ON THE EXTENSION OF SOME PROJECTIVE VARIETIES

FRANCESCO RUSSO

The problem of classifying the extensions in $\mathbb{P}^{N+1}$ of a subvariety $Y \subset \mathbb{P}^N$ in such a way that $Y$ is a hyperplane section of $X$ was considered classically and it was one of the cornerstones in the classification of varieties of small degree, genus, index, etc. etc.

It immediately turned out that some special smooth varieties (for examples Veronese, Segre and Grassmann varieties, not hypersurfaces) cannot be hyperplane sections of smooth varieties and that for many classes there exist only the trivial extensions given by considering cones over $Y$ with vertex a point $p \in \mathbb{P}^{N+1} \setminus \mathbb{P}^N$.

There was a renewed interest in the problem (and in suitable further generalizations) in the modern age of Algebraic Geometry also in connection with the study of some moduli spaces of curves and with the deformation theory of isolated singularities.

Many sufficient conditions for the non-existence of non-trivial extensions of smooth varieties are known. These conditions are usually expressed by suitable vanishing of the first cohomology groups of negative twisting of the tangent bundle of $Y$ (or of its normal bundle in $\mathbb{P}^N$) by the hyperplane section. These results are general and concern a lot of applications, although in some cases the computations of these cohomology groups may be complicated.

In the first part of my talk I will try to survey and explain the above results and then I would like to present a simple geometrical sufficient condition of non-extendability for projective varieties uniruled by lines, using the ABC of the deformation theory of rational curves on algebraic varieties. In particular, we hope to be able to show that in many cases the appearance of the cone as the unique possible extension is very geometric and has a very clear geometrical meaning. Applications to the non-existence of non-trivial extensions of many homogenous varieties of high dimension are then immediate consequences of this condition and of the nesting principle.