GENERIC SINGULAR SPECTRUM FOR ERGODIC SCHRÖDINGER OPERATORS

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ABSTRACT. We consider Schrödinger operators with ergodic potential $V_\omega(n) = f(T^n(\omega))$, $n \in \mathbb{Z}$, $\omega \in \Omega$, where $T : \Omega \to \Omega$ is a non-periodic homeomorphism. We show that for generic $f \in C(\Omega)$, the spectrum has no absolutely continuous component. The proof is based on approximation by discontinuous potentials which can be treated via Kotani Theory.

1. INTRODUCTION

Let Ω be a compact metric space, $T : \Omega \to \Omega$ a homeomorphism, and μ a T-ergodic Borel measure. We will always assume that T is not periodic, that is, μ is non-atomic. For a bounded and measurable function $f : \Omega \to \mathbb{R}$, we consider (line) Schrödinger operators $H_\omega = \Delta + V_\omega$, $n \in \mathbb{Z}$ with potential $V_\omega(n) = f(T^n\omega)$ and the associated Lyapunov exponents $\gamma(z)$, $z \in \mathbb{C}$. By Kunz-Souillard (cf. [5, 12]), there exists a compact set $\Sigma_{\mathrm{ac}}(f) \subseteq \mathbb{R}$ such that $\sigma_{\mathrm{ac}}(H_\omega) = \Sigma_{\mathrm{ac}}(f)$ for μ-a.e. $\omega \in \Omega$. By Pastur-Ishii-Kotani (cf. [5, 9, 10, 14, 15]), $\Sigma_{\mathrm{ac}}(f) = \overline{\{E \in \mathbb{R} : \gamma(E) = 0\}}^{\mathrm{ess}}$.

We shall only consider situations where the potentials V_ω are not periodic. In this case, it is an interesting question whether there can be any absolutely continuous spectrum.

It was observed by Kotani, [11], that $\Sigma_{\mathrm{ac}}(f)$ is empty if f takes only finitely many values. Damanik and Killip, [6], derived the same conclusion under the assumption that f is discontinuous at some point ω_0 (but continuous at all points in the forward orbit of ω_0 under T). Here, we will consider the case of continuous f. It is not in general true in this case that $\Sigma_{\mathrm{ac}}(f)$ is empty: If $\Omega = \mathbb{R}/\mathbb{Z}$, T is a Diophantine irrational rotation, v is analytic and $\lambda > 0$ is small enough, then for $f = \lambda v$, the spectrum is (almost surely) purely absolutely continuous; see, for example, Bourgain and Jitomirskaya [4] and references therein. However, we show that absence of absolutely continuous spectrum is a generic phenomenon:

Theorem 1. *There is a residual set of functions f in $C(\Omega)$ such that $\Sigma_{\mathrm{ac}}(f) = \emptyset$.*

Remark 1.1. A subset of a complete metric space (or more generally a Baire space) is called residual or (Baire) generic if it contains a countable intersection of dense open sets. By Baire's Theorem, such a set is dense.

Remark 1.2. It is well known (see [3] for a more general result) that for generic potentials, the Lyapunov exponent vanishes generically in the spectrum. Frequently (for instance, if $T : \Omega \to \Omega$ has a periodic point in the support of μ), the spectrum contains intervals in a persistent way, and hence has positive Lebesgue measure.

This is not incompatible with our result, since the Lyapunov exponent can be discontinuous.

The method used in the proof of Theorem 1 can be applied to certain one-parameter families of potentials. This has the following consequence.

Theorem 2. *There is a residual set of functions f in $C(\Omega)$ such that $\Sigma_{\mathrm{ac}}(\lambda f) = \emptyset$ for almost every $\lambda > 0$.*

This is particularly striking in the case of quasiperiodic systems. In this context one often expects absolutely continuous spectrum for small coupling, and as we mentioned before, this has been established under strong regularity assumptions on the potential (Bourgain-Jitomirkaya's result). It was not clear whether the assumption on the regularity was an artifact of the known methods. Our result shows that some regularity is indeed necessary: it is not enough to assume only continuity of the potential. It is an interesting open problem to determine the weakest possible regularity assumption on f under which a Bourgain-Jitomirskaya-type result holds.

