epsilon_2} \leq C\delta_1 q. \tag{3.5}$$

Let $\psi$ and $\theta$ be given by Lemma 3.2, and let $v = e^{2\pi i\psi}u$. Then $A(z)v(z) = e^{2\pi i\theta(z)}v(z + p/q)$. Notice that $-C\delta_1 q \leq \ln\|v(z)\| \leq C\delta_1 q$ through $\{|\Im z| < \epsilon_3\}$ and $\|\theta - \hat{\theta}_0\|_{\epsilon_3} \leq e^{-\delta_4 q}$ for some $\delta_4 = \delta_4(\epsilon_2, \epsilon_3) > 0$.

Let $\tilde{B}(z)$ be the matrix with columns $v(z) + \overline{v(\overline{z})}$ and $\frac{1}{i}(v(z) - \overline{v(\overline{z})})$. Then $A(z)\tilde{B}(z) = \tilde{B}(z + p/q)R_{\theta(z)}$. In particular, $b(z + p/q) = b(z)$, where $b = \det\tilde{B}$. Since $\ln\|b\|_{\epsilon_3} \leq C\delta_1 q$, we conclude that $\|b - \hat{b}_0\|_\epsilon \leq e^{-\delta_5 q}$, for some $\delta_5 = \delta_5(\epsilon_3, \epsilon)$, provided $\delta_1$ is sufficiently small.

We claim that $\ln\|b^{-1}\|_\epsilon \leq C\delta_1 q$. Since $b$ has exponentially small oscillation, it suffices to show that $\ln b(x_0)^{-1} \leq C\delta_1 q$ for some $x_0 \in \mathbb{R}$. If this does not hold,

then there exists $\kappa \in \mathbb{C}$ with $|\kappa| = 1$ such that $-\ln \|v(x_0) - \kappa\overline{v(x_0)}\| \gg \delta_1 q$. But since $A_q(x_0)(v(x_0) - \kappa\overline{v(x_0)}) = \lambda(x_0)v(x_0) - \kappa\lambda(x_0)^{-1}\overline{v(x_0)}$, we conclude that $-\ln|\lambda(x_0) - \lambda(x_0)^{-1}| - \ln\|v(x_0)\| \gg \delta_1 q$. Since $-\ln(2 - |t(x_0)|) \leq C\delta_1 q$, this implies that $-\ln\|v(x_0)\| \gg \delta_1 q$, contradiction.

Up to exchanging the roles of $\lambda$ and $\lambda^{-1}$ (which changes $\theta$ to $-\theta$ and $v(z)$ to $\overline{v(\overline{z})}$), we may assume that $b(x) > 0$ for $x \in \mathbb{R}$. Now let $B^{-1} = \frac{1}{b^{1/2}}\tilde{B}$. Then $\|B\|_\epsilon \leq e^{C\delta_1 q}$ and $B(x + p/q)A(x)B(x)^{-1} = R_{\theta(x)}$, as desired.

3.2.2. *Case 2.* Assume now that $|\hat{t}_0| > 2$. Then $t(z) = \lambda(z) + \lambda(z)^{-1}$ with $\lambda \in C^\omega_{\epsilon_1}(\mathbb{R}/\mathbb{Z}, \mathbb{R})$. The argument below is a simple adaptation of that of the previous case.

Applying Lemma 3.1 and Remark 4.1 to $P = A_q - \lambda\,\mathrm{id}$, we obtain $u \in C^\omega_{\epsilon_2}(\mathbb{R}, \mathbb{R}^2)$ with $e^{-C\delta_1 q} \leq \|u(z)\| \leq 1$, $|\Im z| < \epsilon_2$ such that $u(z+1) = \pm u(z)$ and $A_q u = \lambda u$. Notice that $A(z) \cdot u(z)$ is a multiple of $u(z + p/q)$ for every $z$, $A(z)u(z) = \mu(z)u(z + p/q)$. Notice that $\mu$ is 1-periodic. Let $\mu_k$ be as in Lemma 3.2. We clearly have $\|\mu_k\|_{\epsilon_2}, \|\mu_k^{-1}\|_{\epsilon_2} \leq e^{C\delta_1 q}\|A_k\|_{\epsilon_2} \leq e^{C\delta_1 q}$, $1 \leq k \leq q$. Let $\psi$ and $\theta$ be given by Lemma 3.2, and let $v = e^{2\pi i\psi}u$. Then $v$ is real-symmetric and $A(z)v(z) = \gamma(z)v(z + p/q)$, where $\gamma(z)$ is $1/q$-periodic and real-symmetric. Notice that $\gamma$ has the same sign as $\mu$ and $\gamma^q = \lambda$.

