Tutorial 4: Polymer + Random Interface

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The purpose of this tutorial is to explore the variational approach to the model of a polymer pinned at an interface via random charges described in the course.

1 The model

Let $(S = \{S_n\}_{n\in\mathbb{N}_0}, P)$ be a Markov chain on a countable space Υ which contains a marked point $*$. We assume that $P(S_0 = *) = 1$. We introduce the first return time to $*$, namely $\tau := \inf\{n \in \mathbb{N} : \ S_n = *\}$, and we denote by $R(\cdot)$ its distribution:

$$R(n) := P(\tau = n) = P(S_i \neq * \ \forall 1 \le i \le n-1, \ S_n = *) \,, \qquad n \in \mathbb{N} \,.$$

We assume that $\sum_{n\in\mathbb{N}} R(n) = 1$, i.e., the Markov chain is *recurrent*, and as $n \to \infty$

$$\lim_{n\to\infty} \frac{\log R(n)}{\log n} = -(1+a) \,, \qquad \text{with } a \in [0,\infty) \,. \tag{1}$$

This is a way of imposing the asymptotic behavior $R(n) \approx n^{-(1+a)}$ which is more general than requiring that $R(n) = L(n)\, n^{-(1+a)}$ with $L(\cdot)$ slowly varying. We recall that, for a *simple and symmetric* random walk on $\mathbb{Z}$, that is if $P(S_1 = 1) = P(S_1 = -1) = p$ and $P(S_1 = 0) = 1-2p$ with $p \in (0, \frac{1}{2})$, relation (1) holds with $a = \frac{1}{2}$.

The set of allowed polymer configurations is $\mathcal{W}_n = \{w = (i, w_i)_{i=0}^n : \ w_0 = *, \, w_i \in \Upsilon \ \forall 0 < i \le n\}$ on which we define the Hamiltonian

$$H_n^{\beta,h,\omega}(w) = -\sum_{i=0}^{n} (\beta\omega_i - h) 1_{\{w_i = *\}}$$

where $\beta, h \ge 0$ are two parameters that tune the interaction strength and $\omega = \{\omega_n\}_{n\in\mathbb{N}_0}$ is the *random environment*, that is a "typical realization" of a sequence of i.i.d. $\mathbb{R}$-valued random variables with marginal law μ_0. The law of the full sequence ω is therefore $\mathbb{P} := \mu_0^{\otimes\mathbb{N}_0}$. We assume that $M(\beta) := \mathbb{E}(e^{\beta\omega_0}) < \infty$ for all $\beta \in \mathbb{R}$ and we fix $\mathbb{E}(\omega_0) = 0$ and $\mathbb{E}(\omega_0^2) = 1$.

We denote by P_n the projection onto $\mathcal{W}_n$ of the law of S, i.e., $P_n(w) := P(S_i = w_i \ \forall 0 \le i \le n)$ for $w \in \mathcal{W}_n$. This is an *a-priori* law for the free (non-interacting) polymer. We then define our polymer model as the law $P_n^{\beta,h,\omega}$ on $\mathcal{W}_n$ given by

$$P_n^{\beta,h,\omega}(w) = \frac{1}{Z_n(\beta,h,\omega)}\, e^{-H_n^{\beta,h,\omega}(w)}\, P_n(w) \,.$$

The normalizing constant $Z_n(\beta,h,\omega)$ is called *partition function* and it is given by

$$Z_n(\beta,h,\omega) = \sum_{w\in\mathcal{W}_n} e^{-H_n^{\beta,h,\omega}(w)}\, P_n(w) = E_n\left[e^{-H_n^{\beta,h,\omega}(w)}\right] = E\left[e^{\sum_{i=0}^n (\beta\omega_i - h) 1_{\{w_i = *\}}}\right] \,,$$

where the last equality is obtained through the change-of-variable formula. The *quenched free energy* $f^{\text{que}}(\beta,h)$ is defined as the (non random) limit

$$f^{\text{que}}(\beta,h) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\beta,h,\omega) \,, \qquad \mathbb{P}\text{-a.s. and in } L^1(\mathrm{d}\mathbb{P}) \,, \tag{2}$$

which has been shown to exist in the first tutorial. It can be easily shown that $f^{\text{que}}(\beta,h) \geq 0$ which motivates the introduction of a $\mathcal{L}$*ocalized* and a $\mathcal{D}$*elocalized* phase, defined by

$$\mathcal{L} := \{(\beta,h) : f^{\text{que}}(\beta,h) > 0\}, \qquad \mathcal{D} := \{(\beta,h) : f^{\text{que}}(\beta,h) = 0\}.$$