2. A Semi-Continuity Result

We will need some basic facts about Lyapunov exponents and m-functions, see [15]. The Lyapunov exponent is defined by

$$\gamma_f(E) = \lim_{n \to \infty} \frac{1}{n} \int_\Omega \ln \|S_{f,E}^n(\omega)\| \, d\mu(\omega),$$

where

$$S_{f,E}^n(\omega) = S_{f,E}(T^{n-1}(\omega)) \cdots S_{f,E}(\omega),$$

and

$$S_{f,E}(\omega) = \begin{pmatrix} E - f(\omega) & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}).$$

We have that $E \mapsto \gamma_f$ is a non-negative real-symmetric subharmonic function. If $E \in \mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$, we have the formula

$$\gamma_f(E) = \int_\Omega -\Re \ln m_{\omega,f}(E) \, d\mu(\omega),$$

where $m_{\omega,f} : \mathbb{H} \to \mathbb{H}$ is a holomorphic function, called the m-function, which is given almost everywhere by

$$m_{\omega,f}(E) = \lim_{n \to \infty} S_{f,E}^n(T^{-n}(\omega)) \cdot i$$

(here we consider the usual action of $\mathrm{SL}(2, \mathbb{C})$ by Möbius transformations, i.e., $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$).

Define $M(f) = |\{E \in \mathbb{R} : \gamma_f(E) = 0\}|$, where $|\cdot|$ denotes Lebesgue measure. As a consequence of the results of Ishii, Kotani, and Pastur, we have $\Sigma_{\mathrm{ac}}(f) = \emptyset$ if and only if $M(f) = 0$.

Lemma 1. *For every $r > 0$, $\Lambda > 0$, the maps*

(1) $$(L^1(\Omega) \cap B_r(L^\infty(\Omega)), \| \cdot \|_1) \to \mathbb{R}, \quad f \mapsto M(f)$$

and

(2) $$(L^1(\Omega) \cap B_r(L^\infty(\Omega)), \| \cdot \|_1) \to \mathbb{R}, \quad f \mapsto \int_0^\Lambda M(\lambda f) d\lambda$$

are upper semi-continuous.

Proof. It is enough to show that (1) is upper semi-continuous, since this implies that (2) is also upper semi-continuous by Fatou's Lemma.

We have to show that if $(f_n)_{n \in \mathbb{Z}_+}$, f are uniformly bounded in L^∞ and $f_n \to f$ in L^1, then $\limsup M(f_n) \le M(f)$.

Assume otherwise. Then (by passing to a suitable subsequence), there are a constant $C < \infty$ and a sequence (f_n) such that

   (i) $f_n \to f$ in L^1 and pointwise,
   (ii) $\|f_n\|_\infty \le C$, $\|f\|_\infty \le C$,
   (iii) $\liminf M(f_n) \ge M(f) + \varepsilon$ for some $\varepsilon > 0$.

By (i), we have pointwise convergence of the m-functions m_{ω, f_n} in $\mathbb{H}$ for almost every ω. Thus, by dominated convergence and (ii), the associated Lyapunov exponents $\gamma_{f_n}(E)$ converge pointwise in $\mathbb{H}$ to $\gamma_f(E)$.

By (ii), all Lyapunov exponents are positive outside the interval $I = [-2 - C, 2 + C]$. Thus, we can limit our attention to this interval. Consider the region U in $\mathbb{H}$, bounded by the equilateral triangle T with sides I, J, K. Consider a conformal mapping Φ from the unit disk $\mathbb{D}$ to U. By the Schwarz-Christoffel formula (see, e.g., [8]),

(3) $$\Phi'(z) = \text{const} \cdot \prod_{j=1}^3 \left(1 - \tfrac{z}{z_j}\right)^{-2/3},$$

where z_1, z_2, z_3 are the inverse images under Φ of the vertices of T.

The functions $\gamma_{f_n} \circ \Phi$ are harmonic and bounded in $\mathbb{D}$. This yields

$$\gamma_{f_n}(\Phi(0)) = \frac{1}{2\pi} \int_0^{2\pi} \gamma_{f_n}\left(\Phi(e^{i\theta})\right) d\theta,$$

and similarly for γ_f. Since $\gamma_{f_n}(\Phi(0)) \to \gamma_f(\Phi(0))$ as $n \to \infty$, we infer

$$\frac{1}{2\pi} \int_0^{2\pi} \left[\gamma_{f_n}\left(\Phi(e^{i\theta})\right) - \gamma_f\left(\Phi(e^{i\theta})\right)\right] d\theta \to 0.$$

By dominated convergence, the integrals along J and K go to zero individually. Therefore,

$$\int_I [\gamma_{f_n}(E) - \gamma_f(E)] g(E) \, dE \to 0$$

where $g(E) = [\Phi'(\Phi^{-1}(E))]^{-1}$. It follows from (3) that g vanishes at the endpoints of I and is continuous and non-vanishing inside I.