An analogous argument yields a solution $v' \in C^\omega_{\epsilon_3}(\mathbb{R}, \mathbb{R}^2)$ such that $v'(z+1) = \pm v(z)$, $A_q v' = \lambda^{-1}v'$ and $A(z)v'(z) = \gamma(z)^{-1}v(z + p/q)$. Notice that since $\lambda \neq \lambda^{-1}$, $v$ is not colinear with $v'$, so the determinant of the matrix with columns $v(x)$ and $v'(x)$ does not change sign for $x \in \mathbb{R}$. Thus $v$ and $v'$ are both 1-periodic or 1-antiperiodic. Take $\tilde{B}$ as the matrix with columns $v$ and $v'$. Since $A(z)\tilde{B}(z) = \tilde{B}(z + p/q)\begin{pmatrix}\gamma(z) & 0\\ 0 & \gamma(z)^{-1}\end{pmatrix}$, $b = \det\tilde{B}$ is $1/q$-periodic, so that $\|b - \hat{b}_0\|_\epsilon \leq e^{-\delta_5 q}$ (where $\delta_5 > 0$ is independent of $\delta_1$ small). For fixed $x_0 \in \mathbb{R}$, since $A_q(x_0)$ is an $e^{\delta_1 q}$ bounded matrix whose eigenvalues $\lambda(x_0)$ and $\lambda^{-1}(x_0)$ are $e^{-C\delta_1 q}$ apart, the angle between the eigenvectors $v(x_0)$ and $v'(x_0)$ is at least $e^{-C\delta_1 q}$. Thus $|b(x_0)| \geq e^{-C\delta_1 q}$, and hence $|\hat{b}_0| \geq e^{-C\delta_1 q}$. The result follows by taking $B^{-1}$ as the matrix with columns $v$ and $v'/b$. □

## 4. Proof of Theorem 2.1

4.1. **Preliminary estimates.** An important input in our estimates is the polynomial bound on solutions of the Corona problem. Those can already be found in the original work of Carleson [C], but the more precise version given here has been proved using Wolff's approach.

**Theorem 4.1** (Uchiyama [U], see Trent [T])**.** *There exists $C > 0$ with the following property. Let $a_i : \mathbb{D} \to \mathbb{C}$, $1 \leq i \leq k$, be such that $\delta \leq (\sum_{i=1}^k |a_i(x)|^2)^{1/2} \leq 1$, $x \in \mathbb{D}$. Then there exists $\tilde{a}_i : \mathbb{D} \to \mathbb{C}$ such that $(\sum_{i=1}^k |\tilde{a}_i|^2)^{1/2} \leq C\delta^{-2}(1 - \ln\delta)$ and such that $\sum_{i=1}^k a_i\tilde{a}_i = 1$.*

It is easy to see that Corona estimates for the disk imply corresponding ones for the annulus.

**Lemma 4.2.** *Let $a_i \in C^\omega_\delta(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, $1 \leq i \leq k$, be such that $\delta \leq (\sum_{i=1}^k |a_i(z)|^2)^{1/2} \leq 1$ through $\{|\Im z| < \delta\}$. Then there exists $\tilde{a}_i \in C^\omega_\delta(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, $1 \leq i \leq k$, such that $(\sum_{i=1}^k |\tilde{a}_i(z)|^2)^{1/2} \leq C\delta^{-2}(1 + \ln\delta)$ (with $C > 0$ as in the previous theorem) and*

*such that* $\sum_{i=1}^k a_i\tilde{a}_i = 1$. *Moreover, if all the* $a_i$ *are real-symmetric, we can choose all the* $\tilde{a}_i$ *real-symmetric.*

*Proof.* By the previous theorem, there exist $a_i' \in C^\omega_\epsilon(\mathbb{R},\mathbb{C})$ be such that $\sum_{i=1}^k a_i a_i' = 1$ and $(\sum_{i=1}^k |a_i'(z)|^2)^{1/2} \leq C\delta^{-2}(1+\ln\delta)$ (we use that the strip $\{|\Im z| < \delta\}$ is conformally equivalent to $\mathbb{D}$). Let $a_i^{(j)}(z) = \frac{1}{j}\sum_{n=0}^{j-1} a_i'(z+n)$. Let $j_n \to \infty$ be a sequence such that for every $1 \leq i \leq k$, $a_i^{(j_n)}$ converges in the topology of uniform convergence on compact sets, and let $\tilde{a}_i$ be the limits. Then $\tilde{a}_i \in C^\omega_\delta(\mathbb{R}/\mathbb{Z},\mathbb{C})$.