It follows by the convexity and monotonicity properties of the free energy that these phases are separated by a *quenched critical line* $\beta \mapsto h_c^{\text{que}}(\beta) := \inf\{h \in \mathbb{R} : f^{\text{que}}(\beta,h) = 0\}$. The purpose of this tutorial is to get some insight on a *variational formula* for h_c^{que}.

Note that by (2) we can write $f^{\text{que}}(\beta,h) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\log Z_n(\beta,h,\omega))$. Interchanging the expectation $\mathbb{E}$ and the 'log', one obtains the *annealed free energy*:

$$f^{\text{ann}}(\beta,h) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\big(Z_n(\beta,h,\omega)\big) = \lim_{n\to\infty} \frac{1}{n} \log E\big(e^{(\log M(\beta)-h)\sum_{i=0}^{n} 1_{S_i=*}}\big), \tag{3}$$

which is nothing but the free energy $f(\zeta)$ of an homogeneous pinning model, studied in the third tutorial, with $\zeta = \log M(\beta) - h$. Recall that $f(\zeta) > 0$ for $\zeta > 0$ while $f(\zeta) = 0$ for $\zeta \leq 0$. Introducing the *annealed critical line* $h_c^{\text{ann}}(\beta) := \inf\{h \in \mathbb{R} : f^{\text{ann}}(\beta,h) = 0\}$, it then follows that $h_c^{\text{ann}}(\beta) = \log M(\beta)$. Jensen's inequality yields $f^{\text{que}}(\beta,h) \leq f^{\text{ann}}(\beta,h)$ (cf. (3)), whence $h_c^{\text{que}}(\beta) \leq h_c^{\text{ann}}(\beta)$. For this class of models, disorder is said to be *irrelevant* if $h_c^{\text{que}}(\beta) = h_c^{\text{ann}}(\beta)$ and *relevant* if $h_c^{\text{que}}(\beta) < h_c^{\text{ann}}(\beta)$.

2 Some background on large deviations

We give a brief and concise look at some large deviations results. For more details, cf. [3, 2].

2.1 Relative entropy

Let ν, ρ be two probabilities on some measurable space $(\Gamma, \mathcal{G})$, that is, $\nu, \rho \in \mathcal{M}_1(\Gamma)$, the space of probability measures on Γ. If $\nu \ll \rho$ (i.e., ν is absolutely continuous with respect to ρ) we denote by $\frac{\mathrm{d}\nu}{\mathrm{d}\rho}$ the corresponding Radon-Nikodym derivative and we define the *relative entropy* $h(\nu|\rho)$ of ν with respect to ρ by the formula

$$h(\nu|\rho) := \int_\Gamma \log\left(\frac{\mathrm{d}\nu}{\mathrm{d}\rho}\right) \mathrm{d}\nu = \int_\Gamma \left(\frac{\mathrm{d}\nu}{\mathrm{d}\rho}\right) \log\left(\frac{\mathrm{d}\nu}{\mathrm{d}\rho}\right) \mathrm{d}\rho. \tag{4}$$

If $\nu \not\ll \rho$ we simply set $h(\nu|\rho) := +\infty$. Note that the function $g(x) := x\log(x)$ with $g(0) := 0$ is convex (hence continuous) and bounded from below on $[0,\infty)$, therefore the integral that defines $h(\nu|\rho)$ is always well defined in $\mathbb{R} \cup \{+\infty\}$.

- Using Jensen's inequality, show that $h(\nu|\rho) \geq 0$ for all ν, ρ; furthermore, $h(\nu|\rho) = 0$ if and only if $\nu = \rho$.

