Proposição de curso
27^{o} Coloquio Brasileiro de Matematica

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Note: A presente proposição esta escrita em inglês mas o curso pode ser dado em Inglês ou (de preferência) Português.

Title: Intersection homology.

Level: Advanced

Prerequisites: Basic notions of algebraic topology (Homology - Cohomology) and differential geometry (differential forms on a smooth manifold).

Short summary - objective:

Aim of the course is to introduce intersection homology, that is the suitable theory to obtain classical duality theorems in the situation of singular varieties. The course will be characterized by two points: in the one hand numerous examples and pictures, in the other hand introduction to the actual tools and research developments, in particular perverse sheaves and stratified Morse theory. Among famous applications of intersection homology one can mention that it has been used to prove the Kazhdan-Lusztig conjectures and the Riemann-Hilbert correspondence.

Summary - content:

The first step will be to recall the classical Duality Theorems on manifolds (Poincaré, Lefschetz, de Rham) then to provide examples of singular varieties for which these theorems fail. The necessary tools and properties of singular varieties will be defined, in particular stratifications (Whitney stratifications) and triangulations. It is then possible to introduce and define the Goresky-MacPherson Intersection Homology. Various examples, in particular intersection homology of toric varieties will be computed.

The theory allows to recover Duality Theorems in appropriate situations of stratified singular varieties. Poincaré Duality and Lefschetz Duality Theorems for singular varieties will be detailed as well as suitable examples.

Among properties of intersection homology, the local calculus is very important: it motivates and justifies the sheaf definition and introduces the so-called perverse sheaves. Examples and properties of perverse sheaves will be provided as well as a short introduction to hypercohomology. Poincaré duality and factorization of the Poincaré homomorphism will be explained in the language of perverse sheaves, the "mythic" so-called Decomposition Theorem will be given as well as examples of applications.

One course will be devoted to the de Rham Theorem, that deserves to be explicited at two levels: the explicit and elementary level, according to the Whitney proof of de Rham Theorem and the "perverse sheaves level" in order to obtain a general result.

Stratified Morse theory will be the subject of the last course. After recalling Morse Theory in the smooth case, the singular case will be given in the context of intersection homology and the so-called stratified Morse theory. Among applications of the Morse theory, monodromy and Milnor theory will be detailed.

During the course, useful information will be given on related and actual research and problems, such as Hodge structures, equivariant intersection homology, Betti numbers, Witt spaces etc...

Detail of courses - distribution of chapters and courses:

1. Duality Theorems for manifolds (Poincaré, Lefschetz, de Rham...)
   Counter-examples for singular varieties (pinched torus, suspension of the torus).
   Singular varieties: triangulations, stratifications - Whitney stratifications.
   Definition of intersection homology.
   Examples - toric varieties...
2. Properties of intersection homology
   Poincaré Duality - Factorisation of the Poincaré morphism
   Local calculus
3. Sheaf definition
   Perverse sheaves
   Properties of perverse sheaves.
4. de Rham Theorem for manifolds (the Whitney proof).
   de Rham Theorem for singular varieties:
   * Shadow forms (explicit way)
   * Perverse sheaves and de Rham (axiomatic way)
5. Stratified Morse theory.
   More about stratifications
   Morse theorem for stratified singular varieties
   Applications: Monodromy - Milnor fiber

Bibliography:

Basic and classical bibliography.

1. M. Goresky and R. MacPherson, *Intersection homology theory I*, Topology 19 (1980), 135-162.
2. A. Borel, *Intersection Cohomology*, Birkhäuser, 1984.
3. F. Kirwan and J. Woolf, *An Introduction to Intersection Homology Theory*, Second Edition 2007, Chapman & Hall/CRC.
4. J.L. Brylinski, *(Co)-homologie d'intersection et faisceaux pervers*, Séminaire Bourbaki, 24 (1981-1982), Exposé No. 585.
5. M. Goresky and R. MacPherson, *Stratified Morse theory*, Springer, 1988.

Advanced bibliography.

6. J.-P. Brasselet, M. Goresky and R. MacPherson, *Simplicial Differential Forms with Poles*, **Amer. Journal of Maths.**, 113 (1991), 1019-1052.
7. J.-P. Brasselet, *De Rham's theorems for singular varieties*, **Contemporary Mathematics** 161, (1994) 95 - 112.
8. G.Barthel, J.-P. Brasselet, K.H. Fieseler, O. Gabber et L. Kaup, *Relèvement de cycles algébriques et homomorphismes associés en Homologie d'Intersection*, **Annals of Maths** 139 (1994),
9. G.Barthel, J.-P. Brasselet, K.H. Fieseler und L. Kaup, *Poincaré Polynomials and Perverse Sheaves on Fans, a combinatorial framework*, **Tohoku Math. J.** (2) 57, no. 2 (2005), 273292
10. J.-P. Brasselet, *Poincaré-Hopf Theorem for singular varieties.* Proceedings of the Trieste Conference on Singularities, Singularities in geometry and topology, 57–80, World Scientific Publ., Hackensack, NJ, 2007.