For the last statement, notice that if the $a_i$ are real-symmetric then we can substitute each $\tilde{a}_i(z)$ by $\frac{1}{2}(\tilde{a}_i(z) + \overline{\tilde{a}_i(\overline{z})})$. □

**Lemma 4.3.** *There exists* $C > 0$ *with the following property. Consider a function* $P^{(0)} = \begin{pmatrix} a^{(0)} & b^{(0)} \\ c^{(0)} & d^{(0)} \end{pmatrix}$ *with coordinates in* $C^\omega_{\epsilon_0}(\mathbb{R}/\mathbb{Z},\mathbb{C})$. *Assume that* $\delta \leq \|P^{(0)}(z)\| \leq 1$ *through* $\{|\Im z| < \epsilon_0\}$ *and let* $\|\det P^{(0)}\|_{\epsilon_0} = \rho(\delta^2/\ln\delta)^2$. *If* $\rho < C^{-1}$ *then there exists* $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with coordinates* $C^\omega_{\epsilon_0}(\mathbb{R}/\mathbb{Z},\mathbb{C})$ *such that* $\|P^{(0)} - P\| \leq -C\rho(\delta^2/\ln\delta)$ *and* $\det P = 0$. *Moreover, if* $P^{(0)}$ *is real-symmetric then* $P$ *can be chosen real-symmetric.*

*Proof.* Let $K_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be such that $a^{(0)}d + ad^{(0)} - b^{(0)}c - bc^{(0)} = 1$. Let $P^{(1)} = P^{(0)} - K_0 \det P^{(0)}$. Then $\det P^{(1)} = (\det P^{(0)})^2 \det K_0$. Choosing $K_0$ with minimal $\|K_0\|_{\epsilon_0}$, using Lemma 4.2, we get $|\det P^{(1)}| \leq C\rho^2(\delta^2/\ln\delta)^2$, while $\|P^{(1)} - P^{(0)}\|_{\epsilon_0} < -C\rho\delta^2/\ln\delta$. Iterating this procedure we get a sequence $P^{(n)}$ converging to $P$ as desired. □

**Lemma 4.4.** *Let* $w \in C^\omega_\epsilon(\mathbb{R}/\mathbb{Z},\mathbb{C}^2)$ *be such that* $\delta \leq \|w(z)\| \leq 1$ *through* $\{|\Im z| < \epsilon\}$. *If* $w(x)$ *is a multiple of a real vector for* $x \in \mathbb{R}/\mathbb{Z}$, *then there exists* $\tilde{w} \in C^\omega_\epsilon(\mathbb{R},\mathbb{R}^2)$ *such that* $\tilde{w}(x+1) = \pm\tilde{w}(x)$ *for every* $x \in \mathbb{R}$, $w(x)$ *is a multiple of* $\tilde{w}(x)$ *for every* $x \in \mathbb{R}$, *and* $C^{-1}\delta^{3/2} \leq \|\tilde{w}(x)\| \leq C\delta^{-1/2}$ *for* $|\Im x| < \epsilon$.

*Proof.* Let $w = (a,b)$, and let $\tilde{a}(z) = \overline{a(\overline{z})}$, $\tilde{b}(z) = \overline{b(\overline{z})}$. Let $\phi = a/\tilde{a} = b/\tilde{b}$. Then $2^{-1/2}\delta \leq \phi(z) \leq 2^{1/2}\delta^{-1}$ through $\{|\Im z| < \epsilon\}$. Let $\tilde{w}(x) = \phi^{-1/2}w(x)$. □

*Remark* 4.1. If in the statement of Lemma 3.1 we further assume that $P$ is real-symmetric, we can then obtain, using Lemma 4.4, a real-symmetric solution of $Pu = 0$ satisfying the required bounds (with adjusted constants), which is not necessarily 1-periodic, but satisfies $u(x) = \pm u(x+1)$. It is not possible in general to get a 1-periodic solution, as exemplified by $P(x) = \begin{pmatrix} -\sin\pi x\cos\pi x & -\sin^2\pi x \\ \cos^2\pi x & \sin\pi x\cos\pi x \end{pmatrix}$.

Recall the basic convexity estimate (Hadamard Three Circles Theorem),

$$\sup_{\Im z = ta+(1-t)b} \ln|\phi(z)| \leq t\sup_{\Im z = a}\ln|\phi(z)| + (1-t)\sup_{\Im z = b}\ln|\phi(z)|, \tag{4.1}$$

for a 1-periodic analytic function $\phi$ defined in a neighborhood of the strip $\{a \leq \Im z \leq b\}$.