One can show that, for fixed ρ, the functional $\nu \mapsto h(\nu|\rho)$ is convex on $\mathcal{M}_1(\Gamma)$. Note that if Γ is a finite set, say $\Gamma = \{1,\ldots,r\}$ with $r \in \mathbb{N}$, we can write

$$h(\nu|\rho) := \sum_{i=1}^{r} \nu_i \log\frac{\nu_i}{\rho_i}. \tag{5}$$

2.2 Sanov's Theorem in a finite space

Let $Y = \{Y_n\}_{n\in\mathbb{N}}$ be an i.i.d. sequence of random variables taking values in a finite set, that we may identify with $\Gamma = \{1, \ldots, r\}$ with $r \in \mathbb{N}$. We denote by $\rho = \{\rho_i = P(Y_1 = i)\}_{1\le i\le r}$ the marginal law of the sequence: note that $\rho \in \mathcal{M}_1(\Gamma)$, the space of probability measures on Γ, and we assume that $\rho_i > 0$ for all $1 \le i \le r$. For $n \in \mathbb{N}$ we define the *empirical measure*

$$L_n := \frac{1}{n}\sum_{k=1}^{n} \delta_{Y_k}\,, \tag{6}$$

where δ_x denotes the Dirac mass at x. Note that L_n is a *random element* of $\mathcal{M}_1(\Gamma)$, i.e., a random variable taking values in $\mathcal{M}_1(\Gamma)$, which describes the relative frequency of the 'letters' appearing in the sequence $Y_1, \ldots, Y_n$.

Plainly, $\mathcal{M}_1(\Gamma)$ can be identified with the simplex $\{x \in (\mathbb{R}^+)^r : \sum_{i=1}^r x_i = 1\} \subset (\mathbb{R}^+)^r$, hence $\mathcal{M}_1(\Gamma)$ is equipped with the standard Euclidean topology and we can talk about convergence in $\mathcal{M}_1(\Gamma)$ (which is nothing but the convergence of every component). With this identification $L_n = \{L_n(i)\}_{1\le i\le r}$ where $L_n(i)$ is the relative frequency of the symbol i in the sequence $Y_1, \ldots, Y_n$, that is $L_n(i) = \frac{1}{n}\sum_{k=1}^n 1_{\{Y_k=i\}}$.

- Show that the strong law of large numbers yields the a.s. convergence $\lim_{n\to\infty} L_n = \rho$, where the limit is in $\mathcal{M}_1(\Gamma)$.

The purpose of large deviations theory is to quantify the probability that L_n differs from its limit ρ: given a different $\nu \in \mathcal{M}_1(\Gamma)$, what is the probability that L_n is close to ν? Take for simplicity $\nu = \{\nu_i\}_{1\le i\le r}$ of the form $\nu_i = \frac{k_i}{n}$ with $k_i \in \mathbb{N}$ and $\sum_{i=1}^r k_i = n$. (Note that this is the family of laws ν that can be attained by L_n.)

- Prove that for such a ν we have $P(L_n = \nu) = n! \prod_{i=1}^r \frac{{\rho_i}^{k_i}}{k_i!}$.

- Using Stirling's formula $n! = n^n e^{-n+o(n)}$ deduce that $P(L_n = \nu) = e^{-nh(\nu|\rho)+o(n)}$, where $h(\nu|\rho)$ is the relative entropy defined in (5).

In this sense, the relative entropy $h(\nu|\rho)$ gives the rate of exponential decay for the probability that L_n is close to ν instead of ρ.

More generally, one can show that, if O and C are respectively an open and a closed subset of $\mathcal{M}_1(\Gamma)$, setting $I(\nu) := h(\nu|\rho)$, the following relations hold:

$$\liminf_{n\to\infty} \frac{1}{n} \log P(L_n \in O) \;\ge\; -\inf_{\nu\in O} I(\nu)\,, \qquad \limsup_{n\to\infty} \frac{1}{n} \log P(L_n \in C) \;\le\; -\inf_{\nu\in C} I(\nu)\,. \tag{7}$$

Whenever such relations hold, we say that the sequence of random variables $\{L_n\}_{n\in\mathbb{N}}$ satisfies a *large deviations principle* (LDP) with rate function $I(\cdot)$.

