Complete intersection points on affine varieties and zero cycles

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Let $X = \operatorname{Spec} A$ be an irreducible affine variety of dimension d over an algebraically closed field k, so that the coordinate ring A of algebraic functions on X is a finitely generated k-algebra which is an integral domain of Krull dimension d. A *complete intersection point* of X is a point $x \in X$ such that the maximal ideal $\mathfrak{M}_x \subset A$ of functions vanishing at x is generated by d elements $f_1, \ldots, f_d \in \mathfrak{M}_x$. Geometrically, this means that $x \in X$ is a non-singular point, and the hypersurfaces $H_i = \{y \in X | f_i(y) = 0\}$ satisfy $H_1 \cap \cdots \cap H_d = \{x\}$, and the H_i intersect transversally near x.

More generally, if $X = \operatorname{Spec} A$ is a reduced affine k-variety of dimension d, a point x is a complete intersection point if $\mathfrak{M}_x$ has height d, and is generated by d elements. Define a point $x \in X$ to be a non-singular point if its local ring $\mathcal{O}_{X,x}$ is a regular local ring of dimension d; clearly a complete intersection point is non-singular.

We are interested here in characterizing varieties $X = \operatorname{Spec} A$ such that all non-singular points $x \in X$ are complete intersections. This problem turns out to have different flavours, depending on the ground field k, and is related to interesting conjectures in the theory of algebraic cycles, and thereby to algebraic K-theory.

In this talk, I will give an introduction to this topic, dwelling in particular on some results based on the paper [3], and applications to complete intersections in [4].

Here are two results, proved in [4], using the results of [3]. Let $\overline{\mathbb{F}}_p$ denote the algbraic closure of a finite field of char. p, and let $\overline{\mathbb{Q}}$ denote the field of algebraic numbers (algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$).

Theorem 1. *Let A be a finitely generated reduced $\overline{\mathbb{F}}_p$-algebra of dimension $d \geq 2$. Then any non-singular point of $X = \operatorname{Spec} A$ is a complete intersection point.*

Theorem 2. *Let $A = \oplus_{n\geq 0} A_n$ be a reduced, finitely generated $\overline{\mathbb{Q}}$-algebra of dimension $d \geq 2$. Then any non-singular point of $X = \operatorname{Spec} A$ is a complete intersection point.*

This leads to the following interesting examples (of "Bloch-Beilinson type"). Let $A = \mathbb{C}[x, y, z]/(f(x, y, z))$ be the homogeneous coordinate ring of a nonsingular plane curve of degree $d \geq 4$, where f has coefficients in $\overline{\mathbb{Q}}$. A non-singular point $(a, b, c) \in X = \operatorname{Spec} A \subset \mathbb{A}^3_{\mathbb{C}}$ is a complete intersection if and only if the point $[a : b : c] \in \operatorname{Proj} A \subset \mathbb{P}^2_{\mathbb{C}}$ is a $\overline{\mathbb{Q}}$-rational point (i.e., (a, b, c) is proportional to a triple of algebraic numbers).

Some survey articles giving background, detailed references, and explaining further the connections with commutative algebra, are [2] and [1].

References

[1] V. Srinivas, *Zero cycles on singular varieties*, in *The Arithmetic and Geometry of Algebraic Cycles*, ed. B. B. Gordon *et al.*, NATO Science Series Vol. C 548, Kluwer (2000), pp. 347-382.

[2] V. Srinivas, *Some Geometric Methods in Commutative Algebra*, in *Computational Commutative Algebra and Combinatorics (Osaka, 1999)*, Advanced Studies in Pure Math. 33 (2002) 231-276.

[3] Amalendu Krishna, V. Srinivas, *Zero cycles and K-theory on normal surfaces*, Annals of Math. 156 (2002) 155-195.

[4] Amalendu Krishna, V. Srinivas, *Zero cycles on singular varieties*, in J. Nagel, C. Peters, eds., *Algebraic Cycles and Motives, Volume 1*, London Math. Soc. Lect. Note. No. 343, Cambridge U. Press (2007) 264-277.