Advanced Course Proposal for the 27th Brazilian Colloquium of Mathematics: Evolution Equations in Geometry

Bianca Santoro

October 30, 2008

Objectives

This minicourse is intended to serve as an introduction to both graduate students and geometers who are interested in some of the new applications of evolution equations in both Riemannian and Complex Geometry.

We aim to provide an account of the work of Hamilton and Perelman on Ricci Flow, both having profound influence on the study of geometry, and so far the only successful approach to proving the famous Thurston's Geometrization Conjecture.

Also, we will talk about the Ricci Flow in the context of Kähler geometry along with some of its deep implications.

Pre-requisites

Basic knowledge of Riemannian Geometry and Lie Group Theory, as well as some familiarity with the basic techniques of Elliptic and Parabolic Partial Differential Equations. For the part involving Complex Geometry, this course is expected to be self-contained.

Course Description

i) The Geometrization Conjecture

We describe Thurston's Geometrization Conjecture: any closed 3-manifold can canonically be decomposed into pieces in such a way that each piece admits a unique homogeneous geometry.

For that, we need to understand the concepts of prime and irreducible manifolds, the Prime Decomposition Theorem, the Torus Decomposition Theorem, and the classification, via Lie Group Theory, of three-dimensional maximal model geometries.

ii) Ricci Flow on Homogeneous Geometries

In this chapter, we present an outline of what is the current strategy to attack the Geometrization Conjecture. We begin by studying the effect of evolving the Ricci Flow in one example of each isotropy class of model geometry in dimension 3.

Then, we introduce the main obstructions of using Ricci flow in geometrization: in general, formation of singularities will occur in finite time, and its complete understanding is very subtle. This topic will further be developed in Chapter iv.

iii) Ricci Flow on Surfaces

As a Toy Model, we use the Ricci Flow to prove the Uniformization Theorem for Riemann Surfaces, following Hamilton's [H1] and Chow's [CK] works. Even though it does not enclose most of the difficulty of the three-dimensional case, it helps developing intuition for the problem.

iv) Outline of Hamilton's Program to prove Thurston's Geometrization Conjecture

We plan to overview Hamilton's program, and more specifically, to describe the main difficulties in the ground-breaking work by Perelman, who brilliantly completed the program.

Also, fundamental and more basic results, like the short-time existence of solutions (via de Turck's Trick) and the Maximum Principle for Tensors (that have inumerous applications in other problems in geometric analysis) will be studied in further detail.

v) Introduction to Kähler Geometry

In order to study the Kähler version of the Ricci Flow, that evolves metrics to the unique Kähler -Einstein metric in each Kähler class (whose interest goes far beyond Mathematics), we need to review some basic definitions and facts about complex manifolds, following the lines of [We] and [RFV2]. Namely, we specialize the definitions of the main geometric objects (like the Riemann Curvature Tensor and its special contractions) to Kähler geometry, and show some of the implications on symmetry and structure caused by the presence of an integrable complex structure compatible with the metric.

In addition, we explain the relation of our complex geometric problem to the solution of particular Partial Differential Equations on Kähler manifolds.

vi) Kähler Ricci Flow

In this chapter, we introduce the Ricci Flow on Kähler manifolds. We describe the main results in the area, in particular the long time existence result by Cao [C], as well as, if time permits, some of its applications to other problems in complex geometry.

References

[CCCY] *Collected papers on Ricci flow.* Edited by H. D. Cao, B. Chow, S. C. Chu and S. T. Yau. Series in Geometry and Topology, **37**. International Press, Somerville, MA (2003).

[C] Cao, Huai Dong *On Harnack's inequalities for the Kähler -Ricci flow.* Invent. Math. 109 (1992), no. 2, 247–263.

[CK] Chow, Bennett; Knopf, Dan *The Ricci flow: an introduction.* Mathematical Surveys and Monographs, **110**. American Mathematical Society, Providence, RI, 2004.

[RFV2] Chow, Bennett; Chu, Sun-Chin; Glickenstein, David; Guenther, Christine; Isenberg, Jim; Ivey, Tom; Knopf, Dan; Lu, Peng; Luo, Feng; Ni, Lei *The Ricci Flow: Techniques and Applications.* Volume 2 - Part 1: Geometric Aspects. Preprint.

[H1] Hamilton, Richard S., *The Ricci flow on surfaces.* Mathematics and general relativity (Santa Cruz, CA, 1986), 237–262, Contemp. Math., **71**, Amer. Math. Soc., Providence, RI, (1988).

[H2] Hamilton, Richard S. *Three-manifolds with positive Ricci curvature.* J. Differential Geom. **17** (1982), no. 2, 255–306.

[KL] Kleiner, Bruce; Lott, John *Notes on Perelman's papers.* Preprint Arxiv, math.DG/0605667 (2006).

[MT] Morgan, John W.; Tian, Gang *Ricci Flow and the Poincare Conjecture.* Preprint Arxiv, math.DG/0607607 (2006).

[We] Wells, R. O., Jr., *Differential analysis on complex manifolds.* Second edition. Graduate Texts in Mathematics, **65**. Springer-Verlag, New York-Berlin, (1980).

Bianca Santoro (bsantoro@msri.org)
Mathematical Sciences Research Institute
17 Gauss Way, room 217
Berkeley, CA 94720
USA