2.3 Sanov's Theorem in Polish spaces

In the previous subsection we have worked under the assumption that the space Γ is finite, but everything can be generalized to the case when Γ is a *Polish space* (a complete and separable metric space) equipped with the Borel σ-field. Let $Y = \{Y_n\}_{n\in\mathbb{N}}$ be an i.i.d. sequence of random variables taking values in Γ and denote by $\rho \in \mathcal{M}_1(\Gamma)$ the law of Y_1.

We equip the space $\mathcal{M}_1(\Gamma)$ of probability measures on Γ with the topology of weak convergence ($\nu_n \to \nu$ in $\mathcal{M}_1(\Gamma)$ if and only if $\int f \mathrm{d}\nu_n \to \int f \mathrm{d}\nu$ for every bounded and continuous $f : \Gamma \to \mathbb{R}$). This topology turns $\mathcal{M}_1(\Gamma)$ into a Polish space too, that we equip with the corresponding Borel σ-field. We can therefore speak of convergence in $\mathcal{M}_1(\Gamma)$, as well as of random elements of $\mathcal{M}_1(\Gamma)$ (random variables taking values in $\mathcal{M}_1(\Gamma)$).

In particular, the empirical measure L_n introduced in (6) is well defined in this generalized setting as a random element of $\mathcal{M}_1(\Gamma)$. Using the Ergodic Theorem, one can show that, in analogy with the finite Γ case, we have a.s. $\lim_{n\to\infty} L_n = \rho$ in $\mathcal{M}_1(\Gamma)$. Also the large deviations relations in (7) continue to hold, again with $I(\nu) := h(\nu|\rho)$ as defined in (4).

2.4 Process level large deviations

One can take a step further and consider an extended empirical measure, keeping track of 'words' instead of single 'letters'. More precisely, let again $Y = \{Y_n\}_{n\in\mathbb{N}}$ be an i.i.d. sequence of random variables taking values in a Polish space Γ and denote by $\rho \in \mathcal{M}_1(\Gamma)$ the law of Y_1. For $\ell \in \mathbb{N}$ fixed, one can consider the empirical distribution of ℓ consecutive variables ('words' of length ℓ) appearing in the sequence $Y_1, \ldots, Y_n$:

$$L_n^\ell := \frac{1}{n} \sum_{i=1}^{n} \delta_{(Y_i, Y_{i+1}, \ldots, Y_{i+\ell-1})} ,$$

where we use for convenience periodic boundary conditions: $Y_{n+i} = Y_i$ for $i = 1, \ldots, \ell - 1$. Note that L_n^ℓ is a random element of the space $\mathcal{M}_1(\Gamma^\ell)$ of probability measures on Γ^ℓ. One can show that a.s. $\lim_{n\to\infty} L_n^\ell = \rho^{\otimes \ell}$ and one can obtain the large deviations for L_n^ℓ, in analogy with (7), with an explicit rate function $I(\cdot)$.

One can even go beyond, considering the empirical measure associated with 'words' of arbitrary length (up to n) appearing in the sequence $Y_1, \ldots, Y_n$. It is convenient to denote by $(Y_1, \ldots, Y_n)^{\mathrm{per}}$ the *infinite* sequence obtained by repeating periodically $(Y_1, \ldots, Y_n)$, that is $((Y_1, \ldots, Y_n)^{\mathrm{per}})_{mn+j} := Y_j$ for $m \in \mathbb{N}_0$ and $j \in \{1, \ldots, n\}$. Note that $(Y_1, \ldots, Y_n)^{\mathrm{per}}$ takes values in $\Gamma^{\mathbb{N}}$. Denoting by θ the left shift on $\Gamma^{\mathbb{N}}$, that is $(\theta x)_i := x_{i+1}$ for $x = \{x_i\}_{i\in\mathbb{N}}$, we can therefore introduce the *empirical process*

$$R_n := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\theta^i (Y_1, \ldots, Y_n)^{\mathrm{per}}} , \tag{8}$$

which is by definition a random element of the space $\mathcal{M}_1^{\mathrm{inv}}(\Gamma^{\mathbb{N}})$ of shift-invariant probability measures on the Polish space $\Gamma^{\mathbb{N}}$, which is equipped with the product topology and σ-field.