By upper semi-continuity of the Lyapunov exponent and dominated convergence,

$$\int_I \max\{\gamma_{f_n}(E) - \gamma_f(E), 0\} \, g(E) \, dE \to 0,$$

and hence

$$\int_I \min\{\gamma_{f_n}(E) - \gamma_f(E), 0\}\, g(E)\, dE \to 0.$$

Consequently, since $\gamma_f|I$ is bounded and $\gamma_{f_n}|I$ is non-negative,

$$\int_I \min\{\gamma_{f_n}(E) - \gamma_f(E), 0\}\, dE \to 0.$$

Choose $\delta > 0$ such that the set $X = \{E \in I : \gamma_f(E) < \delta\}$ has measure bounded by $M(f) + \frac{\varepsilon}{4}$, with ε from (iii). Then,

$$\int_{I\setminus X} \min\{\gamma_{f_n}(E) - \gamma_f(E), 0\}\, dE \to 0.$$

This shows that for $n \geq n_0$, there exists a set Y_n of measure bounded by $\frac{\varepsilon}{4}$ such that $\gamma_{f_n}(E) \geq \frac{\delta}{2}$ for every $E \in I \setminus (X \cup Y_n)$. Consequently, $\limsup M(f_n) \leq M(f) + \frac{\varepsilon}{2}$, which contradicts (iii). $\qquad\square$

3. Approximation by Discontinuous Potentials

Lemma 2. *There exists a dense subset $\mathcal{Z}$ of $L^\infty(\Omega)$ such that if $s \in Z$, then*

(1) $s(\omega)$, $\omega \in \Omega$, *takes finitely many values,*

(2) $s(T^n(\omega))$, $n \in \mathbb{Z}$, *is not periodic for almost every $\omega \in \Omega$.*

Proof. Let W_k be the closed subspace of functions s taking at most k values. Obviously $W = \cup_{k \geq 2} W_k$ is dense in $L^\infty(\Omega)$. So we only have to show that there is a dense subset $S_k \subset W_k$ of functions satisfying the second property. Given $s \in W$, $\omega \in \Omega$, let $\phi(s, \omega) \in \mathbb{Z}_+ \cup \{\infty\}$ be the period of $s(T^n(\omega))$, $n \in \mathbb{Z}$. Then $\phi(s, \omega)$ is a constant $\Phi(s)$ almost everywhere. Let $W_{k,n} = \{s \in W_k : \Phi(s) \leq n\}$. It is easy to see that $W_{k,n}$ is a closed subset of W_k and $W_k \neq W_{k,n}$. Thus, $W_k \setminus \cup_{n \in \mathbb{Z}_+} W_{k,n}$ is dense in W_k. $\qquad\square$

Lemma 3. *For $f \in C(\Omega)$, $\varepsilon > 0$, $\delta > 0$, $\Lambda > 0$, there exists $\tilde{f} \in C(\Omega)$ such that $\|f - \tilde{f}\|_\infty < \varepsilon$, $M(\tilde{f}) < \delta$, and $\int_0^\Lambda M(\lambda\tilde{f})d\lambda < \delta$.*

Proof. Let $\mathcal{Z}$ be as in Lemma 2 and choose $s \in \mathcal{Z}$ such that $\|f - s\|_\infty < \frac{\varepsilon}{2}$. By the Kotani result, [11], we have $M(\lambda s) = 0$ for every $\lambda > 0$. Next we choose continuous functions f_n, for which we have $\|s - f_n\|_\infty < \frac{\varepsilon}{2}$ for all n and $\|s - f_n\|_1 \to 0$ as $n \to \infty$. For instance, take $f_n(\omega) = \int_\Omega C_n(\omega)^{-1} c_n(\omega, \omega') s(\omega') d\nu(\omega')$, where ν is a probability measure with $\operatorname{supp} \nu = \Omega$, $C_n(\omega) = \int_\Omega c_n(\omega, \omega') d\nu(\omega')$, $c_n(\omega, \omega') = \max\{(n + n_0)^{-1} - \operatorname{dist}(\omega, \omega'), 0\}$, n_0 sufficiently large. Lemma 1 implies $M(f_n), \int_0^\Lambda M(\lambda f_n)d\lambda \to 0$ as $n \to \infty$. Thus, choosing n large enough so that $M(f_n), \int_0^\Lambda M(\lambda f_n)d\lambda < \delta$, we complete the proof. $\qquad\square$