**Lemma 4.5.** *For every* $0 < \epsilon_1 < \epsilon_0$, *there exists* $C_0 > 0$, $\delta_0 > 0$ *such that for every* $0 < \delta < \delta_0$, *if* $q$ *is sufficiently large and* $\phi \in C^\omega_{\epsilon_0}(\mathbb{R}/\mathbb{Z},\mathbb{C})$ *is such that* $\|\phi\|_{\epsilon_0} \leq e^{\delta q}$,

*while there exists $z$ with $|\Im z| < \epsilon_1$ such that $|\phi(z)|, ..., |\phi(z+(q-1)/q)| < e^{-C\delta q}$ with $C > C_0$, then $\|\phi\|_{\epsilon_1} \leq \max\{e^{-C_0^{-1}C\delta q}, e^{-\delta_0 q}\}$.*

*Proof.* Let $\tilde{\phi}(x) = \sum_{k=-[q/2]}^{q-1-[q/2]} \hat{\phi}_k e^{2\pi i k x}$. Then $\|\phi - \tilde{\phi}\|_{\epsilon_1} \leq e^{-\delta_1 q}$ with $\delta_1 = \delta_1(\epsilon_0, \epsilon) > 0$, provided $\delta_0$ is sufficiently small. By Lagrange interpolation,

$$\sup_{\Im x = \Im z} \|\tilde{\phi}(x)\| \leq \sum_{k=0}^{q-1} |\tilde{\phi}(z+k/q)| \leq q(e^{-\delta q} + e^{-\delta_1 q}). \tag{4.2}$$

(Indeed $\psi(x) = e^{2\pi i[q/2]x}\tilde{\phi}(x+z)$ satisfies $\psi(x) = \sum_{k=0}^{q-1} \psi(k/q) c_q(x-k/q)$ where $c_q(x) = \frac{1}{q}\sum_{k=0}^{q-1} e^{2\pi i k x/q}$, so that $\sup_{x\in\mathbb{R}/\mathbb{Z}} |c_q(x)| = 1$.) The result follows by convexity, c.f. (4.1). □

4.2. **Construction of the conjugacy.** Fix $\epsilon < \epsilon' < \epsilon_0$. As in the proof of Theorem 2.2, the hypothesis implies that

$$\|t(x) - \hat{t}_0\|_{\epsilon'} \leq e^{-\delta_3 q} \tag{4.3}$$

for some $\delta_3 = \delta_3(\epsilon', \epsilon_0) > 0$.

There are 2 essentially distinct cases:

(1) $|2 - |\hat{t}_0|| \geq e^{-C_0^2 \delta_1 q}$,
(2) $|2 - |\hat{t}_0|| < e^{-C_0^2 \delta_1 q}$.

Here $C_0 = C_0(\epsilon_0, \epsilon)$ will be some appropriately large constant. The first case is covered by Theorem 2.2, so we concentrate here on the second.

*Remark* 4.2. In the analysis of the second case, we will actually obtain $L_* \in \mathrm{SO}(2,\mathbb{R})$.

We will use, in two distinct situations, the following estimate.

**Lemma 4.6.** *For every $\epsilon < \epsilon_1 < \epsilon_0$, there exists $C_5 > 1$ such that for every $C_3 > 1$ sufficiently large and every $C_4 > 1$, for every $\delta_1 > 0$ is sufficiently small, if $p/q \in \mathbb{Q}$ with $q$ sufficiently large, and $A \in C^\omega_{\epsilon_0}(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ satisfies (2.1), then the following property holds. If there exists $W \in C^\omega_{\epsilon_1}(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ and $R \in \mathrm{SO}(2,\mathbb{R})$ satisfying*

$$\|W(z+p/q)A(z) - RW(z)\|_{\epsilon_1} \leq e^{-C_3 C_4 \delta_1 q}, \tag{4.4}$$

*while*

$$\|W(z)\| \geq e^{-C_4\delta_1 q}, \quad |\Im z| < \epsilon_1, \tag{4.5}$$

*and*

$$\|\det W\|_{\epsilon_1} \leq e^{-C_3 C_4 \delta_1 q}, \tag{4.6}$$

*then there exists $B \in C^\omega_\epsilon(\mathbb{R}/\mathbb{Z}, \mathrm{PSL}(2,\mathbb{R}))$ and a constant diagonal matrix $D \in \mathrm{SL}(2,\mathbb{R})$ such that $\|B\|_\epsilon \leq e^{C_5 C_4 \delta_1 q}$ and $\|B(z+p/q)A(z)B(z)^{-1} - D\|_\epsilon \leq e^{-C_4\delta_1 q}$.*