Again, one can show that a.s. $\lim_{n\to\infty} R_n = \rho^{\otimes\mathbb{N}}$ on $\mathcal{M}_1^{\mathrm{inv}}(\Gamma^{\mathbb{N}})$. Furthermore, in analogy to (7), $\{R_n\}_{n\in\mathbb{N}}$ satisfies a LDP: for every open set O and closed set C in $\mathcal{M}_1^{\mathrm{inv}}(\Gamma^{\mathbb{N}})$

$$\liminf_{n\to\infty} \frac{1}{n} \log P(R_n \in O) \geq - \inf_{\nu\in O} I(\nu) , \qquad \limsup_{n\to\infty} \frac{1}{n} \log P(R_n \in C) \leq - \inf_{\nu\in C} I(\nu) , \tag{9}$$

where the rate function $I(\nu) = H(\nu|\rho^{\otimes\mathbb{N}})$ is the so-called *specific relative entropy*:

$$H(\nu|\rho^{\otimes\mathbb{N}}) := \lim_{n\to\infty} \frac{1}{n} h(\pi_n\nu|\rho^{\otimes n}) , \tag{10}$$

where $h(\,\cdot\,|\,\cdot\,)$ is always the relative entropy defined in (4) and π_n denotes the projection from $\Gamma^{\mathbb{N}}$ to Γ^n onto the first n components. The limit in (10) can be shown to be non-decreasing: in particular, $H(\nu|\rho) = 0$ if and only if $\pi_n\nu = \rho^{\otimes n}$ for every $n \in \mathbb{N}$, that is $\nu = \rho^{\otimes\mathbb{N}}$.

3 Random words cut out from a random letter sequence

We now apply the large deviations theory of the previous section to study the sequence of random words cut out from a random letter sequence according to an independent renewal process. More precisely, our 'alphabet' will be $\mathbb{R}$ while $\widetilde{\mathbb{R}} := \bigcup_{n\in\mathbb{N}} \mathbb{R}^k$ will be the set of finite words drawn from $\mathbb{R}$, which can be metrized to become a Polish space.

We recall that $(\omega = \{\omega_k\}_{k\in\mathbb{N}_0}, \mathbb{P})$ is an i.i.d. sequence of $\mathbb{R}$-valued random variables, with marginal distribution μ_0, and $(S = \{S_n\}_{n\in\mathbb{N}_0}, P)$ is a recurrent Markov chain on the countable space Υ, which contains a marked point $*$. The sequences ω and S are independent. From the sequence of letters ω we now cut out a sequence of words $Y = \{Y_i\}_{i\in\mathbb{N}}$, using the successive excursions of S out of $*$. More precisely, we let T_k denote the epoch of the kth return of S to $*$:

$$T_0 := 0\,, \qquad T_{k+1} := \inf\{m > T_k : S_m = *\}\,, \tag{11}$$

and we set $Y_i := (\omega_{T_{i-1}}, \omega_{T_{i-1}+1}, \ldots, \omega_{T_i-1})$. Note that $Y = \{Y_i\}_{i\in\mathbb{N}} \in \widetilde{\mathbb{R}}^{\mathbb{N}}$.

We then define the empirical process associated to Y in analogy with (8):

$$R_n := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{\tilde{\theta}^i (Y_1,\ldots,Y_n)^{\mathrm{per}}}\,, \tag{12}$$

where we denote for clarity by $\tilde{\theta}$ the shift acting on $\widetilde{\mathbb{R}}$. By definition, R_n is a random element of the space $\mathcal{M}_1^{\mathrm{inv}}(\widetilde{\mathbb{R}}^{\mathbb{N}})$ of shift-invariant probabilities on $\widetilde{\mathbb{R}}^{\mathbb{N}}$.

We may look at Y and R_n in at least two ways: either under the law $P^* := \mathbb{P} \otimes P$ (*annealed way*), that is averaging on both ω and S, or under the law P (*quenched way*), that is for a fixed realization of ω and averaging on S only. We start with the annealed way.