Proof of Theorem 1. For $\delta > 0$, we define

$$M_\delta = \{f \in C(\Omega) : M(f) < \delta\}.$$

By Lemma 1, M_δ is open, and by Lemma 3, M_δ is dense. It follows that

$$\{f \in C(\Omega) : \Sigma_{\mathrm{ac}}(f) = \emptyset\} = \{f \in C(\Omega) : M(f) = 0\} = \bigcap_{\delta > 0} M_\delta$$

is residual. $\qquad\square$

Proof of Theorem 2. For $\Lambda, \delta > 0$, we define

$$M_\delta(\Lambda) = \left\{ f \in C(\Omega) : \int_0^\Lambda M(\lambda f) \, d\lambda < \delta \right\}.$$

By Lemma 1, $M_\delta(\Lambda)$ is open and by Lemma 3, $M_\delta(\Lambda)$ is dense. Thus,

$$\bigcap_{\Lambda, \delta > 0} M_\delta(\Lambda)$$

is residual. It follows that for Baire generic $f \in C(\Omega)$, we have $\Sigma_{\mathrm{ac}}(\lambda f) = \emptyset$ for almost every $\lambda > 0$. $\qquad\square$

4. Concluding Remarks

Remark 4.1. It is possible to improve Lemma 1 to show that $M(f)$ is an upper semi-continuous function of $f \in L^1(\Omega)$. The additional point is that, given $f \in L^1(\Omega)$ and $\varepsilon > 0$, we can choose a bounded interval $I \subset \mathbb{R}$ such that for every $\tilde{f} \in L^1(\Omega)$ close to f, we have $|\{E \in \mathbb{R} \setminus I : \gamma_{\tilde{f}}(E) = 0\}| < \varepsilon$. To see this, one shows first that the integrated density of states $N_f(E) \in L^\infty(\mathbb{R})$ is a continuous function of $f \in L^1(\Omega)$, and then one uses [7] to bound the size of the absolutely continuous spectrum near infinity.

Remark 4.2. By the Wonderland theorem [16] (see also [13]), the set of f's leading to purely singular spectrum is a G_δ set in all metric topologies that imply strong resolvent convergence of the associated operators. This permits one to deduce generic singular spectrum if one can exhibit a dense set with this property. With the Kotani result (combined with Lemma 2) as input, this only gives a generic set in $L^\infty(\Omega)$ and does not imply Theorem 1. It is not clear how to prove Theorem 1 using this strategy, but it would be interesting to find an explicit dense set of continuous functions such that the corresponding operators have empty absolutely continuous spectrum.

Remark 4.3. The result of this paper naturally extends to the context of more general $\mathrm{SL}(2, \mathbb{R})$ cocycles. A possible formulation is the following. Given $A \in C(\Omega, \mathrm{SL}(2, \mathbb{R}))$, one can consider a one-parameter family of cocycles $(T, R_\theta A)$, where $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, and the result is that for generic A and for almost every $\theta \in \mathbb{R}$, the Lyapunov exponent of $(T, R_\theta A)$ is positive. The key point is that the relevant part of Kotani's Theory (which is used in the proof of Lemma 3) can be carried out in this setting (see [2] for related results). (To prove the analogue of Lemma 1, one can use [1] to show that the average Lyapunov exponent of the family $\theta \mapsto (T, R_\theta A)$ depends continuously on A in the L^1 topology.)

Notice that for certain choices of T (say, irrational rotations), there are open sets $U \subset C(\Omega, \mathrm{SL}(2, \mathbb{R}))$ such that the Lyapunov exponent of $(T, R_\theta A)$ is 0 for generic $A \in U$ and for generic $\theta \in \mathbb{R}$, [3]. Based on this, some authors have argued that under weak smoothness requirements (such as continuity), positive Lyapunov exponents are rare. Our result shows in a sense that positive Lyapunov exponents tend to prevail in a mixed topological/measure-theoretic category, even when they are topologically rare.

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