*Proof.* Fix $\epsilon < \epsilon_2 < \epsilon_1$. Apply Lemma 4.3 to $P^{(0)} = W$ to obtain $P$ with $\det P = 0$ such that $\|P - W\|_{\epsilon_1} \leq e^{C^{-1} C_3 C_4 \delta_1 q}$. Using Lemma 3.1 together with Remark 4.1 to get $u \in C^\omega_{\epsilon_2}(\mathbb{R}, \mathbb{R}^2)$ such that $u(z+1) = u(z)$ or $u(z+1) = -u(z)$ such that $Pu = 0$, and satisfying $e^{-CC_4\delta_1 q} \leq \|u(z)\| \leq 1$ through $\{|\Im z| < \epsilon_2\}$. Then

$$\|W(z)\cdot u(z)\|_{\epsilon_2} \leq e^{-C^{-1}C_3C_4\delta_1 q}. \tag{4.7}$$

Using (4.4) we get

$$\|W(z+p/q)\cdot A(z)u(z)\|_{\epsilon_2}\le e^{-C^{-1}C_3C_4\delta_1 q}, \tag{4.8}$$

Using (4.5), it follows that $A(z)\cdot u(z)$ is $e^{-C^{-1}C_3C_4\delta_1 q}$ close to a multiple of $u(z+p/q)$, $|\Im z|<\epsilon_2$. Using Lemma 4.2, define $\tilde B\in C^\omega_{\epsilon_2}(\mathbb{R}/\mathbb{Z},\mathrm{PSL}(2,\mathbb{R}))$ with first column $u$. Then

$$\tilde B(z+p/q)^{-1}A(z)B(z)=\begin{pmatrix}\mu(z) & \tilde s_2(z)\\ \tilde s_3(z) & \mu(z)^{-1}+\tilde s_4(z)\end{pmatrix}, \tag{4.9}$$

with $\mu$ real-symmetric, and $\|\tilde s_3\|_{\epsilon_2},\|\tilde s_4\|_{\epsilon_2}<e^{-C^{-1}C_3C_4\delta_1 q}$. As in the proof of Theorem 2.2, we can apply Lemma 3.2 (with $\epsilon_3=\epsilon$) to obtain $\psi$ and $\theta$ such that $B'=\begin{pmatrix}e^{-\psi} & 0\\ 0 & e^{\psi}\end{pmatrix}\tilde B$ satisfies

$$B'(z+p/q)A(z)B'(z)^{-1}=\begin{pmatrix}e^{2\pi i\theta(z)} & s_2'(z)\\ s_3'(z) & e^{-2\pi i\theta(z)}+s_4'(z)\end{pmatrix}, \tag{4.10}$$

with $\|s_3'\|_\epsilon,\|s_4'\|_\epsilon\le e^{-C^{-1}C_3C_4\delta_1 q}$. We also have the bound $\|B'\|_\epsilon\le e^{CC_4\delta_1 q}$, and hence $\|s_2'\|_\epsilon\le e^{CC_4\delta_1 q}$. Since $\|\theta-\hat\theta_0\|_\epsilon\le e^{-\delta q}$ with $\delta=\delta(\epsilon,\epsilon_2)$, the result follows with $B^{-1}=\begin{pmatrix}d & 0\\ 0 & d^{-1}\end{pmatrix}B'$, $d=e^{10C_4\delta_1 q}(1+\|s_2'\|_\epsilon)$. □

One key consideration when $\hat t_0$ is close to $\pm2$ is whether $\pm A_q$ is close to the identity or not. Fix $\epsilon<\epsilon_1<\epsilon'$. Notice that if $\|A_q\mp\mathrm{id}\,\|_{\epsilon_1}\ge e^{-C_0\delta_1 q}$ then

$$\|A_q(z)\mp\mathrm{id}\,\|\ge e^{-C_1C_0\delta_1 q},\quad |\Im z|<\epsilon_1, \tag{4.11}$$

for an appropriately large constant $C_1$, which does not depend on the choice of $C_0$. Indeed, if this was not the case then there would exists $z$ with $|\Im z|<\epsilon_1$ such that $\|A_q(z+kp/q)\mp\mathrm{id}\,\|\le\|A_k(z)\|^2\|A_q(z)\mp\mathrm{id}\,\|\le e^{-C_2C_0\delta_1 q}$ for $0\le k\le q-1$ with $C_2$ large. Applying Lemma 4.5 to the coefficients of $A_q\mp\mathrm{id}$, that are bounded by $e^{\delta_1 q}+1$ through $\{|\Im z|<\epsilon_0\}$, leads to a contradiction.