- Show that under P^* the sequence Y is i.i.d. with marginal law q_0 given by

$$q_0(\mathrm{d}x_1, \ldots, \mathrm{d}x_n) = R(n)\, \mu_0(\mathrm{d}x_1) \ldots \mu_0(\mathrm{d}x_n)\,.$$

- Conclude from the preceding section that under P^* the sequence $\{R_n\}_{n\in\mathbb{N}}$ satisfies a LDP on $\mathcal{M}_1^{\mathrm{inv}}(\Gamma^{\mathbb{N}})$ with rate function $I^{\mathrm{ann}}(Q) = H(Q|\mu_0^{\otimes\mathbb{N}})$, defined in (10).

Intuitively, the probability under P^* that the first n words that S cuts out of ω, periodically extended to an infinite sequence, have an empirical distribution under shift close to a law $Q \in \mathcal{M}_1^{\mathrm{inv}}(\Gamma^{\mathbb{N}})$ decays exponentially in n with rate $I^{\mathrm{ann}}(Q)$: $P^*(R_n \approx Q) = \exp(-nI^{\mathrm{ann}}(Q) + o(n))$. We stress that $I^{\mathrm{ann}}(Q) \geq 0$ and $I^{\mathrm{ann}}(Q) = 0$ if and only if $Q = \mu_0^{\otimes\mathbb{N}}$.

We now consider the quenched viewpoint, that is we fix ω and we write R_n^ω instead of R_n for clarity. It is intuitively clear that, averaging only on S, it is more difficult to observe a deviation. Therefore, if under P the sequence $\{R_n^\omega\}_{n\in\mathbb{N}}$ satisfies a LDP on $\mathcal{M}_1^{\mathrm{inv}}(\Gamma^{\mathbb{N}})$ with rate function I^{que} — that is, if $P(R_n^\omega \approx Q) = \exp(-nI^{\mathrm{que}}(Q) + o(n))$ —, we should have $I^{\mathrm{que}}(Q) \geq I^{\mathrm{ann}}(Q)$. This is indeed the case: the difference between $I^{\mathrm{que}}(Q)$ and $I^{\mathrm{ann}}(Q)$ can be explicitly quantified, but we do not explore this issue further. For more details, cf. [1].

4 The empirical process of words and the pinning model

We now explore the link between the process of random words Y described in the previous section and our pinning model. Introduce for $z \in [0,1]$ the generating function

$$G(z) := \sum_{n\in\mathbb{N}} z^n \, Z_n^c(\beta,h,\omega)\,,$$

where $Z_n^c(\beta,h,\omega)$ denotes the *constrained* partition function, in analogy with the previous tutorials:

$$Z_n^c(\beta,h,\omega) \;=\; E\left[e^{\sum_{i=0}^{n-1}(\beta\omega_i - h)1_{\{w_i=*\}}}\, 1_{\{S_n=*\}}\right].$$

We recall that $Z_n^c(\beta,h,\omega)$ yields the same free energy as the original partition function $Z_n(\beta,h,\omega)$, that is

$$f^{\mathrm{que}}(\beta,h) = \lim_{n\to\infty}\frac{1}{n}\log Z_n^c(\beta,h,\omega)\,, \qquad \mathbb{P}\text{-a.s. and in } L^1(\mathrm{d}\mathbb{P})\,,$$

- Prove that the radius of convergence $\overline{z}$ of $G(z)$ equals $e^{-f^{\mathrm{que}}(\beta,h)}$.
- In analogy with the third tutorial, show that

$$z^n\, Z_n^c(\beta,h,\omega) \;=\; \sum_{N\in\mathbb{N}}\;\sum_{0=k_0<k_1<\ldots<k_N=n}\;\prod_{i=1}^N z^{k_i-k_{i-1}}\, R(k_i-k_{i-1})\, e^{\beta\omega_{k_{i-1}}-h}\,.$$

- Deduce that $G(z) = \sum_{N\in\mathbb{N}} F_N(\beta,h,\omega,z)$ where

$$\begin{aligned} F_N(\beta,h,\omega,z) \;&:=\; \sum_{0=k_0<k_1<\ldots<k_N<\infty}\;\prod_{i=1}^N z^{k_i-k_{i-1}}\, R(k_i-k_{i-1})\, e^{\beta\omega_{k_{i-1}}-h} \\ &=\; E\left(\prod_{i=1}^N z^{T_i-T_{i-1}}\, e^{\beta\omega_{T_{i-1}}-h}\right) \;=\; E\left(\exp\left(\sum_{i=1}^N\left((\log z)(T_i-T_{i-1}) + \beta\omega_{T_{i-1}} - h\right)\right)\right). \end{aligned}$$

