A Polynomial Algorithm to Find Minimum Crossing Curves

Sóstenes Lins

UFPE

Resumo/Abstract:
Let G be a finite graph embedded into a closed surface S, $G \hookrightarrow S$. In our context, a closed curve means a piecewise linear continuous function from S^1 into S, S^1 being the unit circle. Let D be a closed curve so that its image does not contain vertices of G. This condition implies that the curve D is homotopic to a finite sequence of edges forming a closed path in G^*, the surface dual of G. Use $C \simeq D$ to mean that C and D are freely homotopic on S. Is it possible to effectively find a curve C in S, also missing the vertices of G so that $C \simeq D$ and $|G \cap C|$ is minimum? This problem has been treated by De Graaf and Schrijver (J.Combin.Th.B-1997) who presented a finite algorithm to perform this task. Here I obtain a revised 4-regular graph H embedded into S so that G is a minor of H, the curve D is maintained and the following conditions are satisfied:

• finding $C \simeq D$ so that $|C \cap H|$ is minimum, i.e. a solution for the problem on $(H \cup D \hookrightarrow S)$, induces a solution for the original problem on $(G \cup D \hookrightarrow S)$;
• H is metrically embedded into a geometric version of S, induced by a tripartition (T^*, T, R) of the edge set of H, where: T is a set of edges forming a tree of H, T^* is a set of edges forming a tree of H^*, the dual graph of H in S and R are the remaining edges;
• in U, the universal covering space of the embedding $(H \cup H^* \hookrightarrow S)$, every primal edge is formed by at most six geodesic segments and every dual edge by at most three geodesic segments;
• $(H \cup D \cup H^* \hookrightarrow U)$ is obtained by an $O(n^2)$ algorithm, where n is the number of edges of H.
• the metric of U (which is the hyperbolic plane, if $|R| \geq 3$) induces a combinatorial data structure so that the problem on $(H \cup D \hookrightarrow S)$ is solvable by an algorithm of order $O(n^2)$, where n is the number

of edges of H. The overall complexity of the algorithm depend on how big H is relative to its minor G. An absolute polynomial upper bound for it is easy to establish.

This result relies on an old result of Tutte (Proc. London Math. Soc., 1963) on basic facts about the hyperbolic plane (John G. Ratcliffe-Foundations of Hyperbolic Manifolds, Springer-1994, J.Stillwell, Geometry of Surfaces, Springer-1992) and on a deep theorem conjectured first by V. G. Turaev proved by entirely different methods by J.Hass and P. Scott (Topology-1994) and independently by De Graaf and Schrijver (J.Combin.Th.B-1997).