We will assume below that $C_0$ is chosen much bigger than $C_1$. Then, under the assumption that (4.11) holds, the result follows from Lemma 4.6, with $W=A_q\mp\mathrm{id}$. Notice that in this case the matrix $L_*$ can be taken as $\pm\mathrm{id}$.[3]

Assume not that (4.11) does not hold, so that, as explained above, we must have

$$\|A_q\mp\mathrm{id}\,\|_{\epsilon_1}\le e^{-C_0\delta_1 q}. \tag{4.12}$$

Let us consider a large coefficient of the discrete Fourier transform of the essentially periodic sequence $\{R_{ls/2q}A_s\}_{s=0}^{q-1}$, where $l=0$ if $A_q$ is close to id and $l=1$ is $A_q$ is close to $-\mathrm{id}$. More precisely, take $W_k=\sum_{s=0}^{q-1}R_{ks/q}R_{ls/2q}A_s$, $0\le k\le q-1$. Then

$$W_k(z+p/q)A(z)=R_{-(2k+l)/2q}(W_k(z)\pm A_q(z)-\mathrm{id}), \tag{4.13}$$

so that by (4.12),

$$\|W_k(z+p/q)A(z)-R_{-(2k+l)/2q}W_k(z)\|_{\epsilon_1}\le e^{-C^{-1}C_0\delta_1 q} \tag{4.14}$$

[3]Indeed the diagonal matrix $D$ given by Lemma 4.6 is close to $\pm\mathrm{id}$ since $\mathrm{tr}D^q$ is close to $\mathrm{tr}A_q$ which is close to $\pm2$.

(here and below, we use $C$ for quantities that do not become larger if $C_0$ is taken large, so that we can always assume that $C_0 > C$). Clearly, for every $x \in \mathbb{R}/\mathbb{Z}$ and any unit vector $y \in \mathbb{R}^2$,

$$\sum_{k=0}^{q-1} \|W_k(x) \cdot y\|^2 = q \sum_{s=0}^{q-1} \|A_s(x) \cdot y\|^2 \tag{4.15}$$

(Parseval identity). The average of the right hand side over the circle of unit vectors is

$$q \sum_{s=0}^{q-1} \frac{\|A_s(x)\|^2 + \|A_s(x)\|^{-2}}{2} \geq q^2, \tag{4.16}$$

so for every $x \in \mathbb{R}/\mathbb{Z}$, there exists one unit vector $y$ such that $\sum_{k=0}^{q-1} \|W_k(x) \cdot y\|^2 \geq q^2$. Fix $x_0 \in \mathbb{R}$ and let $W = W_{k_0}$ where $k_0$ is such that such that $\|W_{k_0}(x_0)\|^2$ is maximal. Then $\|W(x_0)\|^2 \geq q$.

We claim that for fixed $\epsilon < \epsilon_1' < \epsilon_1$,

$$-\ln \|W(z)\| \leq C\delta_1 q, \quad |\Im z| < \epsilon_1'. \tag{4.17}$$

Indeed, if $\|W(z)\| \leq e^{-C\delta_1 q}$ with large $C$, then $\max_{0 \leq j \leq q-1} \|W(z+jp/q)\| \leq e^{-C\delta_1 q}$ with large $C$, and Lemma 4.5 implies that $\|W\|_{\epsilon_1'} \leq e^{-C\delta_1 q}$ with large $C$. This contradicts $\|W(x_0)\|^2 \geq q$, so that (4.17) holds.

Let $w = \det W$. Then, by (4.14), $\|w(z + p/q) - w(z)\|_{\epsilon_1} \leq e^{-C^{-1}C_0\delta_1 q}$, so $w$ is $e^{-C^{-1}C_0\delta_1 q}$ close to $\frac{1}{q}\sum_{k=0}^{q-1} w(x + kp/q)$, which is $1/q$-periodic and bounded by $qe^{2\delta_1 q}$ over $\{|\Im z| < \epsilon_1\}$. By convexity, c.f. (4.1), we get that $\|w - \hat{w}_0\|_{\epsilon_1'} \leq e^{-C^{-1}C_0\delta_1 q}$.

Assume that $-\ln |\hat{w}_0| \leq C_0^{1/2}\delta_1 q$. If $w(x) > 0$ for $x \in \mathbb{R}$, then take $B = \frac{1}{w^{1/2}} W$. If $w(x) < 0$ for $x \in \mathbb{R}$, then take $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{(-w)^{1/2}} W$. The result follows (with $L_* = R_{-(2k+l)/2q}$).

Assume that $-\ln |\hat{w}_0| \geq C_0^{1/2}\delta_1 q$. The result follows by applying Lemma 4.6 (with $\epsilon_1'$ instead of $\epsilon_1$). Notice that in this case $L_*$ can be taken as $\pm \mathrm{id}$ (c.f. footnote 3). □

*Remark* 4.3. The analysis above can be refined further to yield considerably more precise estimates.

## 5. Applications

We start with the two corollaries about almost reducibility near constants.