Recall that R_n^ω denotes the empirical process of words defined in (12) and is a random element of the space $\mathcal{M}_1^{\mathrm{inv}}(\widetilde{\mathbb{R}}^{\mathbb{N}})$. The dependence on ω in R_n^ω is to stress that we take the quenched viewpoint, that is, we look at R_n^ω for a fixed (typical) realization of ω. Given a 'sentence' $y = (y_k)_{k\in\mathbb{N}} \in \widetilde{\mathbb{R}}^{\mathbb{N}}$, we denote by $y_1 \in \widetilde{\mathbb{R}}$ its first 'word'; for a 'word' $x \in \widetilde{\mathbb{R}}$, we denote by $\ell(x)$ the length of the 'word' x and by $c(x)$ the first 'letter' in the 'word' x.

- Recalling that $Y_i := (\omega_{T_{i-1}}, \omega_{T_{i-1}+1}, \ldots, \omega_{T_i-1})$, with the T_is defined in (11), prove that

$$m(R_N^\omega) \;:=\; \int_{\widetilde{\mathbb{R}}^{\mathbb{N}}} \ell(y_1)\, R_N^\omega(\mathrm{d}y) \;=\; \frac{1}{N}\sum_{i=1}^N \ell(Y_i) \;=\; \frac{1}{N}\sum_{i=1}^N (T_i - T_{i-1})\,,$$

$$\Phi(R_N^\omega) \;:=\; \int_{\widetilde{\mathbb{R}}^{\mathbb{N}}} c(y_1)\, R_N^\omega(\mathrm{d}y) \;=\; \frac{1}{N}\sum_{i=1}^N c(Y_i) \;=\; \frac{1}{N}\sum_{i=1}^N \omega_{T_{i-1}}\,,$$

  whence

$$F_N(\beta,h,\omega,z) \;=\; E\left(\exp\left(N\left[(\log z) m(R_N^\omega) + \beta\Phi(R_N^\omega)\right]\right)\right) e^{-hN}\,.$$

This shows that $F_N(\beta, h, \omega, z)$ is a function of R_N^ω. It is therefore clear that the properties of the generating function $G(z)$, in particular its radius of convergence $\overline{z}$ (hence the quenched free energy), can be deduced from the asymptotic properties of R_N^ω. Let us set

$$S^{\text{que}}(\beta, z) := \limsup_{N\to\infty} \frac{1}{N} \log E\left(\exp\left(N\left[(\log z) m(R_N^\omega) + \beta\Phi(R_N^\omega)\right]\right)\right) , \tag{13}$$

and $S^{\text{que}}(\beta, 1-) := \lim_{z\uparrow 1} S^{\text{que}}(\beta, z)$.

- Prove that if $h > S^{\text{que}}(\beta, z)$ then $G(z) < \infty$, while if $h < S^{\text{que}}(\beta, z)$ then $G(z) = \infty$.
- Deduce that if $S^{\text{que}}(\beta, 1-) < h$ then $f^{\text{que}}(\beta, h) = 0$, while if $S^{\text{que}}(\beta, 1-) > h$ then $f^{\text{que}}(\beta, h) > 0$. Therefore $h_c^{\text{que}}(\beta) = S^{\text{que}}(\beta, 1-)$.

Finally, using an important tool in large deviations theory (Varadhan's lemma), it can be shown from (13) that

$$h_c^{\text{que}}(\beta) = S^{\text{que}}(\beta, 1-) = \sup_{Q\in\mathcal{M}_1^{\text{inv}}(\widetilde{\mathbb{R}}^{\mathbb{N}})} \left(\beta\Phi(Q) - I^{\text{que}}(Q)\right).$$

This gives an explicit variational characterization of the quenched critical line. An analogous characterization holds for the annealed critical line too.

References

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