*Proof of Corollary 1.2.* Any one-frequency cocycle which is close to constant is either uniformly hyperbolic or subcritical (this is essentially due to [BJ1] and [BJ2], and it is explicitly obtained in [A1] by a different argument). Uniformly hyperbolic cocycles are always almost reducible: they can be conjugated to a diagonal cocycle, which can then be conjugated arbitrarily close to a constant one using approximate solutions of the cohomological equation. If $\alpha$ is exponentially Liouville, the result then follows by Theorem 1.1. The complementary case was established earlier in [A1].[4] □

[4]For $\alpha$ Diophantine, i.e., under the condition $\ln q_{n+1} = O(\ln q_n)$, this was established in [AJ2] Theorem 4.1 (rigorously speaking, [AJ2] only deals with the case of Schrödinger cocycles, but this

*Proof of Corollary 1.3.* Let $(\alpha, A)$ be almost reducible and let $B^{(n)}$ be the sequence of conjugacies as in the definition. Let $(\alpha^{(n)}, A^{(n)})$ be any sequence of non-almost reducible cocycles converging to $(\alpha, A)$. Then there exists a sequence $j_n \to \infty$ such that $\tilde{A}^{(n)}(x) = B^{(n)}(x + \alpha^{(j_n)})A^{(j_n)}(x)B^{(n)}(x)^{-1}$ converges to a constant. By Corollary 1.2, $(\alpha, \tilde{A}^{(n)})$ must be almost reducible for large $n$. Since almost reducibility is conjugacy invariant, $(\alpha^{(j_n)}, A^{(j_n)})$ is almost reducible, contradiction. □

Next we deduce the promised equivalence between almost reducibility and "almost reducibility through coordinate changes isotopic to the identity" for cocycles which are not uniformly hyperbolic:

*Proof of Theorem 1.4.* Assume first thay $\alpha$ is not exponentially Liouville. In this case, the result was previously established in [A1] for cocycles near constant. In general, by definition of almost reducibility we may choose a first conjugacy $\tilde{B}$ that takes the cocycle close to a constant, to which the result applies. By composing the coordinate changes, we obtain a sequence $\tilde{B}^{(n)}$ conjugating the cocycle near a constant $(\alpha, \tilde{R}_*)$ with $\tilde{R}_* \in \mathrm{SO}(2, \mathbb{R})$, but the $\tilde{B}^{(n)}$ are homotopic to $\tilde{B}$, which is not necessarily homotopic to a constant. We can then set $B^{(n)}(x) = R_{-dx/2}\tilde{B}^{(n)}(x)$, where $d$ is the topological degree of $\tilde{B}^{(n)} : \mathbb{R}/\mathbb{Z} \to \mathrm{PSL}(2, \mathbb{R})$, which now provide conjugacies near a constant $(\alpha, R_*)$ with $R_* = R_{-d\alpha/2}\tilde{R}_*$.

Assume now that $\alpha$ is exponentially Liouville. The hypothesis clearly implies subcriticality, and we are going to show that our proof of Theorem 1.1 can be adapted to conclude the stronger form of almost reducibility.

As in the proof of Theorem 1.1, consider an exponentially good periodic approximation (with exponent $\delta' > 0$) and apply Theorem 2.1 to obtain $B$ and $L_*$, satisfying $\|B\|_\epsilon \leq e^{C\delta_1 q}$ and $\|B(z + p/q)A(z)B(z)^{-1} - L_*\|_\epsilon \leq e^{-\delta_1 q}$, where $0 < \delta_1 \ll \delta'$.

Let us first consider the case where $L_* \in \mathrm{SO}(2, \mathbb{R})$. The bound $\|B\|_\epsilon \leq e^{C\delta_1 q}$ implies that the topological degree $d$ of $B : \mathbb{R}/\mathbb{Z} \to \mathrm{PSL}(2, \mathbb{R})$ is at most $C'(\epsilon)C\delta_1 q$.[5] Notice that the function $B'(x) = R_{-dx/2}B(x)$ is then homotopic to a constant, and if $\epsilon' > 0$ is sufficiently small (depending on $\epsilon_0, \epsilon, \delta_1$), we get $\|B'\|_{\epsilon'} < e^{2C\delta_1 q}$ and $\|B'(z + \alpha)A(z)B'(z)^{-1} - L\|_{\epsilon'} \leq e^{-\delta_1 q/2}$, where $L = R_{-dp/2q}R_{-dx/2}L_*R_{dx/2} = R_{-dp/2q}L_*$ (here we use that $L_* \in \mathrm{SO}(2, \mathbb{R})$.

Let us now consider the case where $L_*$ is diagonal. As discussed in Remark 4.2, we may assume that $||\hat{t}_0| - 2| \geq e^{-C_0^2\delta_1 q}$ for some fixed constant $C_0 > 0$. Then Theorem 2.2 gives more precise information: $B(z + p/q)A(z)B(z)^{-1}$ is of the form $R_{\theta(z)}$ (if $|\hat{t}_0| > 2$) or $\begin{pmatrix} e^{\theta(z)} & 0 \\ 0 & e^{-\theta(z)} \end{pmatrix}$ (if $|\hat{t}_0| < 2$) for some $1/q$ periodic $\theta(z)$ which oscillates at most $e^{-\delta_2 q}$ around its mean (for some fixed $\delta_2 > 0$). We can thus assume to be in the second case. Let us show that if $\delta_1 > 0$ is sufficiently small and $q$ is sufficiently large then $(\alpha, A)$ must be uniformly hyperbolic. This is the same as showing that $(\alpha, A')$ is uniformly hyperbolic, where $A'(x) = B(x + \alpha)A(x)B(x)^{-1}$.

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case implies the general one by a simple abstract argument, see Lemma 2.2 of [AJ3]). Under the weaker condition $\ln q_{n+1} = o(q_n)$ (defining the complement of the exponentially Liouville regime), almost reducibility near constants follows from [A1], which provides the necessary estimates for the argument of [AJ2] (see Section 3.7 of [A1]).

[5]Consider a suitable approximation by trigonometric polynomials which is homotopic to $B$ (by truncation of the Fourier series frequencies higher than $C'(\epsilon)C\delta_1 q$), and notice that the resulting bound on the number of zeros of a non-vanishing coordinate also bounds the topological degree of $B$.

Notice that for some $x_0 \in \mathbb{R}/\mathbb{Z}$ we have $|\hat{t}_0| = |e^{q\theta(x_0)} + e^{-q\theta(x_0)}| \geq 2 + e^{-C_0^2\delta_1 q}$, so $|\theta(x_0)| > e^{-C_0^2\delta_1 q}$. This implies that $|\theta(x)| > e^{-C_0^2\delta_1 q} - e^{-\delta_2 q} \geq e^{-C_0^2\delta_1 q}/2$ for $x \in \mathbb{R}/\mathbb{Z}$. In particular, $\theta(x)$ does not change sign.

Assume, for definiteness, that $\theta(x) > 0$ for every $x \in \mathbb{R}/\mathbb{Z}$. We are going to show that for each non-zero vector $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$ with $|a| \geq |b|$ and for each $x \in \mathbb{R}$, applying $\begin{pmatrix} a' \\ b' \end{pmatrix} = A'(x) \cdot \begin{pmatrix} a \\ b \end{pmatrix}$ satisfies $|a'| > |b'|$, which implies uniform hyperbolicity of $(\alpha, A')$ by the usual conefied criterion [Yoc]. We have

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = B(x+\alpha)B(x+p/q)^{-1} \cdot \begin{pmatrix} e^{\theta(x)}a \\ e^{-\theta(x)}b \end{pmatrix}, \tag{5.1}$$

so that $\begin{pmatrix} a' \\ b' \end{pmatrix}$ is obtained by applying a matrix $e^{-\delta' q/2}$ close to the identity to a vector $\begin{pmatrix} a'' \\ b'' \end{pmatrix}$ with $|a''| \geq (1 + e^{-C_0^2\delta_1 q})^2|b'|$. Thus $|a'| \geq (1 + e^{-C_0^2\delta_1 q})|b'|$ as desired. □

As explained in §1.2, Corollary 1.5 follows from Theorem 1.4.

Next we consider the applications to one-frequency Schrödinger operators. The proof of absolute continuity of the i.d.s., Corollary 1.8, was explained in §1.3.

*Proof of Corollary 1.6.* It is shown in [AFK], Theorem 1.2, that for almost every energy $E$, either the Lyapunov exponent is positive or $(\alpha, A^{(E-v)})$ is rotations reducible. Since rotations reducibility implies almost reducibility (as one can approximately solve the cohomological equation), this means that we can split the spectrum into three parts, $\Sigma_-$ (corresponding to almost reducible energies), $\Sigma_+$ (corresponding to energies with a positive Lyapunov exponent), and $\Sigma_0$ (the zero Lebesgue measure complement). By [LS], [K], ac components of spectral measures give zero weight to $\Sigma_+$, and since $\Sigma_0$ has zero Lebesgue measure, they must give full weight to $\Sigma_-$.

On the other hand, Kotani [K] (see also [D], Corollary 1) showed that if in some open set in the spectrum[6] the i.d.s. is absolutely continuous *and* the Lyapunov exponent vanishes, then the restriction of the spectral measures are absolutely continuous for almost every phase. The result follows from Corollary 1.8. □

Corollary 1.7 is an immediate consequence of Corollaries 1.2 and 1.6.

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